Object Reachability via Swaps along a Line

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Abstract

The HOUSING MARKET problem is a widely studied resources allocation problem. In this problem, each agent can only receive a single object and has preferences over all objects. Starting from an initial endowment, we want to reach a certain assignment via a sequence of rational trades. We consider the problem whether an object is reachable for a given agent under a social network, where a trade between two agents is allowed if they are neighbors in the network and no participant has a deficit from the trade. Assume that the preferences of the agents are strict (no tie is allowed). This problem is polynomially solvable in a star-network and NP-complete in a tree-network. It is left as a challenging open problem whether the problem is polynomially solvable when the network is a path. We answer this open problem positively by giving a polynomial-time algorithm. Furthermore, we show that the problem on a path will become NP-hard when the preferences of the agents are weak (ties are allowed).

Introduction

Allocating indivisible objects to agents is an important problem in both computer science and economics. A widely studied setting is that each agent can only receive one single object and each agent has preferences over objects. This problem was previously called ASSIGNMENT problem (Gardenfors 1973; Wilson 1977) and now we prefer to call it HOUSE ALLOCATION problem (Abdulkadiroğlu and Sönmez 1998; Manlove 2013). When each agent is initially endowed with an object and we want to reallocate objects under some conditions without any monetary transfers, the problem is known as HOUSING MARKET problem (Shapley and Scarf 1974). HOUSING MARKET has several real-life applications such as allocation of housings (Abdulkadiroğlu and Sönmez 1999), organ exchange (Roth, Sönmez, and Ünver 2004) and so on. There are two different preference sets for agents. One is *strict*, which is a full ordinal list of all objects, and the other one is *weak*, where agents are allowed to be indifferent between objects. Both preference sets have been widely studied. Under strict preferences, the celebrated *Top Trading Cycle* rule (Shapley and Scarf 1974) has several key desirable properties. Some modifications of *Top Trading Cycle* rule, called *Top Trading Absorbing Sets* rule and *Top Cycles* rule, were introduced for weak preferences (Alcalde-Unzu and Molis 2011; Jaramillo and Manjunath 2012), which also hold some good properties. More studies of HOUSING MARKET under the two preference sets from different aspects can be found in the literature (Jaramillo and Manjunath 2012; Aziz and De Keijzer 2012; Saban and Sethuraman 2013; Ehlers 2014; Sonoda et al. 2014; Ahmad 2017).

Some rules allow a single exchange involving many agents. It is natural and fundamental to consider exchanges being bilateral deals (swaps), i.e., each exchange of objects happens only between two agents. A swap between two agents is allowed when they are mutually beneficial from the exchange. This natural rule for HOUSING MARKET has been studied in the literature (Damamme et al. 2015; Gourvès, Lesca, and Wilczynski 2017).

In some models, it is implicitly assumed that all agents have a tie with others. However, some agents often do not know each other and do not have the capacity to exchange even they can mutually get benefits. So Gourvès, Lesca, and Wilczynski (2017) studied HOUSING MARKET where the agents are embedded in a social network to denote the ability to exchange objects between them. In fact, recently it is a hot topic to study resources allocation problems over social networks and analyze the influences of networks. Abebe, Kleinberg, and Parkes (2017) and Bei, Qiao, and Zhang (2017) introduced social network of agents into the FAIR DIVISION problem of cake cutting. Bredereck, Kaczmarczyk, and Niedermeier (2018) and Beynier et al. (2018) also considered network-based FAIR DIVISION in allocating indivisible resources.

In this paper, we study HOUSING MARKET in a social network with simple trades between pairs of neighbors in the network. In this model, there are the same number of agents and objects and each agent is initially endowed with a single object. Each agent has preferences over all objects. The agents are embedded in a social network which determines

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their ability to exchange their objects. Two agents may swap their items under two conditions: they are neighbors in the social network, and they find it mutually profitable (or no one will become worse under weak preferences). We focus on OBJECT REACHABILITY under this model: to determine whether an object is reachable for a given agent from the initial endowment via swaps. Damamme et al. (2015) firstly proved that the problem is NP-hard even to decide whether any one of a subset of objects is reachable for each agent. Gourvès, Lesca, and Wilczynski (2017) further showed that OBJECT REACHABILITY is polynomially solvable under star-structures and NP-complete under tree-structures. For the network being a path, they solved the special case where the given agent is an endpoint (a leaf) in the path and left it unsolved for the general case. All the above results are under strict preferences. In this paper, we will answer this open problem positively by giving a polynomial-time algorithm for OBJECT REACHABILITY in a path under strict preferences, and also prove that OBJECT REACHABILITY in a path under weak preferences is NP-hard.

Although paths are rather simple graph structures, OBJECT REACHABILITY in a path is not easy at all, as mentioned in (Gourvès, Lesca, and Wilczynski 2017) that "Despite its apparent simplicity, REACHABLE OBJECT (OBJECT REACHABILITY) in a path is a challenging open problem when no restriction on the agent's location is made. We believe that this case is at the frontier of tractability." Our algorithm involves several techniques and needs to call solvers for the 2-SAT problem, but it is still interesting.

The following part of the paper is organized as follows. Section 2 provides some backgrounds. Section 3 tackles the reachability of an object for an agent in a path-network under strict preferences. Section 4 shows the NP-hardness of OBJECT REACHABILITY in a path under weak preferences. The proofs of the lemmas and theorems marked with "$\star$" are omitted due to the limitation of space, which can be found in the full version of this paper.

Background

Model. There are a set $N = \{1, ..., n\}$ of n agents and a set $O = \{o_1, ..., o_n\}$ of n objects. An *assignment* σ is a mapping from N to O, where $\sigma(i)$ is the object held by agent i in σ. We also use $\sigma^T(o_i)$ to denote the agent who holds object o_i in σ. Each agent holds exactly one object at all time. Initially, the agents are endowed with objects, and the initial endowment is denoted by σ_0. We assume w.l.o.g that $\sigma_0(i) = o_i$ for every agent i.

