Towards Optimal Subsidy Bounds for Envy-Freeable Allocations

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Abstract

We study the fair division of indivisible items with subsidies among n agents, where the absolute marginal valuation of each item is at most one. Under monotone valuations (where each item is a good), it is known that a maximum subsidy of $2(n-1)$ and a total subsidy of $2(n-1)^2$ are sufficient to guarantee the existence of an envy-freeable allocation. In this paper, we improve upon these bounds, even in a wider model. Namely, we show that, given an EF1 allocation, we can compute in polynomial time an envy-free allocation with a subsidy of at most $n-1$ per agent and a total subsidy of at most $n(n-1)/2$. Moreover, we present further improved bounds for monotone valuations.

Introduction

We consider the problem of *fairly* dividing items among agents. The notion of fairness that has been extensively studied in the literature is *envy-freeness* (Foley 1967). It requires that no agent wants to swap her bundle with another agent's. When the items to be allocated are divisible, the classical result ensures the existence of an envy-free allocation (Varian 1974). In contrast, when the items are indivisible, envy-freeness is not a reasonable goal. For instance, consider n agents with $n \geq 2$ and a single item valued at 1 by each agent. Allocating the only item to an agent results in envy from the other agents, as they get nothing. Thus, envy-free allocations may not exist when the items are indivisible.

One way to circumvent this issue is to relax the fairness requirement. For example, *envy-freeness up to one item* (EF1) requires that when agent i envies agent j, the envy can be eliminated by either (i) removing one item from agent j's bundle, or (ii) removing one item from agent i's bundle. It is known that an EF1 allocation is guaranteed to exist if each item is either a good or a chore for any agent, i.e., doubly monotone valuations (Lipton et al. 2004; Bhaskar, Sricharan, and Vaish 2021). Moreover, for general valuations, the existence of an EF1 allocation is assured when there are only 2 agents (Bérczi et al. 2020).

Another way to circumvent this issue is monetary compensation (subsidy). Since money is divisible, it can be a powerful tool to achieve envy-freeness. However, since the subsidy payments must be provided by an external agent (e.g., a government or a funding agency), it is desirable that the total subsidy amount is bounded. Thus, in this paper, we study the fair division of indivisible items with limited subsidies.

Most of the existing works on the fair division of indivisible items with limited subsidies focus on some special cases. For example, Maskin (1987) and Klijn (2000) consider the case that the number of agents and the number of items are equal and each agent can be allocated at most one item. Halpern and Shah (2019) consider an extended model where the number of agents and number of items may differ, and each agent can be allocated more than one item, assuming the valuation of each agent is binary additive. Babaioff, Ezra, and Feige (2021) and Goko et al. (2022) consider the case that the valuation of each agent is matroidal (which is not necessarily additive). Barman et al. (2022) examine a broader class of valuations in which the marginal valuation of each item is binary. As far as the authors are aware, the most general model considered so far is monotone valuations (Brustle et al. 2020), where the marginal contribution of each item is non-negative.

In this paper, we study the fair division of indivisible items with limited subsidies when the valuations are not restricted to be monotone. We assume that the valuations are normalized so that the absolute marginal value of each item is at most 1 (i.e., between -1 and 1).

For monotone valuations, which are special cases of our model, Brustle et al. (2020) show that envy-free allocation always exists with a subsidy of amount at most $2(n-1)$ per agent, and the total amount is $2(n-1)^2$. However, the only known lower bound on the total subsidy is $n-1$ (which can be obtained using the case with n agents and one item described at the beginning of this section), it remains an open question whether we can improve on the total subsidy bound of $2(n-1)^2$ for monotone valuations, as mentioned in a recent survey paper (Liu et al. 2023, Open Question 9).

Our Results

In this paper, we present improved upper bounds for the subsidies necessary to achieve envy-freeness. We demonstrate that, given an EF1 allocation, an envy-free allocation with a subsidy can be constructed in polynomial time where each agent receives a subsidy of at most $n-1$ and

$n(n-1)/2$ in total (Theorem 1). To prove these improved bounds, we reveal that the structure of the minimum subsidy vectors satisfies: (i) the minimum subsidy vector remains unchanged irrespective of the maximum weight matching (Lemma 2), and (ii) how the subsidies alter when the weights are changed (Lemma 4). When valuation functions are doubly monotone or there are only two agents, such envy-free allocations with limited subsidies can be computed in polynomial time. Notably, this improves upon the necessary subsidy amount for the existing case of monotone valuations, as monotone valuations are also doubly monotone. Furthermore, when $n = 2$, our obtained bounds are best possible since a subsidy of 1 is indispensable. We also show that from an EFk allocation (i.e., pairwise envy can be eliminated by removing at most k items), we can construct an envy-free allocation with a subsidy of at most $k(n-1)$ per agent and a total subsidy of $k \cdot n(n-1)/2$.

It is worth mentioning that our upper bounds of $n-1$ per agent and $n(n-1)/2$ in total cannot be improved when considering an arbitrary EF1 allocation (Example 1). We overcome this impossibility by making a slight modification of the bundles. To be exact, for three or more agents with monotone valuations, we improve the bounds further to $n - 1.5$ per agent and $(n^2 - n - 1)/2$ in total (Theorem 2).

Related Work

The concept of compensating an indivisible resource allocation with money has been prevalent in classical economics literature (Alkan, Demange, and Gale 1991; Maskin 1987; Klijn 2000; Moulin 2004; Sun and Yang 2003; Svensson 1983; Tadenuma and Thomson 1993). Much of the classical work has focused on the unit-demand case in which each agent is allocated at most one good. Examples include the famous rent-division problem of assigning rooms to several housemates and dividing the rent among them (Su 1999). It is known that, for a sufficient amount of subsidies, an envy-free allocation exists (Maskin 1987) and can be computed in polynomial time (Aragones 1995; Klijn 2000).

Most classical literature, however, has not considered a situation in which the number of items to be allocated exceeds the number of agents, in contrast to the rich body of recent literature on the multi-demand fair division problem. Halpern and Shah (2019) recently extended the model to the multi-demand setting wherein multiple items can be allocated to one agent. Despite the existence of numerous related papers, Halpern and Shah (2019) is the first work to study the asymptotic bounds on the amount of subsidy required to achieve envy-freeness. They showed that an allocation is envy-freeable with a subsidy if and only if the agents cannot increase the utilitarian social welfare by permuting bundles. This characterization implies that an allocation that can be made envy-free with a subsidy needs to satisfy some efficiency condition; hence, an approximately fair allocation, such as an EF1 allocation (Budish 2011), may not be an envy-freeable allocation. It was conjectured in (Halpern and Shah 2019) that, for additive valuations in which the value of each item is at most 1, giving at most 1 to each agent is sufficient to eliminate envies. Brustle et al. (2020) affirmatively settled this conjecture by designing an algorithm that iteratively solves a maximum-matching instance.

