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# Interpolation of a spline developable surface between a curve and two rulings

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**Abstract:** In this paper we address the problem of interpolating a spline developable patch bounded by a given spline curve and the first and the last rulings of the developable surface. In order to complete the boundary of the patch a second spline curve is to be given. Up to now this interpolation problem could be solved, but without the possibility of choosing both endpoints for the rulings. We circumvent such difficulty here by resorting to degree elevation of the developable surface. This is useful not only to solve this problem, but also other problems dealing with triangular developable patches.

**Key words:** Developable surfaces, Spline surfaces, blossoms.
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## 1 Introduction

Developable surfaces have been used extensively in industry for modelling sheets of steel. These surfaces are plane patches that have been curved by isometric transformations, preserving lengths of curves, angles and areas. They mimic the properties of thin steel plates that are transformed by cutting, rolling or folding, but not by stretching or application of heat, which would raise manufacturing costs.

Their inclusion in the NURBS formalism, however, has not been easy. The condition of developability is a non-linear differential equation which translates into non-linear equations for the vertices of the control net of the surface.

To our knowledge the first reference to NURBS developable surfaces arises in technical reports at General Motors (Mancewicz and Frey (1992); Frey and Bindschadler (1993)). One approach has been solving the developability condition for low degrees (Lang and Röschel (1992); Chu and Séquin (2002); Chu *et al.* (2008)).

Another approach to developable surfaces consists in resorting to projective dual geometry. In this geometry "points" are planes and "planes" are points and this is useful to solve the developability condition (Bodduluri and Ravani (1993); Pottmann and Farin (1995); Hu *et al.* (2012)).

One can also construct surfaces which are approximately developable instead (Chalfant and Maekawa (1998); Pottmann and Wallner (1999); Leopoldseder (2001); Peternell (2004); Liu *et al.* (2011); Zeng *et al.* (2012)). A nice review may be found in Pottmann and Wallner (2001). Applications to ship hull design may be found in Kilgore (1967); Pérez and Suárez (2007); Pérez-Arribas *et al.* (2006).

A large family of Bézier developable surfaces was obtained in Aumann (2003, 2004) defining affine transformations between cells of the control net. This result has been extended to spline (Fernández-Jambrina (2007)) and Bézier triangular (Cantón and Fernández-Jambrina (2012)) developable patches. A characterisation of Bézier ruled surfaces is found in Juhász and Róth (2008).

In this paper we make use of the latter constructions to find solutions to interpolation prob-

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lems with developable surfaces. For instance, in Fernández-Jambrina (2007), we were able to draw a developable surface through a given boundary curve and two rulings, but we could not choose both endpoints for these rulings. We would like to solve such an issue and also apply the solution to new problems.

Following Fernández-Jambrina (2007), we first review in Section 2 the main features, definitions and the classification of developable surfaces, whereas in Section 3 we deal with the formalism of B-spline curves. In Section 4 we review the construction of spline developable surfaces grounded on linear relations between vertices of the B-spline net, that was given in Fernández-Jambrina (2007). In Section 5 we use that construction to provide solutions to an interpolation problem between a spline curve and two rulings as in Fernández-Jambrina (2007). Finally, in Section 6 we use degree elevation to provide our new solution to the problem of interpolating a developable patch between a spline curve and segments of the rulings at both ends. This problem could not be solved with just our previous results. This solution is extended to triangular patches in Section 7. A final section of conclusions is included at the end of the paper.

## 2 Developable surfaces

A *ruled surface* patch fills the space between two parametrised curves $c(u)$, $d(u)$,

$$b(u,v) = (1-v)c(u) + vd(u), \ u \in [a,b], \quad (1)$$

for $v \in [0,1]$, by linking with segments, named *rulings*, the points on both curves with the same parameter $u$.

In general, the tangent plane to the ruled surface on a ruling is different for each point on the segment. *Developable surfaces* are the subcase of ruled surfaces for which the tangent plane is constant along each ruling (Struik (1988); Postnikov (1979)).

Let us compute a normal vector at each point of a ruled surface with the derivatives of the parametrisation in Eq. 1,

$$b_u(u,v) = (1-v)c'(u)+vd'(u), \ b_v(u,v) = d(u)-c(u),$$

$$(b_u \times b_v)(u,v) = ((1-v)c'(u) + vd'(u)) \times (d(u) - c(u)),$$

which is linear in the parameter $v$. If we calculate it on both ends of the rulings,

$$(b_u \times b_v)(u,0) = c'(u) \times (d(u) - c(u)),$$

$$(b_u \times b_v)(u,1) = d'(u) \times (d(u) - c(u)),$$

we learn that the three vectors $c'(u)$, $d'(u)$, $d(u) - c(u)$ are to be coplanary in order to have a constant tangent plane along each ruling of the surface.

**Proposition:** A ruled surface parametrised as in Eq. 1 is developable if and only if the vector $\mathbf{w}(u) = d(u) - c(u)$, linking the points $d(u)$, $c(u)$, and the velocities $c'(u)$, $d'(u)$ of the curves at these points are coplanary for every value of $u$.

## 3 B-spline curves

In this section we review the formalism of B-spline curves and their main properties in order to fix the notation, which follows closely the one in Farin (2002). We may define a B-spline curve $c(u)$ of degree $n$ and $N$ pieces on an interval $[u_{n-1}, u_{n+N-1}]$, so that the $I$-th piece of the curve is defined on an interval $[u_{n+I-2}, u_{n+I-1}]$. For this we require an ordered list of values of the parameter $u$, which are named *knots*, $\{u_0, \ldots, u_{2n+N-2}\}$. The actual knots defining the intervals for each piece are the *inner* knots $\{u_{n-1}, \ldots, u_{n+N-1}\}$ whereas the knots $\{u_0, \ldots, u_{n-2}\}$ at the beginning of the list (usually taken to be equal to $u_n$) and $\{u_{n+N}, \ldots, u_{2n+N-2}\}$ at the end (usually taken to be equal to $u_{n+N-1}$) are *auxiliary*.

Points on B-spline curves can be computed using the De Boor's algorithm, $c(u) = c_0^{n)}(u)$, consisting on linear interpolations between consecutive vertices. For a curve of just one piece:

$$\begin{aligned} c_i^{r)}(u) \quad := \quad & \frac{u_{i+n} - u}{u_{i+n} - u_{i+r-1}} c_i^{r-1)}(u) \\ + \quad & \frac{u - u_{i+r-1}}{u_{i+n} - u_{i+r-1}} c_{i+1}^{r-1)}(u), \end{aligned} \quad (2)$$

for $i = 0, \ldots, n-r$, $r = 1, \ldots, n$.