Each agent has preferences regarding objects. A *strict preference* is expressed as a full linear order of all objects. Agent i's preference is denoted by $\succ_i$, and $o_a \succ_i o_b$ indicates the fact that agent i prefers object o_a than object o_b. The whole strict preference profile for all agents is represented by $\succ$. For *weak preferences*, agents are allowed to be indifferent between objects. For two disjoint subsets of objects S_1 and S_2, we use $S_1 \succ_i S_2$ to indicate that all objects in S_1 (resp., S_2) are indifferent for agent i and agent i prefer any object in S_1 than any object in S_2. We use $o_a \succeq_i o_b$ to denote that agent i likes o_a at least as same as o_b. Two relations $o_a \succeq_i o_b$ and $o_b \succeq_i o_a$ together imply that o_a and o_b are indifferent for agent i. We may use $\succeq$ to denote the whole weak preference profile for all agents.

Let $G = (N, E)$ be an undirected graph as the social network among agents, where the edges capture the capability of communication and exchange between two agents. An instance of HOUSING MARKET is a tuple $(N, O, \succ, G, \sigma_0)$ or $(N, O, \succeq, G, \sigma_0)$ according to the preferences being strict or weak.

Dynamics. The approach we take in this paper is dynamic, and we focus on individually rational trades between two agents. A trade is *individual rational* if each participant receives an object at least as good as the one currently held, i.e., for two agents i and j and an assignment σ, the trade between i and j on σ is individual rational if $\sigma(j) \succeq_i \sigma(i)$ and $\sigma(i) \succeq_j \sigma(j)$.

We require that every trade is performed between neighbors in the social network G. Individual rational trades defined according to G are called *swaps*. A swap is an exchange, where two participants have the capability to communicate.

A sequence of swaps can be represented as a sequence of assignments $(\sigma_0, \sigma_1, \sigma_2, ..., \sigma_t)$ such that for any $i \in \{1, ..., t\}$, σ_i results from a swap from σ_{i-1}. An assignment σ' is *reachable* if there exists a sequence of swaps $(\sigma_0, \sigma_1, \sigma_2, ..., \sigma_t)$ such that $\sigma_t = \sigma'$. An object $o \in O$ is *reachable* for an agent $i \in N$ if there is a sequence of swaps $(\sigma_0, \sigma_1, \sigma_2, ..., \sigma_t)$ such that $\sigma_t(i) = o$.

Problems. We consider the problem of checking whether an object is reachable for an agent from the initial endowment via swaps.

OBJECT REACHABILITY
Instance: $(N, O, \succ, G, \sigma_0)$, an agent $k \in N$, and an object $o_l \in O$.
Question: Whether is object o_l reachable for agent k?

When the preferences are strict, we call the problem STRICT OBJECT REACHABILITY. When the preferences are weak, we call the problem WEAK OBJECT REACHABILITY. When the social network is a path P, we call the problem OBJECT REACHABILITY *in a path*. For OBJECT REACHABILITY in a path, an instance will be denoted by $I = (N, O, \succ, P, \sigma_0, k \in N, o_l \in O)$, where we assume w.l.o.g that $l < k$. For path structures, we always assume w.l.o.g that the agents are listed as $1, 2, \ldots, n$ on a line from left to right with an edge between any two consecutive agents. Below is an example for OBJECT REACHABILITY in a path.

Example 1. *There are four agents. The path structure, preference profile and a sequence of swaps are given below.*

P :	1 —	2 —	3 —	4
σ_0 :	$o_1 \leftrightarrow$	o_2	o_3	o_4
σ_1 :	o_2	$o_1 \leftrightarrow$	o_3	o_4
σ_2 :	o_2	o_3	o_1	o_4

$1 : o_2 \succ \boxed{o_1} \succ o_3 \succ o_4$
$2 : o_4 \succ o_3 \succ o_1 \succ \boxed{o_2}$
$3 : o_1 \succ o_4 \succ \boxed{o_3} \succ o_2$
$4 : o_3 \succ o_1 \succ o_2 \succ \boxed{o_4}$

The initial endowments for agents are denoted by squares *within the preferences. After a swap between agents 1 and 2 from σ_0 we get σ_1 and after a swap between agents 2 and 3 from σ_1 we get σ_2. The object o_1 is reachable for agent 3.*

STRICT OBJECT REACHABILITY in a Path

STRICT OBJECT REACHABILITY is known to be NP-complete when the network is a tree and polynomially solvable when the network is a star (Gourvès, Lesca, and Wilczynski 2017). It is left unsolved whether the problem with the network being a path is NP-hard or not. We reveal some properties of STRICT OBJECT REACHABILITY under the path structure and design a polynomial time algorithm for it. In the remaining part of this section, we assume that the preferences are strict and the network is a path.

Recall that the problem is to check whether an object o_l is reachable for an agent k with $l < k$. The main idea of our algorithm is as follows. First, we show that the instance is equivalent to the instance after deleting all agents (and the corresponding endowed objects) on the left of agent l. Thus, we can assume the problem is to check whether object o_1 is reachable for an agent k. Second, we prove that if o_1 is reachable for agent k then there is an object $o_{n'}$ with $n' \geq k$ that should be moved to agent $k-1$ in the final assignment and we can ignore all agents and objects on the right of agent n'. We guess n' by letting it be each possible value between k and n and get at most n candidate instances. These instances are called *neat* $(o_1, o_{n'}, k)$-CONSTRAINED *instances*. A neat $(o_1, o_{n'}, k)$-CONSTRAINED instance contains only n' agents and it is to check whether there is a reachable assignment σ', called *compatible assignment*, such that $\sigma'(k) = o_1$ and $\sigma'(k-1) = o_{n'}$. Third, we are going to find compatible assignments. In a neat $(o_1, o_{n'}, k)$-CONSTRAINED instance, in every compatible assignment each object o_i will be moved to either the left or the right of its original position in the path. We prove that for each direction, there is at most one possible position i_l (or i_r for the right direction) for each object o_i. We can compute i_l and i_r directly in polynomial time. Since there are still two possible final positions for each object, we do not get a feasible assignment yet. Fourth, we reduce the subproblem to 2-SAT and determine which of i_l and i_r should be chose for each agent i by solving a 2-SAT instance. We show that each feasible assignment obtained in this step is corresponding to a reachable assignment for the neat $(o_1, o_{n'}, k)$-CONSTRAINED instance. Finally, we can solve the original problem in polynomial time, because the original instance is a yes-instance if and only if at least one of the candidate neat $(o_1, o_{n'}, k)$-CONSTRAINED instances is a yes-instance.