Babaioff, Ezra, and Feige (2021) and Benabbou et al. (2021) studied the fair allocation of indivisible items with matroidal valuations. The prioritized egalitarian mechanism proposed by Babaioff, Ezra, and Feige (2021) returns an allocation that maximizes the Nash welfare and achieves envy-freeness up to any good and utilitarian optimality. Goko et al. (2022) developed a strategy-proof, polynomial-time implementable mechanism called *subsidized egalitarian* mechanism, which requires a subsidy of the amount at most 1 per agent and $n-1$ in total. Furthermore, Barman et al. (2022) examined a more general model with dichotomous marginals and obtained the same bounds. Recently, Wu, Zhang, and Zhou (2023) examined the upper bound on the total subsidy required to ensure *proportionality*.

Caragiannis and Ioannidis (2022) studied the computational complexity of approximating the minimum amount of subsidies required to achieve envy-freeness. Aziz (2021) considered another fairness requirement, the so-called *equitability*, in conjunction with envy-freeness and characterized an allocation that can be made both equitable and envy-free with a subsidy. Narayan, Suzuki, and Vetta (2021) studied a related but different setting with transfer payments; they analyzed the impact of introducing some amount of transfers on the Nash welfare and utilitarian welfare while achieving envy-freeness.

Preliminaries

We model fair division with a subsidy as follows. For each natural number $k \in \mathbb{N}$, we denote $[k] = \{1, \ldots, k\}$. Let $N = [n]$ be the set of n agents and let $M = \{e_1, \ldots, e_m\}$ be the set of m indivisible items. Each agent i has a *valuation function*, denoted as $v_i \colon 2^M \to \mathbb{R}$, where $\mathbb{R}$ represents the set of real numbers. We assume that the functions v_i's are given as value oracles. In addition, we assume that the maximum marginal contribution of each item is at most one, i.e., $|v_i(X \cup \{e\}) - v_i(X)| \le 1$ for any $i \in N$, $e \in M$, and $X \subseteq M \setminus \{e\}$.

We define an item $e \in M$ as a *good* for agent $i \in N$ if $v_i(X \cup \{e\}) \ge v_i(X)$ for every $X \subseteq M \setminus \{e\}$. Additionally, we define an item $e \in M$ as a *chore* for agent $i \in N$ if $v_i(X \cup \{e\}) \le v_i(X)$ for every $X \subseteq M \setminus \{e\}$, with at least one of these inequalities being strict. An instance is said to be *monotone* if every $e \in M$ is a good for any agent $i \in N$. An instance is said to be *doubly monotone* if every item $e \in M$ is either a good or a chore for any agent $i \in N$.

An *allocation* is an ordered partition $\boldsymbol{A} = (A_1, \ldots, A_n)$ of M, i.e., $\bigcup_{i \in N} A_i = M$ and $A_i \cap A_j = \emptyset$ for any distinct $i, j \in N$. In allocation $\boldsymbol{A}$, each agent i receives the items of bundle A_i. A *subsidy* is a non-negative real vector $\boldsymbol{p} = (p_1, \ldots, p_n) \in \mathbb{R}_+^N$, where p_i is the amount of money given to agent $i \in N$. In an allocation with a subsidy $(\boldsymbol{A}, \boldsymbol{p})$, the utility of each agent i is $v_i(A_i) + p_i$. An allocation with a subsidy $(\boldsymbol{A}, \boldsymbol{p})$ is *envy-free* if $v_i(A_i) + p_i \ge v_i(A_j) + p_j$ for any pair of agents $i, j \in N$. Our goal is to find an envy-free allocation with a subsidy $(\boldsymbol{A}, \boldsymbol{p})$ such that the total subsidy $\sum_{i \in N} p_i$ (or maximum subsidy $\max_{i \in N} p_i$) is minimized.

An allocation $\boldsymbol{A}$ is called *envy-free up to one item (EF1)* if, for all $i, j \in [n]$, it holds that $v_i(A_i \setminus X) \ge v_i(A_j \setminus$

X) for some $X \subseteq A_i \cup A_j$ with $|X| \leq 1$. Similarly, an allocation $\boldsymbol{A}$ is called *envy-free up to k items (EFk)* if, for all $i, j \in [n]$, it holds that $v_i(A_i \setminus X) \geq v_i(A_j \setminus X)$ for some $X \subseteq A_i \cup A_j$ with $|X| \leq k$. It is known that an EF1 allocation always exists and can be found in polynomial time if the valuations are doubly monotone (Lipton et al. 2004; Bhaskar, Sricharan, and Vaish 2021) or $n = 2$ (Bérczi et al. 2020).

Envy-free Structure of Subsidies

An allocation $\boldsymbol{A}$ is called *envy-freeable* if there exists a subsidy vector $\boldsymbol{p}$ such that $(\boldsymbol{A}, \boldsymbol{p})$ is envy-free. We call such a subsidy vector *envy-eliminating*. In this subsection, we describe the structure of envy-eliminating subsidies.

We fix an allocation $\boldsymbol{A} = (A_1, \dots, A_n)$. Let $w \in \mathbb{R}^{[n]\times[n]}$ be a weight matrix such that $w_{i,j} = v_i(A_j)$ for each $i, j \in [n]$. For a permutation σ of $[n]$, let $\boldsymbol{A}^\sigma = (A_{\sigma(1)}, \dots, A_{\sigma(n)})$ and let P^σ be the set of envy-eliminating subsidy vectors:

$$P^\sigma = \left\{ \boldsymbol{p} \in \mathbb{R}^n_+ \,\middle|\, w_{i,\sigma(i)} + p_{\sigma(i)} \geq w_{i,j} + p_j \ (\forall i, j \in [n]) \right\}.$$

We call a permutation σ *maximum weight permutation* for w if it maximizes $\sum_{i=1}^n w_{i,\sigma(i)}$ among permutations. Halpern and Shah (2019) proved that an allocation $\boldsymbol{A}$ is envy-freeable if and only if $\sum_{i=1}^n v_i(A_i) \geq \sum_{i=1}^n v_i(A_{\sigma(i)})$ for any permutation σ of $[n]$. From this result, the polyhedron P^σ is non-empty if σ is a maximum weight permutation. The polyhedron P^σ contains a unique minimal element (i.e., a vector $\boldsymbol{p}$ such that $\boldsymbol{p} \leq \boldsymbol{p}'$ for any $\boldsymbol{p}' \in P^\sigma$) because it is lower bounded by non-negativity and $\boldsymbol{p}, \boldsymbol{p}' \in P^\sigma$ implies that $(\min\{p_i, p'_i\})_{i\in[n]}$ is also in P^σ. The unique minimal vector is called the *minimum subsidy vector* for w with respect to σ. Note that, for $\boldsymbol{p} \in P^\sigma$, the subsidy p_i is associated with the bundle A_i, not the agent i.