A useful construction, named *polarisation* or *blossom* of the parametrisation of the curve, consists of interpolating in each step with a different value $v_i$ of the parameter $u$, $c[v_1, \ldots, v_n] := c_0^{n)}[v_1, \ldots, v_n]$,

$$\begin{aligned} c_i^{r)}[v_1, \ldots, v_r] := & \frac{u_{i+n} - v_r}{u_{i+n} - u_{i+r-1}} c_i^{r-1)}[v_1, \ldots, v_{r-1}] \\ + & \frac{v_r - u_{i+r-1}}{u_{i+n} - u_{i+r-1}} c_{i+1}^{r-1)}[v_1, \ldots, v_{r-1}] \end{aligned} (3)$$

With this notation, $u^{<i>} = \underbrace{u, \ldots, u}_{i \text{ times}}$, we have that $c(u) = c[u^{<n>}]$. Vertices are recovered from the polarisation as $c_i = c[u_i, \ldots, u_{i+n-1}]$.

These expressions are valid for B-spline curves with an arbitrary number of pieces, replacing the interval $[u_{n-1}, u_n]$ of the first piece by the interval of the piece under consideration.

We may summarise some properties of the De Boor algorithm and the polarisation which are relevant for our purposes:

1. The velocity of the curve is

$$\begin{aligned} c'(u) &= \frac{n}{u_n - u_{n-1}} \left( c_1^{n-1)}(u) - c_0^{n-1)}(u) \right) \\ &= \frac{n \left( c[u^{<n-1>}, u_n] - c[u^{<n-1>}, u_{n-1}] \right)}{u_n - u_{n-1}}. \end{aligned} \quad (4)$$

2. The polarisation $c[v_1, \ldots, v_n]$ of the spline curve $c(u)$, is multiaffine and symmetric. That is, if $\lambda + \mu = 1$,

$$c[\lambda v_1 + \mu \tilde{v}_1, \ldots, v_n] = \lambda c[v_1, \ldots, v_n] + \mu c[\tilde{v}_1, \ldots, v_n].$$

Finally, we review two operations with B-spline curves which we shall need later on:

**Insertion of knots:** Given a B-spline curve of degree $n$ with vertices $\{c_0, \ldots, c_L\}$ and knots $\{u_0, \ldots, u_K\}$, we can split into two the piece corresponding to the interval $[u_I, u_{I+1}]$ by inserting a new knot $\tilde{u}$, $u_I < \tilde{u} < u_{I+1}$. The new list of knots is then obviously $\{\tilde{u}_0, \ldots, \tilde{u}_{K+1}\}$,

$$\tilde{u}_i = u_i, \ i = 0, \ldots, I, \ \tilde{u}_{I+1} = \tilde{u}, \ \tilde{u}_i = u_{i-1},$$

for $i = I + 2, \ldots, K + 1$, and, since the curve has not changed, the blossom provides the new sequence of vertices $\{\tilde{c}_0, \ldots, \tilde{c}_{L+1}\}$,

$$\tilde{c}_i = c[\tilde{u}_i, \ldots, \tilde{u}_{i+n-1}], \qquad i = 0, \ldots, L + 1.$$

**Degree elevation:** Formally we may express a B-spline curve $c(u)$ of degree $n$ as a curve of degree $n + 1$. The blossom $c^1$ of the degree-elevated curve is related to the original one in a simple form (Farin (2002)),

$$c^1[v_1, \ldots, v_{n+1}] = \frac{\sum_{i=1}^{n+1} c[v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{n+1}]}{n + 1} \quad (5)$$

and in the list of knots $\{u_0, \ldots, u_K\}$ the multiplicity of inner knots, from $u_{n-1}$ to $u_{n+N-1}$, is increased by one, without modifying the auxiliary knots.

## 4 Spline developable surfaces

The developability condition in Proposition 1 may be readily now adapted to spline curves (Fernández-Jambrina (2007)).

To start, let us consider two B-spline curves of degree $n$ and one segment over a common list of knots $\{u_0, \ldots, u_{2n-1}\}$, defined on the interval $[u_{n-1}, u_n]$. Their respective B-spline polygons are $\{c_0, \ldots, c_n\}$, $\{d_0, \ldots, d_n\}$.

We may draw a simple conclusion using the De Boor algorithm. Using Eq. 4 and the last iteration of Eq. 2, it is clear that the vectors $c'(u)$, $d'(u)$, $d(u) - c(u)$ are coplanary if and only if the four points $c_0^{n-1)}(u)$, $c_1^{n-1)}(u)$, $d_0^{n-1)}(u)$, $d_1^{n-1)}(u)$ are coplanary (see Figure 1).

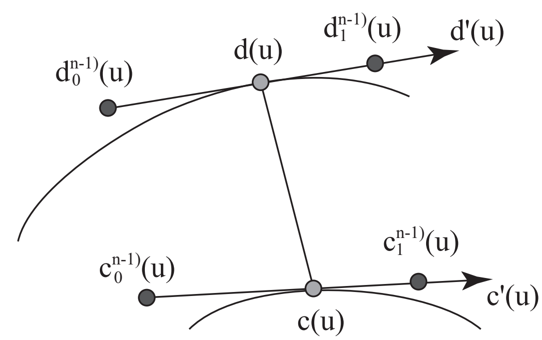


**Fig. 1 Characterisation of developable surfaces**

The developability condition is then equivalent to the possibility of writing one of the points as a barycentric combination of the other ones. For instance,

$$\begin{aligned} d_1^{n-1)}(u) &= \mu_0(u) d_0^{n-1)}(u) + \lambda_0(u) c_0^{n-1)}(u) \\ &+ \lambda_1(u) c_1^{n-1)}(u), \end{aligned}$$

with coefficients $\lambda_0(u)$, $\lambda_1(u)$, $\mu_0(u) = 1 - \lambda_0(u) - \lambda_1(u)$.

We may rewrite this combination in another form, separating the terms related to each curve, also in a barycentric fashion,

$$\begin{aligned} &(1 - \Lambda(u)) c_0^{n-1)}(u) + \Lambda(u) c_1^{n-1)}(u) \\ &= (1 - M(u)) d_0^{n-1)}(u) + M(u) d_1^{n-1)}(u), \end{aligned} \quad (6)$$

$$\Lambda(u) = \frac{\lambda_1(u)}{\lambda_0(u) + \lambda_1(u)}, \quad M(u) = \frac{1}{\lambda_0(u) + \lambda_1(u)},$$

which just excludes the case of parallel vectors $d_1^{n-1)}(u) - d_0^{n-1)}(u)$, $c_1^{n-1)}(u) - c_0^{n-1)}(u)$, which corresponds to a cone. In this sense we use the word *generic*, since the following results will be valid for all developable surfaces, but for this type of cone.