In fact, when the preferences are strict, we have the following observations and lemmas.

Observation 1. *Given a sequence of swaps $(\sigma_0, \sigma_1, \ldots, \sigma_t)$ and an agent $j \in N$. For any $i \in \{0, 1, \ldots, t-1\}$, it holds either $\sigma_{i+1}(j) = \sigma_i(j)$ or $\sigma_{i+1}(j) \succ_j \sigma_i(j)$.*

It implies the following lemma.

Lemma 1. *Given a sequence of swaps $(\sigma_0, \sigma_1, \ldots, \sigma_t)$. For any two integers $i < j$ in $\{0, 1, \ldots, t\}$ and any agent $q \in N$, if $\sigma_i(q) = \sigma_j(q)$, then $\sigma_d(q) = \sigma_i(q)$ for any integer $i \leq d \leq j$.*

Lemma 1 also says that an object will not 'visit' an agent twice. This property is widely used in similar allocation problems under strict preferences.

Next, we analyze properties under the constraint that the social network is a path. In a swap, the moving of an object is on the *right direction* if it is moved from agent i to agent $i+1$, and the moving of an object is on the *left direction* if it is moved from agent i to agent $i-1$. In each swap, one object is moved on the right direction and one object is moved on the left direction. We study the tracks of the objects in a feasible assignment sequence.

Lemma 2. $\star$ *For a sequence of swaps $(\sigma_0, \sigma_1, \ldots, \sigma_t)$, if $\sigma_t^T(o_i) = j$ for an object o_i, then there are exactly $|j-i|$ swaps includes o_i. Furthermore, all the $|j-i|$ movings of o_i are on the right direction if $i < j$, and all the $|j-i|$ movings of o_i are on the left direction if $i > j$.*

Lemma 3. $\star$ *Let $(\sigma_0, \sigma_1, \ldots, \sigma_t)$ be a sequence of swaps, and o_a and o_b be two objects with $a < b$. Let $a' = \sigma_t^T(o_a)$ and $b' = \sigma_t^T(o_b)$. If $a' \leq a$ or $b' \geq b$, then $a' < b'$.*

Lemma 1 shows that any object can only move on one direction. Lemma 3 shows that when an object moves on the right direction, all objects initially allocated on the left of it can not move to the right of it at any time; when an object moves on the left direction, all objects initially allocated on the right of it can not move to the left of it at any time.

In fact, if we want to move an object o_l to an agent k with $k > l$, we may not need to move any object on the left of o_l, i.e., objects $o_{l'}$ with $l' < l$. Equipped with Lemma 3, we can prove

Lemma 4. $\star$ *If object o_l is reachable for agent k, then there is a feasible assignment sequence $(\sigma_0, \sigma_1, \ldots, \sigma_t)$ such that $\sigma_t^T(o_l) = k$, and $\sigma_t(i) = \sigma_0(i)$ for all $i < l$ if $l \leq k$ and for all $i > l$ if $l \geq k$.*

For an instance $I = (N, O, \succ, P, \sigma_0, k, o_l)$ of STRICT OBJECT REACHABILITY in a path with $l < k$, let $I' = (N', O', \succ', P', \sigma_0', k, o_l)$ denote the instance obtained from I by deleting agents $\{1, 2, \ldots, l-1\}$ and objects $\{o_1, o_2, \ldots, o_{l-1}\}$. In other words, we let $N' = \{l, l+1, \ldots, n\}$, $O' = \{o_l, o_{l+1}, \ldots, o_n\}$, and $\succ', P'$ and σ_0' be the corresponding subsets of $\succ, P$ and σ_0.

Lemma 5. *Object o_l is reachable for agent k in the instance I if and only if object o_l is reachable for agent k in the instance I'.*

By Lemma 5, we can always assume that the instance of STRICT OBJECT REACHABILITY in a path is to check whether the object o_1 is reachable for an agent k.

Assume that object o_1 is reachable for agent k. For a sequence of swaps $(\sigma_0, \sigma_1, \ldots, \sigma_t)$ such that $\sigma_t(o_1) = k$, there are exactly $k-1$ swaps including o_1 which are moving o_1 on the right direction according to Lemma 2. The last swap including o_1 will be happened between agent $k-1$ and agent k. Let $o_{n'}$ denote the other object included in the last swap. In other words, after the last swap, agent $k-1$ will get the object $o_{n'}$ and agent k will get the object o_1. Note that the moving of $o_{n'}$ in this swap is in the left direction. By Lemma 2, we know that all movings of $o_{n'}$ in the sequence of swaps are in the left direction. Therefore, we have the following observation.

Observation 2. *It holds that $n' \geq k$.*

Our idea is to transform our problem to the following constrained problem: to determine whether there is a reachable assignment σ such that $\sigma(k-1) = o_{n'}$ and $\sigma(k) = o_1$, where $n' \geq k$. We do not know the exact value of n'. So we search by letting n' be each value in $\{k, k+1, \ldots, n\}$. This will only increase the running time bound by a factor of n. We denote the above constrained problem by $(o_1, o_{n'}, k)$-CONSTRAINED problem.

Lemma 6. *An instance $I = (N, O, \succ, P, \sigma_0, k, o_1)$ is yes if and only if at least one of the $(o_1, o_{n'}, k)$-CONSTRAINED instances for $n' \in \{k, k+1, \ldots, n\}$ is yes.*

For an $(o_1, o_{n'}, k)$-CONSTRAINED instance I, we use $I_{-n'}$ to denote the instance obtained from I by deleting agents $\{n'+1, n'+2, \ldots, n\}$ and objects $\{o_{n'+1}, o_{n'+2}, \ldots, o_n\}$.