For the pair (w, σ) of a weight w and a permutation σ of $[n]$, we define the *envy graph* $G^{w,\sigma} = (V, E; \gamma)$ as a weighted directed complete graph in which each agent is a vertex (i.e., $V = [n]$), and each edge $(i, j) \in E$ $(= \{(i', j') \mid i', j' \in [n], i' \neq j'\})$ has a weight $\gamma_{i,j} = w_{i,\sigma(j)} - w_{i,\sigma(i)}$. Note that $\gamma_{i,j}$ represents the envy from i towards j in allocation $\boldsymbol{A}^\sigma$. The minimum subsidy vector can be characterized by using this envy graph.

Lemma 1 (Halpern and Shah (2019, Theorem 2)). *For any maximum weight permutation σ, the minimum subsidy $p_{\sigma(i)}$ is the maximum length of any path in $G^{w,\sigma}$ starting at i.*

It should be noted that the envy graph $G^{w,\sigma}$ does not contain any positive-weight directed cycle if σ is a maximum weight permutation. Hence, we only need to consider simple paths. Although there may exist several maximum weight permutations, the following lemma states that the corresponding minimum subsidy vectors are identical.

Lemma 2. *Let σ and σ' be maximum weight permutations for w. Also, let $\mathbf{p}$ and $\mathbf{p}'$ be the minimum subsidy vectors for w with respect to σ and σ', respectively. Then, $\boldsymbol{p} = \boldsymbol{p}'$.*

Proof. It is sufficient to prove that $\boldsymbol{p} \in P^{\sigma'}$ and $\boldsymbol{p}' \in P^\sigma$. In addition, by symmetry, it is sufficient to show only the former.

Define a vector $\boldsymbol{q} \in \mathbb{R}^n$ as $q_i = w_{i,\sigma(i)} + p_{\sigma(i)}$ for each $i \in [n]$ and a weight $w' \in \mathbb{R}^{[n]\times[n]}$ as $w'_{i,j} = w_{i,j} + p_j - q_i$ for each $i, j \in [n]$. By definition of $\boldsymbol{p}$, we have $w'_{i,j} = (w_{i,j} + p_j) - (w_{i,\sigma(i)} + p_{\sigma(i)}) \leq 0$ $(\forall i, j \in [n])$ and $w'_{i,\sigma(i)} = 0$ $(\forall i \in [n])$.

For any permutation π of $[n]$, the difference between the total weights of w and w' is

$$\sum_{i\in[n]} w_{i,\pi(i)} - \sum_{i\in[n]} w'_{i,\pi(i)} = \sum_{i\in[n]} p_i - \sum_{i\in[n]} q_i,$$

which is a constant independent of π. Thus, σ and σ' are maximum weight permutations for w', and hence the total weight of σ' for w' is $\sum_{i\in[n]} w'_{i,\sigma'(i)} = \sum_{i\in[n]} w'_{i,\sigma(i)} = 0$. As w' is nonpositive, it holds that $w'_{i,\sigma'(i)} = 0$ for every $i \in [n]$. Thus, for any $i, j \in [n]$, we have

$$\begin{aligned} w_{i,\sigma'(i)} + p_{\sigma'(i)} &= w'_{i,\sigma'(i)} + q_i = q_i \\ &\geq w'_{i,j} + q_i = w_{i,j} + p_j, \end{aligned}$$

where the inequality holds by $w'_{i,j} \leq 0$. Hence, $\boldsymbol{p}$ must be in $P^{\sigma'}$. □

From this lemma, the minimum subsidy vector is determined for an allocation without specifying a maximum weight permutation.

It should be noted that the subsidy vector $\boldsymbol{p}$ and the vector $\boldsymbol{q} \in \mathbb{R}^n$ defined in the proof of Lemma 2 can be viewed as dual variables of an assignment problem. Here, the primal of the assignment problem is

$$\begin{array}{lll} \max & \sum_{i=1}^n \sum_{j=1}^n w_{i,j} x_{i,j} & \\ \text{s.t.} & \sum_{j=1}^n x_{i,j} = 1 & (\forall i \in [n]), \\ & \sum_{i=1}^n x_{i,j} \geq 1 & (\forall j \in [n]), \\ & x_{i,j} \geq 0 & (\forall i, j \in [n]), \end{array}$$

and its dual is

$$\begin{array}{lll} \min & \sum_{i=1}^n q_i - \sum_{j=1}^n p_j & \\ \text{s.t.} & w_{i,j} + p_j - q_i \leq 0 & (\forall i, j \in [n]), \\ & p_j \geq 0 & (\forall j \in [n]). \end{array}$$

By the complementary slackness, we have $x_{i,j}(q_i - w_{i,j} - p_j) = 0$ $(\forall i, j \in [n])$ if $\boldsymbol{x}$ and $(\boldsymbol{p}, \boldsymbol{q})$ are optimal solutions.

The maximum weight permutation can be computed in polynomial time via a maximum-weight bipartite perfect matching algorithm. The minimum subsidy vector for w can be computed in polynomial time by a shortest path algorithm (e.g., the Floyd–Warshall algorithm).

Bounding Subsidy for EF1 Allocations

In this section, we prove the following key lemma.

Lemma 3. *Let $w \in \mathbb{R}^{[n]\times[n]}$ be a weight matrix. We denote the sequence of numbers in descending order of $(\max_{j\in[n]}(w_{i,j} - w_{i,i}))_{i\in[n]}$ as $(\beta_1, \dots, \beta_n)$, i.e., $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$. Let p^* be the minimum subsidy vector for w. Then, the rth largest value among $p^*_1, \dots, p^*_n$ is at most $\sum_{\ell=1}^{n-r} \beta_\ell$ for $r = 1, 2, \dots, n$.*

We remark that the identical permutation id may not be a maximum weight permutation. Thus, this lemma cannot be directly derived from Lemma 1 because $G^{w,\mathtt{id}}$ can contain a positive directed cycle that results in an infinitely long path.

For an EF1 allocation $\boldsymbol{A} = (A_1, \dots, A_n)$, if we set the weight matrix as $w = (v_i(A_j))_{i,j\in[n]}$, then the numbers β_i ($i \in [n]$) in Lemma 3 are nonnegative and at most 1 because $v_i(A_i) \geq v_i(A_j) - 1$ for all $i, j \in [n]$. Thus, the rth largest value in the minimum subsidy vector is at most $n-r$. Recall that an EF1 allocation $\boldsymbol{A}$ can be found in polynomial time if the valuations are doubly monotone (Bhaskar, Sricharan, and Vaish 2021) or $n = 2$ (Bérczi et al. 2020). Additionally, a maximum weight permutation σ and the minimum subsidy vector can be computed in polynomial time. Thus, from Lemma 3, we can obtain the following theorem.