Using blossoms and taking into account that these are multiaffine Eq. (5),

$$
\begin{aligned}
&(1-\Lambda(u))c_0^{n-1)}(u)+\Lambda(u)c_1^{n-1)}(u)\\
&=(1-\Lambda(u))c[u^{<n-1>},u_{n-1}]+\Lambda(u)c[u^{<n-1>},u_n]\\
&=c\left[u^{<n-1>},(1-\Lambda(u))u_{n-1}+\Lambda(u)u_n\right],
\end{aligned}
$$

the coplanarity condition (Eq. 6) may be written in a more compact expression,

$$c[u^{<n-1>},\Lambda^*(u)]=d[u^{<n-1>},M^*(u)], \qquad (7)$$

$$\Lambda^*(u)=(1-\Lambda(u))u_{n-1}+\Lambda(u)u_n,$$

$$M^*(u)=(1-M(u))u_{n-1}+M(u)u_n.$$

This expression is valid for B-spline curves with arbitrary number of pieces, replacing the interval $[u_{n-1},u_n]$ of the first piece by the interval of the piece under consideration.

The higher the degree of $\Lambda^*(u)$, $M^*(u)$, the larger the number of conditions imposed by Eq. 7. Hence, we restrict now to the case with constant $\Lambda^*$, $M^*$, which produces the families of developable surfaces in Aumann (2003); Fernández-Jambrina (2007). In this case expressions on both sides of Eq. 7 may be viewed as parametrisations of curves of degree $n-1$ and therefore this condition is equivalent to the same one for their blossoms, since a blossom is uniquely determined by its parametrisation:

**Theorem 1** Two B-spline curves of degree $n$ and $N$ pieces with the same list of knots $\{u_0,\ldots,u_K\}$ define a developable surface on the interval $[u_{n-1},u_{n+N-1}]$ if their blossoms are related by

$$c[v_1,\ldots,v_{n-1},\Lambda^*]=d[v_1,\ldots,v_{n-1},M^*],$$

for some values $\Lambda^*$, $M^*$.

We may obtain relations between the B-spline polygons of both curves by applying the previous expression to lists of correlative knots, $\{u_{i+1},\ldots,u_{i+n-1}\}$, taking into account that blossoms are multiaffine,

$$
\begin{aligned}
&c[u_{i+1},\ldots,u_{i+n-1},\Lambda^*]\\
&=c\left[u_{i+1},\ldots,u_{i+n-1},\frac{u_{i+n}-\Lambda^*}{u_{i+n}-u_i}u_i+\right.\\
&\left.\frac{\Lambda^*-u_i}{u_{i+n}-u_i}u_{i+n}\right]\\
&=\frac{u_{i+n}-\Lambda^*}{u_{i+n}-u_i}c\left[u_i,\ldots,u_{i+n-1}\right]\\
&+\frac{\Lambda^*-u_i}{u_{i+n}-u_i}c\left[u_{i+1},\ldots,u_{i+n}\right]\\
&=\frac{u_{i+n}-\Lambda^*}{u_{i+n}-u_i}c_i+\frac{\Lambda^*-u_i}{u_{i+n}-u_i}c_{i+1},
\end{aligned}
$$

since $c_i=c[u_i,\ldots,u_{i+n-1}]$.

**Corollary 1:** Two B-spline curves of degree $n$ with the same list of knots $\{u_0,\ldots,u_K\}$ and B-spline polygons $\{c_0,\ldots,c_L\}$, $\{d_0,\ldots,d_L\}$ define a developable surface if the cells of the B-spline net of the surface are plane and their vertices are related by

$$
\begin{aligned}
&(u_{i+n}-\Lambda^*)c_i+(\Lambda^*-u_i)c_{i+1}\\
&=(u_{i+n}-M^*)d_i+(M^*-u_i)d_{i+1}, \qquad (8)
\end{aligned}
$$

for some values $\Lambda^*$, $M^*$ and $i=0,\ldots,L-1$.

This family of spline developable surfaces has the advantage of being defined by linear relations between vertices, in spite of the non-linearity of the condition of null gaussian curvature.

The data for this construction are the B-spline polygon $\{c_0,\ldots,c_L\}$, the list of knots $\{u_0,\ldots,u_K\}$ and, for instance, the first plane cell of the net, given by either $d_0$, $d_1$ or $d_0$ and the parameters $\Lambda^*$, $M^*$.

Since this construction is based on blossoms of curves, it is compatible with algorithms for B-spline curves, grounded on blossoms, such as, for instance, the knot insertion algorithm for subdivision of B-spline curves. That is, if we split into two pieces the interval $[u_I,u_{I+1}]$ by inclusion of a new knot $\tilde{u}$, so that the new list is $\{u_0,\ldots,u_I,\tilde{u},u_{I+1},\ldots,u_K\}$ and we compute the new B-spline polygons $\{\tilde{c}_0,\ldots,\tilde{c}_{L+1}\}$ and $\{\tilde{d}_0,\ldots,\tilde{d}_{L+1}\}$, these new vertices satisfy Eq. 8.

However, this construction is not compatible with degree elevation of B-spline curves. The degree-elevated B-spline developable surface through two B-spline curves does not coincide with the B-spline developable surface through the corresponding degree-elevated curves. See, for instance, in Figure 3 a developable surface and the control polygons of the

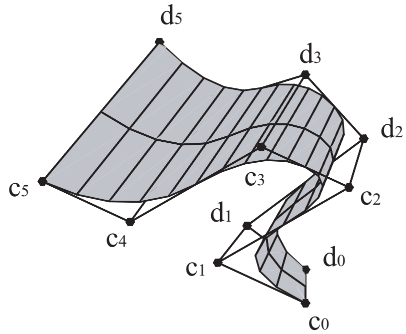


**Fig. 2 Developable B-spline surface of 4 pieces of degree 2**

degree-elevated boundary curves (denoted by tildes): the central cell of the degree-elevated surface is not even planar.

We show it explictly with a simple example:

**Example 1** Find a developable surface patch of degree two and just one piece, bounded by two curves, $c(u)$ and $d(u)$, with polygons,

$$c_0 = (0,0,0),\ c_1 = (3,3,0),\ c_2 = (4,3,0);$$

$$d_0 = (0,0,2),\ d_1 = (2,2,3),$$

and knots $\{0,0,1,1\}$.

From Eq. 8 applied to the first cell of the B-spline net, $i=0$,

$$(u_2-\Lambda^*)c_0+(\Lambda^*-u_0)c_1 = (u_2-M^*)d_0+(M^*-u_0)d_1,$$

with $n=2$, $u_0=0$, $u_2=1$, we get

$$\begin{aligned}(1-\Lambda^*)(0,0,0)+\Lambda^*(3,3,0) &= (1-M^*)(0,0,2)\\ &+ M^*(2,2,3),\end{aligned}$$

and hence $\Lambda^* = -4/3$ and $M^* = -2$.

We lack the vertex $d_2$, but for the second cell of the net,

$$(u_3-\Lambda^*)c_1+(\Lambda^*-u_1)c_2 = (u_3-M^*)d_1+(M^*-u_1)d_2,$$

$$\frac{7}{3}(3,3,0)-\frac{4}{3}(4,3,0) = 3(2,2,3)-2d_2,$$

we conclude $d_2 = (13/6, 3/2, 9/2)$.