Lemma 7. $\star$ *An $(o_1, o_{n'}, k)$-CONSTRAINED instance I is yes if and only if the instance $I_{-n'}$ is yes.*

By Lemma 7, we can ignore all agents on the right of n' in an $(o_1, o_{n'}, k)$-CONSTRAINED instance. An $(o_1, o_{n'}, k)$-CONSTRAINED instance is called *neat* if n' is the last agent. We may simply consider neat $(o_1, o_{n'}, k)$-CONSTRAINED instances only. For any two integers a and b, we use $[a, b]$ to denote the set of integers between a and b (including a and b).

Lemma 8. $\star$ *Let $(\sigma_0, \sigma_1, \ldots, \sigma_t)$ be a sequence of swaps, and o_a and o_b be two objects with $a < b$. Let $a' = \sigma_t^T(o_a)$, $b' = \sigma_t^T(o_b)$ and $Q = [a, a'] \cap [b, b']$. Assume that $Q \neq \emptyset$.*
***(a)** If $a' > a$ and $b' > b$, it holds that $o_a \succ_q o_b$ for all $q \in Q$.*
***(b)** If $a' < a$ and $b' < b$, it holds that $o_b \succ_q o_a$ for all $q \in Q$.*

See Figure 1 for an illustration for Lemma 8(a).

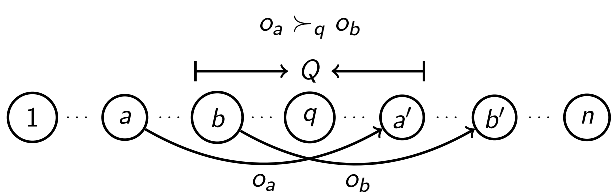

Figure 1: An illustration for Lemma 8(a)

Lemma 9. $\star$ *Let $(\sigma_0, \sigma_1, \ldots, \sigma_t)$ be a sequence of swaps for a neat $(o_1, o_{n'}, k)$-CONSTRAINED instance such that $\sigma_t^T(o_1) = k$ and $\sigma_t^T(o_{n'}) = k-1$, and o_a and o_b be two objects with $a < b$. Let $a' = \sigma_t^T(o_a)$ and $b' = \sigma_t^T(o_b)$. Assume that $a' > a$, $b' < b$ and $Q = [a, a'] \cap [b, b'] \neq \emptyset$. Let $Q' = [a+1, a'] \cap [b, b']$.*
***(a)** There is a swap including o_a and o_b which happens between agent $c-1$ and c, where $c = a' + b' - k + 1 \in Q'$.*
***(b)** It holds that $o_b \succ_q o_a$ for all $\max(a, b') \leq q < c$, and $o_a \succ_q o_b$ for all $c \leq q \leq \min(a', b)$.*

See Figure 2 for an illustration for Lemma 9.

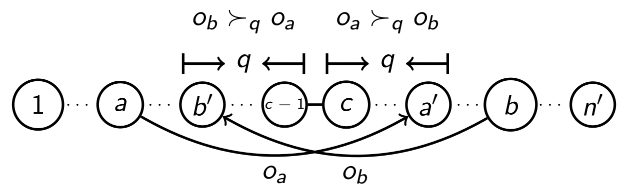

Figure 2: An illustration for Lemma 9

Given a neat $(o_1, o_{n'}, k)$-CONSTRAINED instance and an assignment σ_t such that $\sigma_t^T(o_1) = k$ and $\sigma_t^T(o_{n'}) = k-1$. We show some conditions for σ_t to be a feasible assignment. For any two objects o_a and o_b, we let $a' = \sigma_t^T(o_a)$ and $b' = \sigma_t^T(o_b)$. We say o_a and o_b are *intersected* if $Q = [a, a'] \cap [b, b']$ is not an empty set. There are three kinds of intersections, which are corresponding to Lemma 8(a), Lemma 8(b) and Lemma 9. We say a pair of objects o_a and o_b ($a < b$) are *compatible* in assignment σ_t if there are either not intersected or intersected and satisfying one of the follows:
1. when $a' > a$ and $b' > b$, it holds that $a' < b'$ (corresponding to Lemma 3) and $o_a \succ_q o_b$ for all agents $q \in Q$ (corresponding to Lemma 8(a));
2. when $a' < a$ and $b' < b$, it holds that $a' < b'$ (corresponding to Lemma 3) and $o_b \succ_q o_a$ for all agents $q \in Q$ (corresponding to Lemma 8(b));
3. when $a' > a$ and $b' < b$, it holds that $c = a' + b' - k + 1 \in Q' = Q \setminus \{a\}$, $o_b \succ_q o_a$ for all $\max(a, b') \leq q < c$, and $o_a \succ_q o_b$ for all $c \leq q \leq \min(a', b)$ (corresponding to Lemma 9).
An assignment σ is *compatible* if it holds $\sigma(i) \neq o_i$ for any agent i and any pair of objects in it are compatible.

Lemma 3, Lemma 8 and Lemma 9 directly imply that

Lemma 10. *If σ_t is a reachable assignment for a neat $(o_1, o_{n'}, k)$-CONSTRAINED instance such that $\sigma_t^T(o_1) = k$ and $\sigma_t^T(o_{n'}) = k-1$, then σ_t is compatible.*

Lemma 11. $\star$ *Let σ_t be a compatible assignment for a neat $(o_1, o_{n'}, k)$-CONSTRAINED instance such that $\sigma_t^T(o_1) = k$ and $\sigma_t^T(o_{n'}) = k-1$. For any two objects o_{x-1} and o_x, if $\sigma_t^T(o_{x-1}) > x-1$ and $\sigma_t^T(o_x) < x$, then the swap between agents $x-1$ and x in σ_0 is feasible. Let σ_1 denote the assignment after the swap between $x-1$ and x in σ_0. Then σ_t is still compatible by taking σ_1 as the initial endowment.*[1]

Based on Lemma 11 we will prove the following lemma.