Theorem 1. *If the valuations are doubly monotone or $n = 2$, there exists an envy-free allocation with a subsidy $(\boldsymbol{A}, \boldsymbol{p})$ such that $\max_{i=1}^n p_i \leq n-1$ and $\sum_{i=1}^n p_i \leq \sum_{\ell=1}^n (n-\ell) = n(n-1)/2$. Moreover, such an envy-free allocation with a subsidy can be computed in polynomial time.*

Lemma 3 also implies that, even when the valuations are general and we only have an EFk allocation, we can still derive an envy-free allocation with a subsidy of at most $k(n-1)$ per agent and $k \cdot n(n-1)/2$ in total.

We first observe that the bound of Theorem 1 cannot be improved even if the valuations are monotone and additive as long as an arbitrary EF1 allocation is considered.

Example 1. *Consider an instance with $M = \{e_{i,j} \mid i \in [n],\, j \in [n+1]\}$. The valuation of agent $i \in [n]$ for an item $e_{i',j}$ ($i' \in [n], j \in [n+1]$) is*

$$v_i(\{e_{i',j}\}) = \begin{cases} n/(n+1) & \text{if } i' = i, \\ 1 & \text{if } i' = i-1, \\ 0 & \text{otherwise}. \end{cases}$$

Let $A_i = \{e_{i,1}, \dots, e_{i,n+1}\}$ for each $i \in [n]$. Then, $\boldsymbol{A} = (A_1, \dots, A_n)$ is an envy-freeable EF1 allocation. Indeed, the weight matrix $w = (v_i(A_j))_{i,j\in[n]}$ is

$$\begin{array}{cc} & \begin{array}{ccccc} A_1 & A_2 & \cdots & A_{n-1} & A_n \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ n-1 \\ n \end{array} & \begin{pmatrix} n & & & & \\ n+1 & n & & & \\ & n+1 & \ddots & & \\ & & \ddots & n & \\ & & & n+1 & n \end{pmatrix} \end{array}.$$

It is not difficult to see that the identity permutation id is a maximum weight one, and a path that visits $n, n-1, \dots, 2, 1$ in this order is the longest one in $G^{w,\mathtt{id}}$. Thus, the minimum subsidy vector for $\boldsymbol{A}$ is $\boldsymbol{p} = (0, 1, \dots, n-1)$. Hence, $\max_{i\in[n]} p_i = n-1$ and $\sum_{i\in[n]} p_i = n(n-1)/2$ hold.

In what follows, we will prove Lemma 3. Let $\boldsymbol{A} = (A_1, \dots, A_n)$ be an allocation and let $\boldsymbol{p}^*$ be the minimum subsidy vector for the weight matrix $w = (v_i(A_j))_{i,j\in[n]}$. Let σ^* be a maximum weight permutation for w. Note that the envy graph G^{w,σ^*} may have an edge with a large weight. Thus, the proof is not straightforward in such a case.

To prove the theorem, we use the following notations. For each $i \in [n]$, we denote $q_i = \max_{j\in[n]}(v_i(A_j) + p^*_j)$, i.e., the maximum valuation for agent i over bundles including subsidies, and $r_i = -(v_i(A_i) + p^*_i - q_i)$, i.e., the maximum envy for agent i. Also, let us define

$$\hat{w}_{i,j} = \begin{cases} v_i(A_i) + r_i & \text{if } i = j, \\ v_i(A_j) & \text{if } i \neq j \end{cases} \tag{1}$$

for each $i, j \in [n]$. Note that $r_i \geq 0$ and $\hat{w}_{i,j} \geq w_{i,j}$ for any $i, j \in [n]$. We write $\hat{\boldsymbol{p}}$ for the minimum subsidy vector for $\hat{w}$. We demonstrate that $\hat{\boldsymbol{p}} = \boldsymbol{p}^*$ holds.

Lemma 4. *The identity permutation is a maximum weight permutation for $\hat{w}$. Moreover, the minimum subsidy vectors $\boldsymbol{p}^*$ and $\hat{\boldsymbol{p}}$ are the same.*

Proof. Since $(\boldsymbol{A}^{\sigma^*}, \boldsymbol{p}^*)$ is envy-free by Lemma 2, we have $q_i = \max_{j\in[n]}(v_i(A_j) + p^*_j) = v_i(A_{\sigma^*(i)}) + p^*_{\sigma^*(i)}$. Thus,

$$w_{i,j} + p^*_j - q_i = v_i(A_j) + p^*_j - q_i \leq 0 \tag{2}$$

for all $i, j \in [n]$, and

$$w_{i,\sigma^*(i)} + p^*_{\sigma^*(i)} - q_i = v_i(A_{\sigma^*(i)}) + p^*_{\sigma^*(i)} - q_i = 0 \tag{3}$$

for all $i \in [n]$. By the definition of $\hat{w}$, we have

$$\hat{w}_{i,j} + p^*_j - q_i = w_{i,j} + p^*_j - q_i \leq 0 \tag{4}$$

for all $i, j \in [n]$ with $i \neq j$ and

$$\hat{w}_{i,i} + p^*_i - q_i = w_{i,i} + p^*_i - q_i + r_i = 0, \tag{5}$$

for all $i \in [n]$, where the inequality holds by (2). In addition, for any $i \in [n]$ with $i \neq \sigma^*(i)$, we have

$$\hat{w}_{i,\sigma^*(i)} + p^*_{\sigma^*(i)} - q_i = w_{i,\sigma^*(i)} + p^*_{\sigma^*(i)} - q_i = 0 \tag{6}$$

by (3). Thus, for each $i \in [n]$, we obtain

$$\hat{w}_{i,\sigma^*(i)} = q_i - p^*_{\sigma^*(i)} = w_{i,\sigma^*(i)} \tag{7}$$

since the first equality holds by (5) and (6) and the second equality holds by (3).

By (4) and (5), the total weight $\sum_{i=1}^n \hat{w}_{i,\sigma(i)}$ is at most $\sum_{i\in[n]} q_i - \sum_{j\in[n]} p^*_j$ for any permutation σ. Thus, σ^* and the identical permutation id are maximum weight permutations for $\hat{w}$ since the total weight of σ^* and id for $\hat{w}$ are $\sum_{i\in[n]} q_i - \sum_{j\in[n]} p^*_i$ by (5) and (6). Moreover, $\boldsymbol{p}^*$ is an envy-eliminating subsidy for $\hat{w}$ (with respect to σ^*) since $\hat{w}_{i,\sigma^*(i)} + p^*_{\sigma^*(i)} = q_i \geq \hat{w}_{i,j} + p^*_j$ for any $i, j \in [n]$ by (4), (5), and (6).

As $\boldsymbol{p}^*$ is an envy-eliminating subsidy for $\hat{w}$, we have $\hat{\boldsymbol{p}} \leq \boldsymbol{p}^*$. To prove that $\hat{\boldsymbol{p}} = \boldsymbol{p}^*$, what is left is to show that $\hat{\boldsymbol{p}}$ is an envy-eliminating subsidy vector for w.