If we formally elevate the degree of both curves to three, the list of knots extends to $\{0,0,0,1,1,1\}$ and the new polygons obtained with Eq. 5,

$$\begin{aligned}
\tilde c_0 &= \tilde c[0,0,0] = c[0,0] = c_0 = (0,0,0)\\
\tilde c_1 &= \tilde c[0,0,1] = \frac{c[0,0]+2c[0,1]}{3} = \frac{c_0+2c_1}{3}\\
&= (2,2,0)\\
\tilde c_2 &= \tilde c[0,1,1] = \frac{2c[0,1]+c[1,1]}{3} = \frac{2c_1+c_2}{3}\\
&= (10/3,3,0)\\
\tilde c_3 &= \tilde c[1,1,1] = c[1,1] = c_2 = (4,3,0)
\end{aligned}$$

$$\begin{aligned}
\tilde d_0 &= \tilde d[0,0,0] = d[0,0] = d_0 = (0,0,2)\\
\tilde d_1 &= \tilde d[0,0,1] = \frac{d[0,0]+2d[0,1]}{3} = \frac{d_0+2d_1}{3}\\
&= (4/3,4/3,8/3)\\
\tilde d_2 &= \tilde d[0,1,1] = \frac{2d[0,1]+d[1,1]}{3} = \frac{2d_1+d_2}{3}\\
&= (37/18,11/6,7/2)\\
\tilde d_3 &= \tilde d[1,1,1] = d[1,1] = d_2 = (13/6,3/2,9/2)
\end{aligned}$$

correspond to a developable surface with non constant $\Lambda^*(u) = -2-u/2$, $M^*(u) = -3-u/2$ and it is easy to check that the four points that form the second cell, $\tilde c_1$, $\tilde c_2$, $\tilde d_1$, $\tilde d_2$ do not lie on a plane.

This feature, however, will be shown to be useful for solving interpolation problems, as it will be apparent in the following sections.

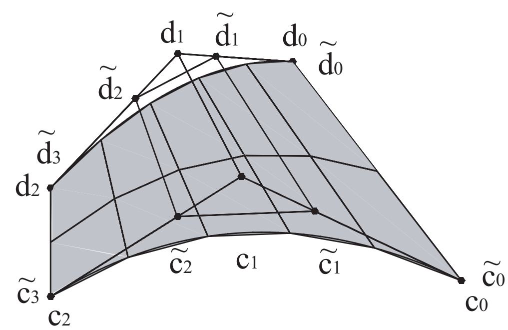


**Fig. 3 Degree-elevated developable surface of one piece of degree 2**

## 5 Interpolation of B-spline developable surfaces

Let us consider the following interpolation problem:

**Problem 1:** Given a spline curve $c(u)$ of degree $n$, $N$ pieces, B-spline polygon $\{c_0,\ldots,c_L\}$ and list of knots

$\{u_0, \ldots, u_K\}$, $u \in [a, b]$, $a = u_{n-1}$, $b = u_{n+N-1}$, and two straight lines $l_a$ and $l_b$ through the endpoints of $c(u)$ with respective director vectors $\mathbf{v}$, $\mathbf{w}$, find a developable surface $b(u, v)$ such that $c(u, 0) = c(u)$ and $l_a$ and $l_b$ are the first and last rulings of the surface, that is, $l_a: \ c(a, v)$, $l_b: \ c(b, v)$.

The special case of Bézier curves of degree $n$ was solved by Aumann (2003), making use of his family of developable surfaces. His solution is extended to spline curves in Fernández-Jambrina (2007), solving the recursion in Eq. 8 for the B-spline net. We review here this construction in order to extend it to solve new problems in next sections.

We focus on the general case of crossing rulings $l_a$ and $l_b$, since the particular cases of parallel or intersecting rulings may be solved in a simpler fashion resorting to cylinders and cones respectively.

As in Fernández-Jambrina (2007), the last ruling of the developable surface can be written in terms of the B-spline net of the curve $c(u)$, the list of knots and the coefficients $\Lambda^*$, $M^*$,

$$\begin{aligned}
d_L - c_L =& \prod_{i=0}^{L-1} \frac{M^* - u_{i+n}}{M^* - u_i}(d_0 - c_0) \\
&+ \frac{\Lambda^* - M^*}{M^* - u_{L-1}}\big(c_L - a(M^*)\big), \\
a(M^*) =& \frac{M^* - u_{L-1}}{M^* - u_0} \prod_{i=1}^{L-1} \frac{M^* - u_{i+n}}{M^* - u_i} c_0 \\
&+ \sum_{i=1}^{L-1} \frac{u_{i+n} - u_{i-1}}{M^* - u_{i-1}} \left( \prod_{j=i}^{L-2} \frac{M^* - u_{n+j+1}}{M^* - u_j} \right) c_i . \quad (9)
\end{aligned}$$

From this expression we learn that the vectors along the first and last rulings, $d_0 - c_0 = \sigma\mathbf{v}$, $d_L - c_L = \tau\mathbf{w}$, and the vector, $c_L - a(M^*)$ have to be linearly dependent and this will happen for any solution $M_0^*$ of the algebraic equation

$$\det(a(M^*) - c_L, \mathbf{v}, \mathbf{w}) = 0. \quad (10)$$

This allows us to write the linear combination in terms of a basis $\{\mathbf{v}, \mathbf{w}, \mathbf{n}\}$, $\mathbf{n} = \mathbf{v} \times \mathbf{w}$,

$$a(M_0^*) = c_L + \alpha\mathbf{v} + \beta\mathbf{w} + 0\mathbf{n},$$

where the coefficients are readily obtained by Cramer's rule,

$$\alpha = \frac{\det(a(M_0^*) - c_L, \mathbf{w}, \mathbf{n})}{\det(\mathbf{v}, \mathbf{w}, \mathbf{n})},$$

$$\beta = \frac{\det(\mathbf{v}, a(M_0^*) - c_L, \mathbf{n})}{\det(\mathbf{v}, \mathbf{w}, \mathbf{n})}.$$

Since $M^*$ is fixed by the coplanarity condition in Eq. 10, if we wish, we can modify the length of the rulings through either $\sigma$ or $\tau$ just with the parameter $\Lambda^*$, which remains free so far,

$$\begin{aligned}
\sigma &= \alpha \frac{\Lambda^* - M_0^*}{M_0^* - u_{L-1}} \prod_{i=0}^{L-1} \frac{M_0^* - u_i}{M_0^* - u_{i+n}}, \\
\tau &= \beta \frac{M_0^* - \Lambda^*}{M_0^* - u_{L-1}}. \quad (11)
\end{aligned}$$

Hence, we have solved the interpolation problem and we can use $\Lambda^*$ for fixing either $d_0$ or $d_L$, but we cannot choose both ends of the rulings. An example of this construction is shown in Figure 4