Lemma 12. $\star$ *Let σ_t be an assignment for a neat $(o_1, o_{n'}, k)$-CONSTRAINED instance such that $\sigma_t^T(o_1) = k$ and $\sigma_t^T(o_{n'}) = k-1$. If σ_t is compatible, then σ_t is a reachable assignment.*

By Lemma 12, to solve a neat $(o_1, o_{n'}, k)$-CONSTRAINED instance, we only need to find a compatible assignment.

[1]If the swap is happened between agents 1 and 2, then object o_1 will be moved to agent 2 in σ_1 and we will may not be able to get an $(o_1, o_{n'}, k)$-CONSTRAINED instance by letting σ_1 be the initial endowment. For this case, we will delete agent 1 and object $\sigma_1(1)$ to keep object o_1 in the first agent. More details will be given in the full version.

Computing Compatible Assignments

In a compatible assignment, object o_1 will be assigned to agent k and object $o_{n'}$ will be assigned to agent $k-1$. We consider other objects o_i for $i \in \{2, 3, \ldots, n'-1\}$. In a compatible assignment, object o_i will not be assigned to agent i since each agent will attend in at least one swap including object o_1 or $o_{n'}$. There are two possible cases: o_i is assigned to agent i' such that $i' < i$; o_i is assigned to agent i' such that $i' > i$. We say that o_i is moved to the *left side* for the former case and moved to the *right side* for the latter case. We will show that for each direction, there is only one possible position for each object o_i in a compatible assignment.

First, we consider $i \in \{2, 3, \ldots, k-1\}$. Assume that object o_i is moved to the left side in a compatible assignment. Thus, o_1 and o_i are intersected and the intersection is of the case in Lemma 9. We find the index i' such that $i' \leq i$, $o_i \succ_{i'-1} o_1$ and $o_1 \succ_j o_i$ for each $j \in \{i', i'+1, \ldots, i\}$. We can see that i' is the only possible agent for object o_i to make o_1 and o_i are compatible if o_i is moved to the left side. We use i_l to denote this agent if it exists for i. Assume that object o_i is moved to the right side in a compatible assignment. Since o_1 will be moved to agent k and o_i will be moved to the right side, by Lemma 3 we know that o_i will be moved to the right of o_1, i.e., an agent i'' with $i'' > k$. Thus, o_i and $o_{n'}$ are intersected and the intersection is of the case in Lemma 9. We find the index i' such that $i' > k$, $o_i \succ_{i'} o_{n'}$ and $o_{n'} \succ_j o_i$ for each $j \in \{k-1, k, \ldots, i'-1\}$. We can see that i' is the only possible agent for object o_i to make $o_{n'}$ and o_i are compatible if o_i is moved to the right side. We use i_r to denote this agent if it exists for i.

Second, we consider $i \in \{k, k+1, \ldots, n'-1\}$. In fact, the structure of neat $(o_1, o_{n'}, k)$-CONSTRAINED instances is symmetrical. We can rename the agents on the path from left to right as $\{n', n'-1, \ldots, 1\}$ instead of $\{1, 2, \ldots, n'\}$ and then this case becomes the above case. We can compute i_l and i_r for each $i \in \{k, k+1, \ldots, n'-1\}$ in a similar way. Therefore, we can compute i_l and i_r for all $i \in \{2, 3, \ldots, n'-1\}$ if they exist.

If none of i_l and i_r exists for some i, then this instance is a no-instance. If only one of i_l and i_r exists, then object o_i must be assigned to this agent in any compatible assignment. The hardest case is that both of i_l and i_r exist, where we may not know which agent the object will be assigned to in the compatible assignment. For this case, we will rely on algorithms for 2-SAT to find possible solutions.

For each agent j, we will use R_j to store all possible objects that may be assigned to agent j in a compatible assignment. We use the following procedure to maintain R_j. Initially, we let $R_{k-1} = \{o_{n'}\}$, $R_k = \{o_1\}$, and $R_i = \emptyset$ for all other agent i. Then for each $i \in \{2, 3, \ldots, n'-1\}$, we compute i_l and i_r and add o_i into R_{i_l} and R_{i_r}. After this, we can do the following to make the size of each R_j at most 2. While there is a set R_{j_0} becoming an empty set, stop and report the instance is a no-instance; while there is a set R_{j_0} containing only one object o_{i_0} and the object o_{i_0} appears in two sets R_{j_0} and $R_{j_0'}$, delete o_{i_0} from $R_{j_0'}$.

The correctness of the third step is based on the fact that agent j_0 should get one object. If there is only one candidate object o_{i_0} for agent j_0, then o_{i_0} can only be assigned to agent j_0, no possible to agent j_0'.

We also analyze the running time of the above procedure to compute R_j. For each object o_i, we can compute i_l and i_r in $O(n)$. Therefore, we use $O(n^2)$ time to set the values for all sets R_j in the first two steps. To update R_i, we may execute at most n iterations in the third step and each iteration can be executed in $O(n)$. Therefore, the procedure running times in $O(n^2)$ time.

Lemma 13. * *After the above procedure, either the instance is a no-instance or it holds that* $1 \leq |R_j| \leq 2$ *for each* $j \in \{1, 2, \ldots, n'\}$.

For a set R_j containing only one object o_i, we know that the object o_i should be assigned to agent j in any compatible assignment. For sets R_j containing two objects, we still need to design which object is assigned to this agent such that we can get a compatible assignment.

We will reduce the remaining problem to 2-SAT. The instance contains n' variables $\{x_1, x_2, \ldots, x_{n'}\}$ corresponding to the n' objects. When $x_i = 1$, it means that the object o_i moves on the right direction, i.e, we will assign it to the agent i_r in the compatible assignment. When $x_i = 0$, it means that the object o_i moves on the left direction and we will assign it to the agent i_l in the compatible assignment. We have two kinds of clauses, called agent clauses and compatible clauses.