Define $\hat{q}_i = \max_{j\in[n]}(\hat{w}_{i,j} + \hat{p}_j)$ for each $i \in [n]$. Since $\hat{\boldsymbol{p}}$ is the minimum subsidy vector for $\hat{w}$ with respect to σ^* by Lemma 2, we have

$$\hat{w}_{i,\sigma^*(i)} + \hat{p}_{\sigma^*(i)} - \hat{q}_i = 0 \quad (\forall i \in [n]). \tag{8}$$

Thus, for each $i \in [n]$, we have

$$\begin{aligned} w_{i,\sigma^*(i)} + \hat{p}_{\sigma^*(i)} &= \hat{w}_{i,\sigma^*(i)} + \hat{p}_{\sigma^*(i)} = \hat{q}_i \\ &= \max_{j\in[n]}(\hat{w}_{i,j} + \hat{p}_j) \geq \max_{j\in[n]}(w_{i,j} + \hat{p}_j), \end{aligned}$$

where the first equality holds by (7), the second equality holds by (8), and the last inequality holds by $\hat{w}_{i,j} \geq w_{i,j}$ for any $i, j \in [n]$. Therefore, $\hat{\boldsymbol{p}}$ is an envy-eliminating subsidy vector for w, which completes the proof. □

By definition, the weight of each edge in $G^{\hat{w},\mathtt{id}}$ is at most that of the corresponding edge in $G^{w,\mathtt{id}}$ because

$$\hat{w}_{i,j} - \hat{w}_{i,i} = w_{i,j} - (w_{i,i} + r_i) \leq w_{i,j} - w_{i,i}$$

for any $i, j \in [n]$ with $i \neq j$. By combining Lemma 4 with Lemma 1, we prove Lemma 3.

Proof of Lemma 3. Recall that $\hat{\boldsymbol{p}} = \boldsymbol{p}^*$ and id is a maximum weight permutation by Lemma 4. For each $i \in [n]$, let $P_i \subseteq E$ be a longest path in $G^{\hat{w},\mathtt{id}}$ starting from i. By Lemma 1, $\hat{p}_i$ $(= p^*_i)$ is the length of P_i in $G^{\hat{w},\mathtt{id}}$. Note that P_i is a simple path. As the weight of each edge in $G^{\hat{w},\mathtt{id}}$ is at most that of the corresponding edge in $G^{w,\mathtt{id}}$, we have

$$\begin{aligned} p^*_i &= \textstyle\sum_{(s,t)\in P_i} (\hat{w}_{s,t} - \hat{w}_{s,s}) \\ &\leq \textstyle\sum_{(s,t)\in P_i} (w_{s,t} - w_{s,s}) \leq \sum_{\ell=1}^{|P_i|} \beta_\ell \end{aligned} \tag{9}$$

for each $i \in [n]$.

What is left is to provide upper bounds of the numbers of edges in the longest paths $P_1, \dots, P_n$. Let $S = \bigcup_{i\in[n]} P_i$. Without loss of generality, we may assume that, if two paths P_i and P_j share a common vertex, all of the edges that follow the vertex in these two paths are identical. Then, S is a directed forest and $|S| \leq n - 1$. We relabel the vertices as $|P_1| \geq |P_2| \geq \cdots \geq |P_n|$. Then, $|P_1| \leq n - 1$ since $P_1 \subseteq S$ and $|S| \leq n - 1$. As P_1 is longest among $P_1, \dots, P_n$, every path in $P_2, \dots, P_n$ does not use vertex 1. Indeed, if P_i $(i > 1)$ passes vertex 1, then by the assumption, we have $|P_i| > |P_1|$. Hence, $\bigcup_{i=2}^n P_i$ forms a directed forest that does not contain vertex 1, and $|P_2| \leq n-2$. By repeatedly applying the same argument, we have $|P_i| \leq n - i$ for $i = 1, 2, \dots, n$.

Therefore, for $r = 1, 2, \dots, n$, the rth largest value in p^* is at most $\sum_{\ell=1}^{|P_r|} \beta_\ell \leq \sum_{\ell=1}^{n-r} \beta_\ell$ by (9). □

Improved Bounds for Monotone Valuations

In this section, we provide an improved upper bound of subsidy when the valuations are monotone. As observed in Example 1, a maximum subsidy of $n - 1$ is required to guarantee envy-freeness for an EF1 allocation. We demonstrate that the upper bound can be improved by slightly modifying a given EF1 allocation. Formally, we present the following theorem.

Theorem 2. *If $n \geq 3$ and the valuations are monotone, there exists an envy-free allocation with a subsidy $(\boldsymbol{A}, \boldsymbol{p})$ such that $\max_{i\in[n]} p_i \leq n - 1.5$ and $\sum_{i\in[n]} p_i \leq (n^2 - n - 1)/2$. Moreover, such an envy-free allocation with a subsidy can be computed in polynomial time.*

Note that if $n = 2$, Theorem 1 implies that there exists an envy-free allocation with a subsidy where only one agent receives a subsidy of at most 1. This bound cannot be improved even when there is one item with a value of 1 for each agent.

In what follows, we assume that $n \geq 3$. We describe that we can obtain in polynomial time an EF1 allocation $\boldsymbol{A}$ satisfying $\sum_{i\in[n]} v_i(A_i) \geq \sum_{i\in[n]} v_i(A_{\sigma(i)})$ for any permutation σ such that $\boldsymbol{A}^\sigma$ is EF1. We first compute an EF1 allocation $\boldsymbol{X}$ in polynomial time using the envy-cycles algorithm (Lipton et al. 2004). Next, we modify $\boldsymbol{X}$ as follows. Construct a bipartite graph $([n], [n]; E)$ where an edge $(i, j) \in E$ exists if and only if the EF1 criterion still holds for agent i when we swap bundles of agents i and j, i.e., $v_i(X_j) \geq \min_{Y\subseteq X_k : |Y|\leq 1} v_i(X_k \setminus Y)$ for all $k \in [n]$. We assign the weight of an edge $(i, j) \in E$ as $v_i(X_j)$. Then we find the maximum weight perfect matching on the bipartite graph. By permutating bundles according to the matching, we can obtain the desired allocation $\boldsymbol{A}$.

Define $w = (v_i(A_j))_{i,j\in[n]}$. Let $\boldsymbol{p}^*$ be the minimum subsidy vector for w. In addition, let $q_i = \max_j (v_i(A_j) + p^*_j)$ and $r_i = -(v_i(A_i) + p^*_i - q_i)$ for each $i \in [n]$. Let $\hat{w}$ be the weights defined as (1).