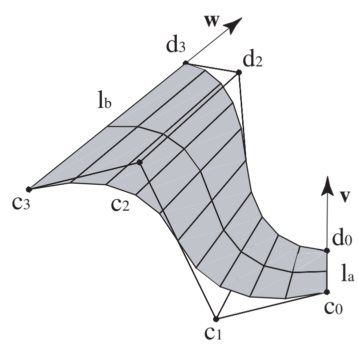


**Fig. 4 Developable surface of degree 2 and 2 pieces**

The procedure for solving the problem is clear:

1. Write the algebraic equation 10 with the B-spline polygon for $c(u)$, vectors $\mathbf{v}$, $\mathbf{w}$ and the list of knots and obtain a solution $M_0^*$. For any value of $\Lambda^*$ the resulting developable surface will have $c(u)$ as part of the boundary and the first and last rulings will be straight lines with respective directions $\mathbf{v}$, $\mathbf{w}$.
2. Fix $\Lambda_0^*$ by choosing either $d_0$ or $d_L$ in Eq. 11.
3. Use the recursivity relation in Eq. 8 for computing the vertices $d_i$ for $d(u)$.
4. The B-spline polygons $\{c_0, \ldots, c_L\}$, $\{d_0, \ldots, d_L\}$ form the B-spline net for the developable patch complying with the prescription.

We illustrate this with an example, which will be useful as a first step for following sections:

**Example 2** Consider a spline curve of degree three and three pieces with B-spline polygon

$$c_0 = (0,0,0),\ c_1 = (2,3,0),\ c_2 = (4,3,0),$$

$$c_3 = (5,0,0),\ c_4 = (7,2,1),\ c_5 = (9,-1,3),$$

and list of knots $\{0,0,0,0.3,0.7,1,1,1\}$, not uniformly spaced. For the first ruling we choose direction $\mathbf{v} = (0,0,2)$ and for the last ruling we choose $\mathbf{w} = (-1,0,1)$. Find a developable surface patch bounded by $c(u)$ and the rulings defined by $\mathbf{v}$, $\mathbf{w}$.

We calculate the determinant in Eq. 10,

$$\det(a(M^*) - c_L, \mathbf{v}, \mathbf{w})$$
$$= \frac{2(M^{*4} + 6.2M^{*3} - 12.3M^{*2} + 9.3M^* - 2.1)}{M^{*3}(M^* - 0.3)(M^* - 0.7)},$$

and we ensure developability by choosing the parameter $M^*$ as one of the real solutions of

$$M^{*4} + 6.2M^{*3} - 12.3M^{*2} + 9.3M^* - 2.1 = 0,$$

which are $M^* = -7.91,\ 0.37$.

We further choose $d_0 = c_0 + \mathbf{v} = (0,0,2)$ along the first ruling, which amounts to choosing $\sigma = 1$ in Eq. 11, to obtain the respective values of the parameter $\Lambda^* = -6.18,\ 0.61$. We perform the calculations for the first pair of parameters, $\Lambda^* = -6.18$, $M^* = -7.91$.

We may use now Corollary 1 to obtain the B-spline polygon of the other boundary curve of the developable patch through $c(u)$ with prescribed rulings,

$$d_{i+1} = \frac{(u_{i+n} - \Lambda^*)c_i + (\Lambda^* - u_i)c_{i+1} + (M^* - u_{i+n})d_i}{M^* - u_i}$$

for $i = 0 \dots L-1$,

$$\begin{aligned}
d_1 &= \frac{(u_3 - \Lambda^*)c_0 + (\Lambda^* - u_0)c_1 + (M^* - u_3)d_0}{M^* - u_0} \\
&= (1.56, 2.34, 2.08) \\
d_2 &= \frac{(u_4 - \Lambda^*)c_1 + (\Lambda^* - u_1)c_2 + (M^* - u_4)d_1}{M^* - u_1} \\
&= (3.09, 2.29, 2.26) \\
d_3 &= \frac{(u_5 - \Lambda^*)c_2 + (\Lambda^* - u_2)c_3 + (M^* - u_5)d_2}{M^* - u_2} \\
&= (3.75, -0.15, 2.55) \\
d_4 &= \frac{(u_6 - \Lambda^*)c_3 + (\Lambda^* - u_3)c_4 + (M^* - u_6)d_3}{M^* - u_3} \\
&= (5.22, 1.42, 3.55) \\
d_5 &= \frac{(u_7 - \Lambda^*)c_4 + (\Lambda^* - u_4)c_5 + (M^* - u_7)d_4}{M^* - u_4} \\
&= (6.76, -1.00, 5.24).
\end{aligned}$$

and check that in fact $d_5$ lies on the last ruling since

$$d_5 - c_5 = (-2.24, 0.00, 2.24),$$

which is a vector proportional to $\mathbf{w}$. The resulting patch is shown in Figure 5.

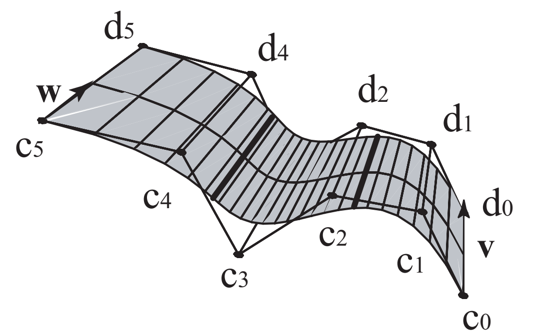


**Fig. 5 Developable surface of degree 3 and 3 pieces**

Another way to look at this developable surface would be to split the spline curve into three cubic Bézier curves, $\{C_0, C_1, C_2, C_3\}$, $\{C_3, C_4, C_5, C_6\}$, $\{C_6, C_7, C_8, C_9\}$, by knot insertion,

$$C_0 = (0,0,0),\ C_1 = (2,3,0),\ C_2 = (2.86,3,0),$$

$$C_3 = (3.48, 2.61, 0),\ C_4 = (4.3, 2.1, 0),$$

$$C_5 = (4.7, 0.9, 0),\ C_6 = (5.52, 1.04, 0.33),$$

$$C_7 = (6.14, 1.14, 0.57),\ C_8 = (7,2,1),$$

$$C_9 = (9, -1, 3).$$

If we also split by knot insertion the other boundary curve in three cubic pieces, $\{D_0, D_1, D_2, D_3\}$, $\{D_3, D_4, D_5, D_6\}$, $\{D_6, D_7, D_8, D_9\}$, by knot insertion,

$$D_0 = (0,0,2),\ D_1 = (1.56, 2.34, 2.08),$$

$$D_2 = (2.21, 2.32, 2.15),$$

$$D_3 = (2.67, 1.99, 2.24),\ D_4 = (3.29, 1.56, 2.35),$$

$$D_5 = (3.55, 0.58, 2.46),\ D_6 = (4.15, 0.68, 2.84),$$

$$D_6 = (4.15, 0.68, 2.84),\ D_7 = (4.59, 0.75, 3.12),$$

$$D_8 = (5.22, 1.42, 3.55),\ D_9 = (6.76, -1.00, 5.24),$$

it is easy to check that the three pieces of the composite ruled surface are in fact independent developable surfaces on their respective intervals $[0, 0.3]$, $[0.3, 0.7]$, $[0.7, 1]$, with the same parameters $\Lambda^* = -6.18$, $M^* = 0.61$. The boundary rulings of these Bézier developable surfaces have been marked in Figure 5.