For each set R_j, we associate $|R_j|$ literals with it. If there is an object o_i such that $i_l = j$, we associate literal $\overline{x_i}$ with R_j; if there is an object o_i such that $i_r = j$, we associate literal x_i with R_j. For each set R_j of size 1 (let the associated literal be ℓ_j), we construct one clause containing only one literal $c_j : \ell_j$. For each set R_j of size 2 (let the associated literals be ℓ_j^1 and ℓ_j^2), we construct two clauses $c_{j1} : \ell_j^1 \vee \ell_j^1$ and $c_{j2} : \overline{\ell_j^1} \vee \overline{\ell_j^2}$. These clauses are called *agent clauses*. The agent clauses are to guarantee that exactly one object is assigned to each agent.

For each pair of sets R_j and R_i, we construct several clauses according to the definition of compatibility. For any two objects $o_{j'} \in R_j$ (the corresponding literal associated to R_j is ℓ_j) and $o_{i'} \in R_i$ (the corresponding literal associated to R_i is ℓ_i), we say that ℓ_j and ℓ_i are *compatible* if $o_{j'}$ and $o_{i'}$ are compatible when $o_{j'}$ is assigned to agent j and $o_{i'}$ is assigned to agent i in the assignment. If ℓ_j and ℓ_i are not compatible, then either $o_{j'}$ cannot be assigned to agent j or $o_{i'}$ cannot be assigned to agent i in any compatible assignment. So we construct one *compatible clause*: $\overline{\ell_j} \vee \overline{\ell_i}$ for each pair of incompatible pair ℓ_j and ℓ_i. Since each set contains at most two objects, for each pair of sets R_j and R_i, we will create at most $2 \times 2 = 4$ clauses. In fact, when there are four clauses for a pair, the instance will become a no-instance, since no matter what objects assigned to agents j and i, there is no compatible assignment. In the following example, we will illustrate the construct of agent clauses and compatible clauses.

We can see that each clause contains at most two literals and then the instance is a 2-SAT instance. The construction of 2-SAT instances implies that

Lemma 14. *The* 2-SAT *instance has a feasible assign-*

ment if and only if the corresponding neat $(o_1, o_{n'}, k)$-
CONSTRAINED *instance has a compatible assignment.*

The main steps of the whole algorithm to solve STRICT OBJECT REACHABILITY in a path are listed in Algorithm 1. The correctness of the algorithm follows from Lemma 5, Lemma 6, Lemma 7 and Lemma 14. Next, we analyze the running time bound of it. The dominating part of the running time is taken by the computation for neat $(o_1, o_{n'}, k)$-CONSTRAINED instances. Next, we consider the running time used for solving each neat $(o_1, o_{n'}, k)$-CONSTRAINED instance. By the above analysis, we use $O(n^2)$ time to compute the final values for all sets R_j. To construct 2-SAT instance, we construct at most $2n$ agent clauses in $O(n)$ time and construct at most $4\binom{n}{2}$ compatible clauses, each of which will take $O(n)$ time to check the compatibility. So the 2-SAT instance can be constructed in $O(n^3)$ time. We use the $O(n+m)$-time algorithm for 2-SAT (Aspvall, Plass, and Tarjan 1979) to solve our instance, where $m = O(n^2)$. There are at most n neat $(o_1, o_{n'}, k)$-CONSTRAINED instances. In total, we use $O(n^4)$ time.

Theorem 1. STRICT OBJECT REACHABILITY *in a path can be solved in $O(n^4)$ time.*

Algorithm 1: Main steps to solve STRICT OBJECT REACHABILITY in a path

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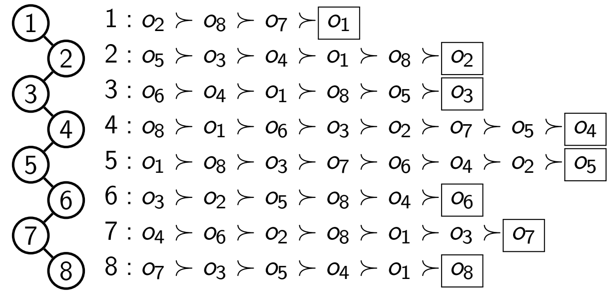
Input: An instance $(N, O, \succ, P, \sigma, k \in N, o_1 \in O)$
Output: To determine whether o_1 is reachable for k
1 **for** $k \le n' \le n$ **do**
2 Construct the neat $(o_1, o_{n'}, k)$-CONSTRAINED instance by deleting agent i and object o_i for all $n' < i \le n$;
3 Compute i_r and i_l for all $1 \le i \le n'$ if it exist according to Lemma 9;
4 Construct the set R_j of all possible objects that may be assigned to each agent j according to the result in the above step, where $R_{k-1} = \{o_{n'}\}$ and $R_k = \{o_1\}$;
5 Iteratively update R_j according to the procedure before Lemma 13;
6 Construct a 2-SAT instance as follows: construct a variable for each object, construct agent clauses according to R_j, and construct compatible clauses for all incompatible pairs;
7 Determine whether the 2-SAT instance is satisfiable;
8 **if** *the* 2-SAT *instance is* ***yes*** **then**
9 **return yes**;
10 **return no**.

We give an example to show the steps to compute a compatible assignment for a neat $(o_1, o_{n'}, k)$-CONSTRAINED instance.

Example 2. *Consider a neat $(o_1, o_{n'}, k)$-*CONSTRAINED *instance with $n' = 8$ and $k = 5$ as the top right figure. We compute i_r and i_l for all $1 \le i \le n'$ by the above procedure, the values of which are listed in the top right table.*



agent i	1	2	3	4	5	6	7	8
i_l	-	1	2	3	2	3	-	4
i_r	5	6	6	7	6	7	8	-

We construct R_j for each agent j according to this table, and then update them as doing in the procedure. After the update, it holds that $1 \le |R_j| \le 2$ for all $1 \le j \le n'$. We reduce the remaining problem to 2-SAT. *The agent clauses for each set R_j are given in the last column of the table.*