A main task in the proof of Theorem 2 is to show that there exists an allocation with a subsidy $(\boldsymbol{A}'', \boldsymbol{p}'')$ such that $\max_{i\in[n]} p''_i \leq n - 1.5$ by modifying $(\boldsymbol{A}, \boldsymbol{p}^*)$. We assume that $\max_{i\in[n]} p^*_i > n - 1.5$ since otherwise (i.e., $\max_{i\in[n]} p^*_i \leq n - 1.5$) we have $\sum_{i\in[n]} p^*_i \leq \sum_{k=1}^{n-1} \min\{k,\, n - 1.5\} = (n^2 - n - 1)/2$ by Lemma 3. Here, $\beta_i \leq 1$ $(\forall i \in [n])$ in the lemma since $\boldsymbol{A}$ is EF1. By Lemma 4, $\boldsymbol{p}^*$ is the minimum subsidy vector for $\hat{w}$, and id is a maximum weight permutation for $\hat{w}$. Since $r_i \geq 0$ $(\forall i \in [n])$, the weight of edge (i, j) in $G^{\hat{w},\mathtt{id}}$ is

$$\begin{aligned} \hat{w}_{i,j} - \hat{w}_{i,i} &= v_i(A_j) - (v_i(A_i) + r_i) \\ &\leq v_i(A_j) - v_i(A_i) \leq 1. \end{aligned} \tag{10}$$

By Lemma 1, the length of a longest path in $G^{\hat{w},\mathtt{id}}$ is $\max_{i\in[n]} p^*_i$ $(> n - 1.5)$. As each edge weight is at most 1 by (10), the longest path must contain all the vertices and $n - 1$ positive weight edges. Without loss of generality, assume that $(n, n-1, \dots, 1)$ is the longest path (see Figure 1).

Let $s_i = 1 + r_i + v_i(A_i) - v_i(A_{i-1})$ for $i = 2, 3, \dots, n$. Note that $1 - s_i$ is the positive weight of $(i, i - 1)$, and

$$0 \leq r_i \leq s_i \leq 1. \tag{11}$$

For each $i \in [n]$, the path $(i, i - 1, \dots, 1)$ must be a longest path starting at i in $G^{\hat{w},\mathtt{id}}$. This is because, if there was a longer path starting at i, we could replace the subpath of the longest path $(n, n - 1, \dots, 1)$ starting from i with this longer path, thereby creating a longer path, contradicting the assumption that $(n, n-1, \dots, 1)$ is a longest path. Thus, for each $i \in [n]$, it holds that $p^*_i = \sum_{j=2}^{i} (1 - s_j)$. In addition, since $(n, n - 1, \dots, 1)$ is a longest path, $\max_{i\in[n]} p^*_i$ is achieved by $i = n$. Then since $\sum_{i=2}^n (1 - s_i) = \max_{i\in[n]} p^*_i > n - 1.5$, we have

$$0 \leq \textstyle\sum_{i=2}^n r_i \leq \sum_{i=2}^n s_i < 0.5. \tag{12}$$

Next, we observe that the weight of each edge, from a vertex with a lower index to a higher index, is small. This observation will be used to evaluate modified allocations.

Lemma 5. *For $i, j \in [n]$ with $i < j$, it holds that*

$$v_i(A_j) - v_i(A_i) - r_i \leq -\textstyle\sum_{k=i+1}^{j} (1 - s_k).$$

Proof. As the identity permutation is a maximum weight permutation for $\hat{w}$ by Lemma 4, envy graph $G^{\hat{w},\mathtt{id}}$ contains

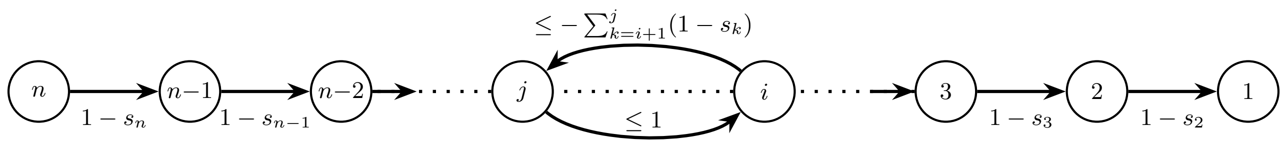

Figure 1: The envy graph $G^{\hat{w},\mathtt{id}}$

no positive-weight directed cycle. Hence, for $i, j \in [n]$ with $i < j$, we have

$$(\hat{w}_{i,j} - \hat{w}_{i,i}) + \textstyle\sum_{k=i+1}^{j}(\hat{w}_{k,k-1} - \hat{w}_{k,k}) \leq 0.$$

Since $\hat{w}_{k,k-1} - \hat{w}_{k,k} = 1 - s_k$, we obtain $v_i(A_j) - v_i(A_i) - r_i = \hat{w}_{i,j} - \hat{w}_{i,i} \leq -\sum_{k=i+1}^{j}(1-s_k)$. $\square$

For $i = 1$, we can also obtain the following bound.

Lemma 6. *For every $j \in \{2, 3, \dots, n\}$, we have*

$$v_1(A_j) \leq \max\{v_1(A_1) - (1 - s_2),\ \min_{e \in A_1} v_1(A_1 \setminus \{e\})\}.$$

Proof. Let $j^* \in \arg\max_{j \in \{2,3,\dots,n\}} v_1(A_j)$ and $\boldsymbol{A}^{(j^*)} = (A_{j^*}, A_1, A_2, \dots, A_{j^*-1}, A_{j^*+1}, \dots, A_n)$. It is sufficient to prove that $v_1(A_{j^*}) \leq v_1(A_1) - (1 - s_2)$ under the assumption that $v_1(A_{j^*}) > \min_{e \in A_1} v_1(A_1 \setminus \{e\})$.

In the allocation $\boldsymbol{A}^{(j^*)}$, each agent $j \in \{2, 3, \dots, n\}$ does not get worse than $\boldsymbol{A}$ because $1 - s_j \geq 0$, and thus the EF1 criterion is still satisfied for agent j. By the choice of j^*, we have $v_1(A_{j^*}) \geq v_1(A_j)$ for all $j \in \{2, 3, \dots, n\}$. Hence, the allocation $\boldsymbol{A}^{(j^*)}$ is EF1 if $v_1(A_{j^*}) > \min_{e \in A_1} v_1(A_1 \setminus \{e\})$.