## 6 Degree elevation of developable surfaces

We have seen how to interpolate a spline developable surface bounded by a spline curve and two rulings, but we cannot choose both endpoints for such rulings. This is a limitation of the procedure in Fernández-Jambrina (2007) described in the previous sections. A way to deal with this problem is to try to find a solution of higher degree.

As it is pointed out in Aumann (2004), degree elevation may be used for enlarging a developable patch by modifying the length of the ruling segments of the patch. The idea is simple. We may modify the length of the director vector

$$\mathbf{w}(u) = d(u) - c(u),$$

of each ruling by multiplication by a function $f(u)$,

$$\tilde{\mathbf{w}}(u) = f(u)\mathbf{w}(u) = \tilde{d}(u) - c(u),$$

and as a consequence the boundary of the surface patch changes. For instance the new second curve $\tilde{d}(u)$ starts at $\tilde{d}_0 = c_0 + f(u_{n-1})(d_0 - c_0)$ and ends at $\tilde{d}_L = c_L + f(u_{n+N-1})(d_L - c_L)$.

It is clear that this transformation just changes the patch of the developable surface that is covered by the parametrisation and it allows us to change the endpoints $d_0$ and $d_L$ of the first and last rulings. The only problem is that the curve $\tilde{d}(u)$ is no longer a spline of degree $n$. The simplest choice for the factor is an affine function $f(u) = au + b$, and in this case the new surface patch

$$\tilde{b}(u, v) = (1 - u)c(u) + v\tilde{d}(u)$$

will be of degree $(n + 1, 1)$. An example is shown in Figure 6.

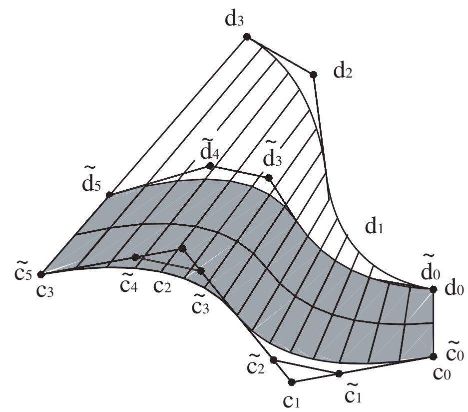


**Fig. 6 Developable surface of degree 2 and 2 pieces stretched to a patch of degree 3**

The next step will be the calculation of the B-spline polygon of the new boundary of the extended surface patch.

First, we obtain the blossom of the new parametrised curve,

$$\tilde{d}(u) = (1 - f(u))\, c(u) + f(u)d(u).$$

The blossom is a $(n + 1)$-affine symmetric form $\tilde{d}[u_0, \ldots, u_n]$ for which

$$\tilde{d}(u) = \tilde{d}[u^{<n+1>}].$$

Since $f(u)$ is an affine function, it is already its own blossom, $f[u] = f(u)$. For the product $h(u) = f(u)d(u)$ it is simple to produce an $(n+1)$-affine form $\hat{h}$ satisfying $\hat{h}[u^{<n+1>}] = h(u)$,

$$\hat{h}[u_0, \ldots, u_n] = f(u_0)d[u_1, \ldots, u_n],$$

but this form is clearly non-symmetric.

However, we may obtain a symmetric form just by permuting the argument of the function $f$,

$$h[u_0, \ldots, u_n] = \frac{\displaystyle\sum_{i=0}^{n} f(u_i)d[u_0, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n]}{n + 1}.$$

This form $h$ is $(n + 1)$-affine, symmetric and clearly $h[u^{<n+1>}] = h(u)$. Hence, it is the blossom of the parametrisation $h(u)$.

We may use this result to conclude that the blossom of $\tilde{d}(u)$ is given by

$$\tilde{d}[u_0, \ldots, u_n] = \frac{\displaystyle\sum_{i=0}^{n} f(u_i)d[u_0, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n]}{n + 1} + \frac{\displaystyle\sum_{i=0}^{n} \big(1 - f(u_i)\big)c[u_0, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n]}{n + 1}. \quad (12)$$

The degree of the curve $c(u)$ must be formally elevated to $n + 1$ in order to complete the B-spline net of the surface patch of degree $(n+1, 1)$. It can be computed by taking $f \equiv 1$ in the previous formula for $\tilde{d}$. The degree-elevated blossom for $c(u)$ is

$$\tilde{c}[u_0, \ldots, u_n] = \frac{1}{n + 1}\sum_{i=0}^{n} c[u_0, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n].$$

The list of knots of the degree-elevated curves (Farin (2002)) is also modified by increasing by one the multiplicity of the inner knots $u_{n-1},\dots,u_{n+N-1}$,

$$\{u_0,\dots,u_{n-1},u_{n-1},\dots,u_{n+N-1},u_{n+N-1},\dots,u_K\}.$$

Then the new B-spline polygons of the curves $c(u)$ and $\tilde{d}(u)$ will be $\{\tilde{c}_0,\dots,\tilde{c}_{L'}\}$, $\{\tilde{d}_0,\dots,\tilde{d}_{L'}\}$,

$$\tilde{c}_i=\tilde{c}[\tilde{u}_i,\dots,\tilde{u}_{i+n}],\ \tilde{d}_i=\tilde{d}[\tilde{u}_i,\dots,\tilde{u}_{i+n}], \qquad (13)$$

for $i=0,\dots,L'$. The list of knots has been renumbered as $\{\tilde{u}_0,\dots,\tilde{u}_{K'}\}$ in order to have correlative indices.

This construction is useful to solve the following interpolation problem:

**Problem 2:** Given a spline curve $c(u)$ of degree $n$, $N$ pieces, B-spline polygon $\{c_0,\dots,c_L\}$ and list of knots $\{u_0,\dots,u_K\}$, $u\in[a,b]$, $a=u_{n-1}$, $b=u_{n+N-1}$, and two points $d_0$, $d_L$, find a developable surface $b(u,v)$ such that $c(u,0)=c(u)$, $c(a,1)=d_0$, $c(b,1)=d_L$.

The procedure for solving this problem is clear:

1. Write the algebraic equation 10 with the B-spline polygon for $c(u)$, the list of knots and vectors for the rulings $\overrightarrow{c_0d_0}$, $\overrightarrow{c_Ld_L}$ and obtain a solution $M_0^*$.