Set	*Initial*	*Updated*	*Agent clauses*
R_1	$\{o_2\}$	$\{o_2\}$	$\overline{x_2}$
R_2	$\{o_3, o_5\}$	$\{o_3, o_5\}$	$x_3 \vee x_5, \overline{x_3} \vee \overline{x_5}$
R_3	$\{o_4, o_6\}$	$\{o_4, o_6\}$	$x_4 \vee x_6, \overline{x_4} \vee \overline{x_6}$
R_4	$\{o_8\}$	$\{o_8\}$	$\overline{x_8}$
R_5	$\{o_1\}$	$\{o_1\}$	x_1
R_6	$\{o_2, o_3, o_5\}$	$\{o_3, o_5\}$	$x_3 \vee x_5, \overline{x_3} \vee \overline{x_5}$
R_7	$\{o_4, o_6\}$	$\{o_4, o_6\}$	$x_4 \vee x_6, \overline{x_4} \vee \overline{x_6}$
R_8	$\{o_7\}$	$\{o_7\}$	x_7

Next, we construct compatible clauses for all incompatible pairs. We check all pairs of objects and find that there are only two incompatible cases: o_4 and o_5 are incompatible if o_4 and o_5 are moved to agent 3 and agent 2, respectively; o_4 and o_5 are incompatible if o_4 and o_5 are moved to agent 7 and agent 6, respectively. The compatible clauses are

$$x_4 \vee x_5 \quad \text{and} \quad \overline{x_4} \vee \overline{x_5}.$$

By using the $O(n + m)$ time algorithm for 2-SAT *(Aspvall, Plass, and Tarjan 1979), we get a feasible variables assignment $(1, 0, 0, 0, 1, 1, 1, 0)$ for the* 2-SAT *instance. The corresponding swap sequence constructed via variables assignment above is given below.*

Remark: Although Step 4 of our algorithm will compute possible values i_l and i_r for each i, it does not mean that object o_i must be reachable for i_l or i_r. In the above example, object o_2 is not reachable for agent $2_r = 6$ since agent 3 prefers its initial object o_3 to o_2 and then o_2 cannot go to the right direction. In our algorithm, the compatible clauses can avoid assigning an object to an unreachable value i_l or i_r. In the above example, if object o_2 goes to agent 6, then some object o_i with $i > 2$ will go to agent 1 or 2 and we will get an incompatible pair o_2 and o_i.

P:	1	2	3	4	5	6	7	8
σ_0:	$o_1 \Leftrightarrow$	o_2	o_3	o_4	o_5	o_6	o_7	o_8
σ_1:	o_2	$o_1 \Leftrightarrow$	o_3	o_4	o_5	o_6	o_7	o_8
σ_2:	o_2	o_3	$o_1 \Leftrightarrow$	o_4	o_5	o_6	o_7	o_8
σ_3:	o_2	o_3	o_4	o_1	o_5	o_6	$o_7 \Leftrightarrow$	o_8
σ_4:	o_2	o_3	o_4	o_1	o_5	$o_6 \Leftrightarrow$	o_8	o_7
σ_5:	o_2	o_3	o_4	o_1	$o_5 \Leftrightarrow$	o_8	o_6	o_7
σ_6:	o_2	o_3	o_4	$o_1 \Leftrightarrow$	o_8	o_5	o_6	o_7
σ_7:	o_2	o_3	o_4	o_8	o_1	o_5	o_6	o_7

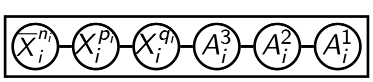
WEAK OBJECT REACHABILITY in a Path

We have proved that STRICT OBJECT REACHABILITY in a path is polynomially solvable. Next, we show that WEAK OBJECT REACHABILITY in a path is NP-hard. One of the most important properties is that Lemma 1 will not hold for WEAK OBJECT REACHABILITY and an object may 'visit' an agent more than one time. Our proof is a modification of the reduction in (Gourvès, Lesca, and Wilczynski 2017) to prove the NP-hardness of STRICT OBJECT REACHABILITY in a tree.

Theorem 2. WEAK OBJECT REACHABILITY *is NP-hard even when the network is a path.*

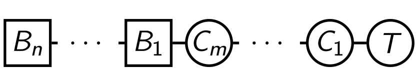
We give a reduction from the known NP-complete problem 2P1N-SAT (Yoshinaka 2005). In a 2P1N-SAT instance, we are given a set $V = \{v_1, v_2, \ldots, v_n\}$ of variables and a set $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ of clauses over V such that every variable occurs 3 times in $\mathcal{C}$ with 2 positive literals and 1 negative literal. The question is to check whether there is an variable assignment satisfying $\mathcal{C}$. For a 2P1N-SAT instance I_{SAT}, we construct an instance I_{WOR} of WEAK OBJECT REACHABILITY on a path such that I_{SAT} is a yes-instance if and only if I_{WOR} is a yes-instance.

The instance I_{WOR} contains $6n + m + 1$ agents and objects, which are constructed as follows. There is an agent named T. For each clause C_i ($i \in \{1, \ldots, m\}$), we introduce an agent also named C_i. For each variable v_i, we add six agents, named as $\overline{X}_i^{n_i}, X_i^{p_i}, X_i^{q_i}, A_i^3, A_i^2$ and A_i^1. They form a path of length 5 in the order as showed below. We call the path a *block* and denote it by B_i. The names of the six agents have certain meaning: $\overline{X}_i^{n_i}$ means that the negative literal of v_i appears in the clause C_{n_i}; $X_i^{p_i}$ and $X_i^{q_i}$ mean that the positive literals of v_i appear in the two clauses C_{p_i} and C_{q_i}; A_i^3, A_i^2 and A_i^1 are three auxiliary agents.



B_i

The whole path is connected in the order showed below.



In the initial assignment σ_0, object t is assigned to agent T, object c_i is assigned to agent C_i for $i \in \{1, 2, \ldots, m\}$, object a_i^j is assigned to agent A_i^j for each $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, 3\}$, and $\overline{o}_i^{n_i}$ (resp., $o_i^{p_i}$ and $o_i^{q_i}$) is assigned to agent $\overline{X}_i^{n_i}$ (resp., $X_i^{p_i}$ and $X_i^{q_i}$) for each $i \in \{1, 2, \ldots, n\}$.