By the definition of $\boldsymbol{A}$ and the fact that $\boldsymbol{A}^{(j)}$ is an EF1 allocation, we have $\sum_{j \in [n]} v_j(A_j) \geq \sum_{j \in [n]} v_j(A_j^{(j^*)})$. This implies that

$$\begin{aligned} v_1(A_1) - v_1(A_{j^*}) &\geq \textstyle\sum_{j=2}^{j^*}(v_j(A_{j-1}) - v_j(A_j)) \\ &= \textstyle\sum_{j=2}^{j^*}(1 - s_j + r_j) \geq 1 - s_2, \end{aligned}$$

by (12). Consequently, we obtain that $v_1(A_j) \leq v_1(A_{j^*}) \leq v_1(A_1) - (1 - s_2)$ for every $j \in \{2, 3, \dots, n\}$. $\square$

As $\boldsymbol{A}$ is an EF1 allocation and $1 - s_2 > 0$, we choose $e^* \in A_1$ such that $v_2(A_2) \geq v_2(A_1 \setminus \{e^*\})$. Define

$$\boldsymbol{A}' = (A_1 \setminus \{e^*\}, A_2, A_3, \dots, A_{n-1}, A_n \cup \{e^*\})$$

and let w' be the weights such that $w'_{i,j} = v_i(A'_j)$ $(i, j \in [n])$. We demonstrate that the minimum subsidy vector $\boldsymbol{p}'$ for w' satisfies the conditions that $\max_{i \in [n]} p'_i \leq n - 1.5$.

Before we proceed to the proof, we observe the effect of this modification to the minimum subsidy vector for the instance in Example 1.

Example 2. *Consider the instance observed in Example 1. Then, $\boldsymbol{A}$ in the example is an EF1 allocation such that $\sum_{i \in [n]} v_i(A_i) = n^2 \geq \sum_{i \in [n]} v_i(A_{\sigma(i)})$ for any permutation σ. Let $e^* = e_{1,1}$ and consider $\boldsymbol{A}' = (A_1 \setminus \{e^*\}, A_2, A_3, \dots, A_{n-1}, A_n \cup \{e^*\})$. Then the valuations for the bundles are*

$$(v_i(A'_j))_{i,j \in [n]} = \begin{array}{cc} & \begin{array}{ccccc} A'_1 & A'_2 & \cdots & A'_{n-1} & A'_n \end{array} \\ \begin{array}{c} 1 \\ 2 \\ \vdots \\ n-1 \\ n \end{array} & \begin{pmatrix} \frac{n^2}{n+1} & & & & \frac{n}{n+1} \\ n & n & & & \\ & n+1 & \ddots & & \\ & & \ddots & n & \\ & & & n+1 & n \end{pmatrix} \end{array}$$

and the minimum subsidy vector for this allocation is

$$(0, 0, 1, \dots, n-2).$$

Let us now proceed with the proof. To provide upper bounds with Lemma 3, we analyze the structure of $G^{w',\mathtt{id}}$.

Lemma 7. *For each $i, j \in [n]$, the weight of edge (i, j) in $G^{w',\mathtt{id}}$ is*

$$\begin{aligned} w'_{i,j} - w'_{i,i} &= v_i(A'_j) - v_i(A'_i) \\ &\leq \begin{cases} \max\{0.5,\ v_1(A'_n) - v_1(A'_1)\} & \textit{if } i = 1, \\ 0.5 & \textit{if } i = 2, \\ 1 & \textit{if } i \geq 3. \end{cases} \end{aligned}$$

Proof. Let $i, j \in [n]$. If $i = j$, the claim clearly holds as $w'_{i,j} - w'_{i,i} = 0$. Hence, we assume that $i \neq j$.

Case 1. $i = 1$. If $2 \leq j < n$, we have

$$\begin{aligned} w'_{i,j} - w'_{i,i} &= v_1(A_j) - v_1(A'_1) \\ &\leq \max\left\{ \begin{array}{l} v_1(A_1) - (1 - s_2), \\ \min_{e \in A_1} v_1(A_1 \setminus \{e\}) \end{array} \right\} - v_1(A_1 \setminus \{e^*\}) \\ &\leq \max\{s_2, 0\} = s_2 < 0.5 \end{aligned}$$

by Lemma 6 and (12). If $j = n$, we have $w'_{i,j} - w'_{i,i} = v_1(A'_n) - v_1(A'_1)$.

Case 2. $i = 2$. If $j = 1$, we have $w'_{i,j} - w'_{i,i} = v_2(A_1 \setminus \{e^*\}) - v_2(A_2) \leq 0$ by the choice of e^*. If $2 < j < n$, we have

$$\begin{aligned} w'_{i,j} - w'_{i,i} &= v_2(A_j) - v_2(A_2) \leq -\sum_{k=3}^{j}(1 - s_k) + r_2 \\ &\leq -(1 - s_3) + s_2 = -1 + s_3 + s_2 \leq 0 \end{aligned}$$

by Lemma 5, (11) and (12). If $j = n$, we have

$$\begin{aligned} w'_{i,j} - w'_{i,i} &= v_2(A'_n) - v_2(A_2) \leq v_2(A_n) - v_2(A_2) + 1 \\ &\leq -\sum_{k=3}^{n}(1 - s_k) + 1 + r_2 \leq s_2 + s_3 < 0.5. \end{aligned}$$

by Lemma 5, (12), and $n \geq 3$.

Case 3. $i \geq 3$. If $j < n$, we have $w'_{i,j} - w'_{i,i} = v_i(A'_j) - v_i(A'_i) \leq v_i(A_j) - v_i(A_i) \leq 1$ by $A'_i \supseteq A_i$, $A'_j = A_j$, and (10). If $j = n$ (and hence $i \neq n$), we have

$$\begin{aligned} w'_{i,j} - w'_{i,i} &= v_i(A'_n) - v_i(A_i) \leq v_i(A_n) - v_i(A_i) + 1 \\ &\leq -\textstyle\sum_{k=i+1}^{n}(1 - s_k) + 1 + r_i \\ &\leq -(1 - s_n) + 1 + s_i = s_i + s_n < 0.5. \end{aligned}$$

by Lemma 5 and (12). $\square$

Lemma 7 implies that each edge has a small weight except $(1, n)$. When the weight of $(1, n)$ is large, we show that

$$\boldsymbol{A}'' = (A'_n, A'_1, A'_2, \ldots, A'_{n-1})$$

induces small edge weights. Let w'' be the weights such that $w''_{i,j} = v_i(A''_j)$ $(i, j \in [n])$. Note that the minimum subsidy vector for w'' is $\boldsymbol{p}'' = (p'_n, p'_1, p'_2, \ldots, p'_{n-1})$.

Lemma 8. *For each $i, j \in [n]$, the weight of edge (i, j) in $G^{w'',\mathtt{id}}$ is*

$$\begin{aligned} w''_{i,j} - w''_{i,i} &= v_i(A''_j) - v_i(A''_i) \\ &\leq \begin{cases} 0.5 + \max\{0,\, v_1(A'_1) - v_1(A'_n)\} & \text{if } i = 1, \\ s_2 + s_3 & \text{if } i = 2, \\ s_i & \text{if } i \geq 3. \end{cases} \end{aligned}$$

Proof. Let $i, j \in [n]$. The proof is clear when $i = j$, and thus we assume that $i \neq j$.