2. Fix $\Lambda_0^*$ by choosing $d_0$ in Eq. 11 ($\sigma=1$, but $\tau\neq 1$ in general).

3. Use the recursivity relation in Eq. 8 for computing the vertices of $d(u)$.

4. Increase by one the multiplicity of the inner knots of the boundary curves.

5. Formally raise the degree of $c(u)$ and compute the new B-spline vertices $\tilde{c}_i$ with Eq. 5.

6. Choose $f(u)$ so that $f(a)=1$, $f(b)=1/\tau$,
$$f(u)=\frac{b-u}{b-a}+\frac{1}{\tau}\frac{u-a}{b-a}. \qquad (14)$$

7. Use this function to compute the B-spline vertices $\tilde{d}_i$ for the new boundary curve $\tilde{d}(u)$ with Eq. 13 and Eq. 12.

8. The B-spline polygons $\{\tilde{c}_0,\dots,\tilde{c}_{L'}\}$, $\{\tilde{d}_0,\dots,\tilde{d}_{L'}\}$ form the B-spline net for the developable patch complying with the prescription.

We go back now to Example 2:

**Example 3** Consider a spline curve of degree three and three pieces with B-spline polygon

$$c_0=(0,0,0),\ c_1=(2,3,0),\ c_2=(4,3,0),$$

$$c_3=(5,0,0),\ c_4=(7,2,1),\ c_5=(9,-1,3),$$

and list of knots $\{0,0,0,0.3,0.7,1,1,1\}$. For the first ruling we choose direction $\mathbf{v}=(0,0,2)$ and for the last ruling we choose $\mathbf{w}=(-1,0,1)$. Find a developable surface patch bounded by $c(u)$, an unknown curve $\tilde{d}(u)$ and the rulings defined by $\mathbf{v}$, $\mathbf{w}$, such that $\tilde{d}(0)=c_0+\mathbf{v}=(0,0,2)$, $\tilde{d}(1)=c_5+\mathbf{w}=(8,-1,4)$.

We already have obtained that the spline curve with B-spline polygon

$$d_0=(0,0,2),\ d_1=(1.56,2.34,2.08),$$

$$d_2=(3.09,2.29,2.26),\ d_3=(3.75,-0.15,2.55),$$

$$d_4=(5.22,1.42,3.55),\ d_5=(6.76,-1.00,5.24),$$

and the same list of knots provides a developable surface patch with the required prescription except that $d_5$ lies on the final ruling, but it is not $(8,-1,4)$. In fact, $d_5=c_5+\tau\mathbf{w}$ with $\tau=2.24$.

In order to shorten the surface patch so that the final vertex of the new boundary curve $\tilde{d}(u)$ is $(8,-1,4)$, we have to raise the degree of the curves from three to four.

Increasing the multiplicity of the inner knots 0, 0.3, 0.7, 1, we get the new list of knots for the degree-elevated curves,

$$\{0,0,0,0,0.3,0.3,0.7,0.7,1,1,1,1\}.$$

We calculate first the B-spline polygon for $c(u)$ as a curve of formal degree four with Eq. 5. The auxiliary points are computed in Appendix A.

$$\begin{aligned}
\tilde{c}_0 &= \tilde{c}[0,0,0,0]=c[0,0,0]=(0,0,0)\\
\tilde{c}_1 &= \tilde{c}[0,0,0,0.3]=\frac{c[0,0,0]+3c[0,0,0.3]}{4}\\
&= (1.5,2.25,0)\\
\tilde{c}_2 &= \tilde{c}[0,0,0.3,0.3]=\frac{c[0,0,0.3]+c[0,0.3,0.3]}{2}\\
&= (2.43,3,0)\\
\tilde{c}_3 &= \tilde{c}[0,0.3,0.3,0.7]\\
&= \frac{c[0,0.3,0.3]+2c[0,0.3,0.7]+c[0.3,0.3,0.7]}{4}\\
&= (3.79,2.78,0)
\end{aligned}$$

$$
\begin{aligned}
\tilde{c}_4 &= \tilde{c}[0.3,0.3,0.7,0.7] \\
&= \frac{c[0.3,0.3,0.7]+c[0.3,0.7,0.7]}{2} = (4.5,1.5,0) \\
\tilde{c}_5 &= \tilde{c}[0.3,0.7,0.7,1] \\
&= \frac{c[0.3,0.7,0.7]+2c[0.3,0.7,1]+c[0.7,0.7,1]}{4} \\
&= (5.21,0.51,0.14) \\
\tilde{c}_6 &= \tilde{c}[0.7,0.7,1,1] = \frac{c[0.7,0.7,1]+c[0.7,1,1]}{2} \\
&= (6.57,1.57,0.79) \\
\tilde{c}_7 &= \tilde{c}[0.7,1,1,1] = \frac{3c[0.7,1,1]+c[1,1,1]}{4} \\
&= (7.5,1.25,1.5) \\
\tilde{c}_8 &= \tilde{c}[1,1,1,1] = c[1,1,1] = (9,-1,3).
\end{aligned}
$$

Now we have to move the curve $d(u)$ over the developable surface patch so that the new boundary curve $\tilde{d}(u)$ goes through the endpoints of both rulings, shortening the director vector $\mathbf{w}(u)$ by a factor $f(u)$ as in Eq. 14,

$$f(u) = (1-u) + \frac{u}{2.24}.$$

Finally, we use Eq. 12 to compute the B-spline polygon of the new boundary curve of degree four that goes through the endpoints of both rulings,

$$
\begin{aligned}
\tilde{d}_0 &= \tilde{d}[0,0,0,0] = f(0)d[0,0,0] + (1-f(0))c[0,0,0] \\
&= d_0 = (0,0,2) \\
\tilde{d}_1 &= \tilde{d}[0,0,0,0.3] = \frac{f(0.3)d[0,0,0]+3f(0)d[0,0,0.3]}{4} \\
&+ \frac{(1-f(0.3))c[0,0,0]3(1-f(0)c[0,0,0.3]}{4} \\
&= (1.17,1.76,1.97) \\
\tilde{d}_2 &= \tilde{d}[0,0,0.3,0.3] = \frac{f(0.3)d[0,0,0.3]+f(0)d[0,0.3,0.3]}{2} \\
&+ \frac{(1-f(0.3))c[0,0,0.3]+(1-f(0)c[0,0.3,0.3]}{2} \\
&= (1.93,2.39,1.94) \\
\tilde{d}_3 &= \tilde{d}[0,0.3,0.3,0.7] = \frac{f(0.7)d[0,0.3,0.3]}{4} \\
&+ \frac{2f(0.3)d[0,0.3,0.7]+f(0)d[0.3,0.3,0.7]}{4} \\
&+ \frac{(1-f(0.7))c[0,0.3,0.3]}{4} \\
&+ \frac{(1-f(0.3))c[0,0.3,0.7]}{2} \\
&+ \frac{(1-f(0))c[0.3,0.3,0.7]}{4} = (3.06,2.24,1.86)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{d}_4 &= \tilde{d}[0.3,0.3,0.7,0.7] = \frac{f(0.3)d[0.3,0.7,0.7]}{2} \\
&+ \frac{f(0.7)d[0.3,0.3,0.7]+(1-f(0.3))c[0.3,0.7,0.7]}{2} \\
&+ \frac{(1-f(0.7))c[0.3,0.3,0.7]}{2} = (3.71,1.20,1.74) \\
\tilde{d}_5 &= \tilde{d}[0.3,0.7,0.7,1] = \frac{f(1)d[0.3,0.7,0.7]}{4} \\
&+ \frac{2f(0.7)d[0.3,0.7,1]+f(0.3)d[0.7,0.7,1]}{4} \\
&+ \frac{(1-f(1))c[0.3,0.7,0.7]}{4} \\
&+ \frac{(1-f(0.7))c[0.3,0.7,1]}{2} \\
&+ \frac{(1-f(0.3))c[0.7,0.7,1]}{4} = (4.38,0.35,1.73) \\
\tilde{d}_6 &= \tilde{d}[0.7,0.7,1,1] = \frac{f(1)d[0.7,0.7,1]}{2} \\
&+ \frac{f(0.7)d[0.7,1,1]+(1-f(1))c[0.7,0.7,1]}{2} \\
&+ \frac{(1-f(0.7))c[0.7,1,1]}{2} = (5.68,1.30,2.14) \\
\tilde{d}_7 &= \tilde{d}[0.7,1,1,1] = \frac{3f(1)d[0.7,1,1]+f(0.7)d[1,1,1]}{4} \\
&+ \frac{3(1-f(1))c[0.7,1,1]+(1-f(0.7))c[1,1,1]}{4} \\
&= (6.56,1.05,2.70) \\
\tilde{d}_8 &= \tilde{d}[1,1,1,1] = f(1)d[1,1,1] \\
&+ (1-f(1))c[1,1,1] = (8,-1,4).
\end{aligned}
$$