Next, we define the preference profile $\succeq$. We only show the objects that each agent prefers at least as its initial one and all other objects can be put behind its initial endowment in any order. The initial endowment is denoted by a square in the preference. Let L_i be the set of the objects associated with the literals in clause C_i, i.e., L_i is the set of objects $\overline{o}_a^{n_a}$, $o_b^{p_b}$ and $o_c^{q_c}$ with $n_a = i$, $p_b = i$ or $q_c = i$. For each variable v_i, we define a set of objects $W_i = \{c_1, \ldots, c_m\} \cup \{\overline{o}_j^{n_j} : j > i\} \cup \{o_j^{p_j} : j \neq i\} \cup \{o_j^{q_j} : j > i\} \cup \{a_j^l : j < i, l = 1, 2, 3\}$. We are ready to give the preferences for the agents.

First, we consider the preferences for T and C_i. The following preferences ensure that when C_i holds an object in L_i for each $i \in \{1, \ldots, m\}$, object t is reachable for agent C_m via a sequence of m swaps between C_i and C_{i-1} for $i = 1, \ldots, m$, where $C_0 = T$.

$T : L_1 \succ \boxed{t}$;

$C_i : L_{i+1} \succ t \succ L_i \succ c_1 \succ L_{i-1} \succ \ldots \succ c_{i-1} \succ L_1 \succ \boxed{c_i}$, for all $1 \le i \le m-1$;

$C_m : t \succ L_m \succ c_1 \succ L_{m-1} \succ \ldots \succ c_{m-1} \succ L_1 \succ \boxed{c_m}$.

Next, we consider the preferences for the agents in each block B_i. The following preferences ensure that at most one of $\overline{o}_i^{n_i}$ and $\{o_i^{p_i}, o_i^{q_i}\}$ can be moved to the right of the block, which will indicate the corresponding variable is either true or false. If $\overline{o}_i^{n_i}$ is moved to the right of the block, we will assign the corresponding variable false; if some of $\{o_i^{p_i}, o_i^{q_i}\}$ is moved to the right of the block, we will assign the corresponding variable true. We use the preference of A_i^3 to control this. Furthermore, we use A_i^1, A_i^2 and A_i^3 to (temporarily) hold $\overline{o}_i^{n_i}$ (or $o_i^{p_i}$ and $o_i^{q_i}$) if they do not need to be moved to the right of the block.

$\overline{X}_i^{n_i} : W_i \cup \{a_i^1, a_i^2, a_i^3, o_i^{p_i}, o_i^{q_i}\} \succ \boxed{\overline{o}_i^{n_i}}$,

$X_i^{q_i} : W_i \cup \{a_i^1, a_i^2, a_i^3, \overline{o}_i^{n_i}, o_i^{p_i}\} \succ \boxed{o_i^{q_i}}$,

$X_i^{p_i} : W_i \cup \{a_i^1, a_i^2, a_i^3, o_i^{q_i}, \overline{o}_i^{n_i}\} \succ \boxed{o_i^{p_i}}$,

$A_i^1 : W_i \cup \{\boxed{a_i^1}, a_i^2, a_i^3, o_i^{p_i}, o_i^{q_i}, \overline{o}_i^{n_i}\}$,

$A_i^2 : W_i \cup \{a_i^1, \boxed{a_i^2}, a_i^3, o_i^{p_i}, o_i^{q_i}, \overline{o}_i^{n_i}\}$,

$A_i^3 : W_i \cup \{a_i^1, a_i^2, o_i^{p_i}, o_i^{q_i}\} \succ \overline{o}_i^{n_i} \succ \boxed{a_i^3}$, for all $1 \le i \le n$.

The instance I_{WOR} is to determine whether object t is reachable for agent C_m.

Lemma 15. *A 2P1N-SAT instance I_{SAT} is yes if and only if the corresponding instance I_{WOR} of* WEAK OBJECT REACHABILITY *in a path is yes.*

Proof Sketch. If t is reachable for c_m, there are m swaps including t which happen between C_i and C_{i+1} for $i \in \{0, 1, \ldots, m-1\}$, where $C_0 = T$. Note that the swap between C_i and C_{i+1} (where C_i holds the object t) can happen if and only if C_{i+1} holds an object $a \in L_{i+1}$. We can let the literal corresponding to the object $a \in L_{i+1}$ to be true for all agents C_i to get a satisfying assignment for I_{SAT} because the construction of each block B_i does not allow both $\overline{o}_i^{n_i}$ and one of $o_i^{p_i}$ and $o_i^{q_i}$ moving to the right of the block according to the construction of the block B_i the preference of A_i^3.

On the other hand, if there is a satisfied assignment τ for I_{SAT}, we can construct a reachable assignment for I_{WOR}. For each variable v_i, if it is true in τ, we move $o_i^{p_i}$ and $o_i^{q_i}$ to agents A_i^3 and A_i^2; if it is false in τ, we move $\overline{o}_i^{n_i}$ to agent A_i^2. These objects are called *true objects*. All these happen within each block. After this procedure, we move true objects $\overline{o}_a^{n_a}$, $o_b^{p_b}$ and $o_c^{q_c}$ to agent C_j with $j = n_a$, $j = p_b$ or $j = q_c$ one by one in an order where C_j with smaller j first gets its object in L_j. During this procedure, once another true object is moved out of its position A_i^3 or A_i^2 (this may happen when the true object is on the moving path of another true object to C_j), we will simply move it back by one swap. So in each iteration only one true object is moved out of its current position A_i^3 or A_i^2 and it is moved to its final position C_j directly. □

Conclusion

In this paper, we mainly investigate OBJECT REACHABILITY that asks whether an object is reachable for a given agent. We show that when the network is a path, WEAK OBJECT REACHABILITY is NP-hard but STRICT OBJECT REACHABILITY is polynomially solvable. In the literature, more problems under the HOUSING MARKET model with a different objective have been investigated, such as the reachability of a whole assignment, finding Pareto efficient assignments and so on (Gourvès, Lesca, and Wilczynski 2017). Some of our results can be extended to these problems with some modifications. We will give the details in the full version of this paper.

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