Case 1. $i = 1$. If $j = 2$, we have

$$w''_{i,j} - w''_{i,i} = v_1(A'_1) - v_1(A'_n).$$

If $j > 2$, we have

$$\begin{aligned} w''_{i,j} - w''_{i,i} &\leq v_1(A_{j-1}) - v_1(A'_n) \\ &\leq \max\left\{\begin{matrix} v_1(A_1) - (1 - s_2), \\ \min_{e \in A_1} v_1(A_1 \setminus \{e\}) \end{matrix}\right\} - v_1(A'_n) \\ &\leq \max\{v_1(A'_1) + s_2,\, v_1(A'_1)\} - v_1(A'_n) \\ &\leq 0.5 + v_1(A'_1) - v_1(A'_n) \end{aligned}$$

by Lemma 6 and (12).

Case 2. $i = 2$. In this case, since $r_2 \geq 0$, we have

$$w''_{i,i} = v_2(A''_2) = v_2(A'_1) \geq v_2(A_1) - 1 \geq v_2(A_2) - s_2.$$

If $j = 1$, since $A''_1 = A'_n = A_n \cup \{e^*\}$, we have

$$\begin{aligned} w''_{i,j} - w''_{i,i} &\leq v_2(A''_1) - v_2(A_2) + s_2 \\ &\leq 1 + v_2(A_n) - v_2(A_2) + s_2 \\ &\leq -\textstyle\sum_{k=3}^{n}(1 - s_k) + 1 + s_2 \quad (\because \text{Lemma 5}) \\ &\leq -(1 - s_3) + 1 + s_2 = s_2 + s_3. \end{aligned}$$

If $j \geq 3$, we have

$$\begin{aligned} w''_{i,j} - w''_{i,i} &\leq v_2(A'_{j-1}) - v_2(A_2) + s_2 \\ &= v_2(A_{j-1}) - v_2(A_2) + s_2 \\ &\leq -\textstyle\sum_{k=3}^{j-1}(1 - s_k) + s_2 \leq s_2. \end{aligned}$$

Case 3. $i \geq 3$. Recall that $w''_{i,i} = v_i(A''_i) = v_i(A_{i-1}) = v_i(A_i) + (1 - s_i)$.

If $j = 1$, we have

$$\begin{aligned} w''_{i,j} - w''_{i,i} &= v_i(A_n \cup \{e^*\}) - v_i(A_i) - (1 - s_i) \\ &\leq 1 + v_i(A_n) - v_i(A_i) - (1 - s_i) \\ &\leq -\textstyle\sum_{k=i+1}^{n}(1 - s_k) + r_i + s_i \\ &\leq -1 + s_n + 2s_i \leq 0, \end{aligned}$$

where the second inequality holds by Lemma 5 and the last inequality holds by (12). If $j \geq 2$, we have

$$\begin{aligned} w''_{i,j} - w''_{i,i} &= v_i(A''_j) - v_i(A''_i) = v_i(A'_{j-1}) - v_i(A'_{i-1}) \\ &\leq v_i(A_{j-1}) - v_i(A_{i-1}) \\ &= v_i(A_{j-1}) - v_i(A_i) - (1 - s_i) \\ &\leq 1 - (1 - s_i) = s_i, \end{aligned}$$

where the last inequality holds by (10). $\square$

Now, we are ready to prove Theorem 2.

Proof of Theorem 2. In what follows, suppose that $n \geq 3$. If $\max_{i \in [n]} p^*_i \leq n - 1.5$, the total subsidy $\sum_{i \in [n]} p^*_i$ is at most $\sum_{k=1}^{n-1} \min\{k,\, n-1.5\} = (n^2-n-1)/2$ by Lemma 3. Therefore, we assume that $\max_{i \in [n]} p^*_i > n - 1.5$. We show that in this case, $\boldsymbol{p}'$ and $\boldsymbol{p}''$ defined before are desired ones.

For $\boldsymbol{p}'$, by Lemmas 3 and 7, we have

$$\begin{aligned} \max_{i \in [n]} p'_i &\leq n - 2 + \max\{0.5, v_1(A'_n) - v_1(A'_1)\} \\ &\leq (n - 1.5) + \max\{0, v_1(A'_n) - v_1(A'_1)\}. \quad (13) \end{aligned}$$

and

$$\begin{aligned} \sum_{i \in [n]} p'_i &\leq \sum_{i=1}^{n-2}(n - i)\cdot 1 + 1\cdot \max\{0.5, v_1(A'_n) - v_1(A'_1)\} \\ &\leq \tfrac{n^2-n-1}{2} + \max\{0, v_1(A'_n) - v_1(A'_1)\}. \quad (14) \end{aligned}$$

For $\boldsymbol{p}''$, by Lemmas 3 and 8, it holds that

$$\begin{aligned} \max_{i \in [n]} p''_i &\leq s_3 + \sum_{i=2}^{n} s_i + 0.5 + \max\{0, v_1(A'_1) - v_1(A'_n)\} \\ &\leq 1.5 + \max\{0, v_1(A'_1) - v_1(A'_n)\} \\ &\leq (n - 1.5) + \max\{0, v_1(A'_1) - v_1(A'_n)\}. \quad (15) \end{aligned}$$

and

$$\begin{aligned} \sum_{i \in [n]} p''_i &\leq \sum_{i=1}^{n-2}(n - i)\cdot \tfrac{1}{2} + \tfrac{1}{2} + \max\{0, v_1(A'_n) - v_1(A'_1)\} \\ &\leq \tfrac{n^2-n-1}{2} + \max\{0, v_1(A'_1) - v_1(A'_n)\}. \quad (16) \end{aligned}$$

Thus, if $v_1(A'_n) \leq v_1(A'_1)$, then $\boldsymbol{p}'$ satisfies the requirements of this theorem, i.e., $\max_{i \in [n]} p'_i \leq n - 1.5$ by (13) and $\sum_{i \in [n]} p'_i \leq (n^2 - n - 1)/2$ by (14); otherwise, $\boldsymbol{p}''$ satisfies the requirements by (15) and (16). However, recall that $\boldsymbol{A}''$ and $\boldsymbol{p}''$ are rearrangements of $\boldsymbol{A}'$ and $\boldsymbol{p}'$, respectively. Thus, both $\boldsymbol{p}'$ and $\boldsymbol{p}''$ satisfy the requirements. Hence, $(\hat{\boldsymbol{A}}, \hat{\boldsymbol{p}}) = ((A'_{\tau^*(1)}, \ldots, A'_{\tau^*(n)}), (p'_{\tau^*(1)}, \ldots, p'_{\tau^*(n)}))$ is a desired envy-free allocation with a subsidy, where τ^* is a maximum weight permutation for w'. In addition, $(\hat{\boldsymbol{A}}, \hat{\boldsymbol{p}})$ can be computed in polynomial time via computing τ^*. $\square$

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