The degree-elevated B-spline net for the new surface patch, complying with the requirements of the example can be seen in Figure 7.

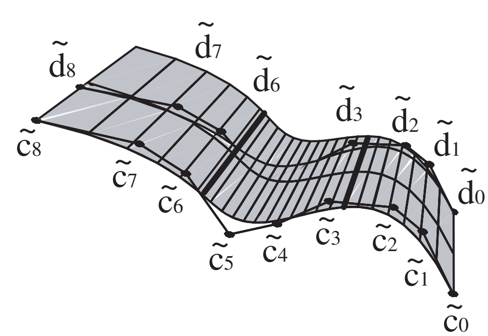


**Fig. 7 Degree-elevation and restriction of the developable surface patch in Figure 5**

We could also have split the original curve $c(u)$ in three cubic Bézier pieces and raise the degree of each of them to obtain curves of formally degree four with control points,

$$\tilde{C}_0 = (0,0,0),\ \tilde{C}_1 = (1.5, 2.25, 0),\ \tilde{C}_2 = (2.43, 3, 0),$$

$$\tilde{C}_3 = (3.01, 2.90, 0),\ \tilde{C}_4 = (3.48, 2.61, 0),$$

$$\tilde{C}_5 = (4.09, 2.23, 0),\ \tilde{C}_6 = (4.5, 1.5, 0),$$

$$\tilde{C}_7 = (4.910.93, 0.08),\ \tilde{C}_8 = (5.52, 1.04, 0.33),$$

$$\tilde{C}_9 = (5.99, 1.12, 0.51),\ \tilde{C}_{10} = (6.57, 1.57, 0.79),$$

$$\tilde{C}_{11} = (7.5, 1.25, 1.5),\ \tilde{C}_{12} = (9, -1, 3),$$

and use the construction in Aumann (2004) to extend each Bézier developable surface patch to comply with the prescription of endpoints, by multiplication by the same factor $f(u)$. One reaches the same result as applying insertion of knots $0, 0.3, 0.7, 1$ to $\tilde{d}(u)$,

$$\tilde{D}_0 = (0,0,2),\ \tilde{D}_1 = (1.17, 1.76, 1.97),$$

$$\tilde{D}_2 = (1.93, 2.39, 1.94),\ \tilde{D}_3 = (2.41, 2.32, 1.91),$$

$$\tilde{D}_4 = (2.81, 2.10, 1.87),\ \tilde{D}_5 = (3.34, 1.79, 1.81),$$

$$\tilde{D}_6 = (3.71, 1.20, 1.74),\ \tilde{D}_7 = (4.09, 0.71, 1.74),$$

$$\tilde{D}_8 = (4.68, 0.82, 1.86),\ \tilde{D}_9 = (5.12, 0.89, 1.96),$$

$$\tilde{D}_{10} = (5.68, 1.30, 2.14),\ \tilde{D}_{11} = (6.56, 1.05, 2.70),$$

$$\tilde{D}_{12} = (8, -1, 4).$$

The boundary rulings of the quartic Bézier developable surfaces have been marked in Figure 7.

## 7 Triangular developable surfaces

We may pose another interpolation problem in which the first ruling collapses to a point, $c(a) = d(a)$,

$$b(u,v) = (1-v)c(u) + vd(u), \quad u \in [a,b].$$

The resulting developable patch is triangular in the sense that it is bounded by two curves and just one straight segment. Instead of the first point of the unknown curve of the boundary, we may give as datum its initial velocity $d'(a)$.

**Problem 3:** Given a spline curve $c(u)$ of degree $n$, $N$ pieces, B-spline polygon $\{c_0, \dots, c_L\}$ and list of knots $\{u_0, \dots, u_K\}$, $u \in [a,b]$, $a = u_{n-1}$, $b = u_{n+N-1}$, a point $d_L$ and a vector $d'(a)$, find a triangular developable surface $b(u,v)$ through $c(u)$, such that $c(u,0) = c(u)$, $c(a,v) = c_0$ for all $v$, $c(b,1) = d_L$, $c_u(a,1) = d'(a)$.

We do not know the first ruling of the surface, but we may use previous constructions to compute a spline developable patch through the curve $c(u)$ and use $d_L$ to fix the last ruling,

$$b(u,v) = c(u) + v\mathbf{w}(u), \quad \mathbf{w}(u) = d(u) - c(u).$$

In order to collapse the first ruling to a point, we shorten the patch along the rulings,

$$\hat{b}(u,v) = c(u) + vf(u)\mathbf{w}(u), \quad f(u) = \frac{u-a}{b-a}, \quad (15)$$

so that $\hat{c}(a,v) = c_0$ for all $v$.

We compute the velocity,

$$\hat{b}_u(u,v) = c'(u) + \frac{v}{b-a}\mathbf{w}(u) + vf(u)\mathbf{w}'(u),$$

of the boundary curve $d(u)$ at $u = a$, making use of Eq. 4

$$\begin{aligned} \hat{d}'(a) &= \hat{c}_u(a,1) = c'(a) + \frac{\mathbf{w}(a)}{b-a} \\ &= n\frac{c_1 - c_0}{u_n - u_{n-1}} + \frac{d_0 - c_0}{b-a} \end{aligned}$$

and from this expression we get the vertex $d_0$ that is necessary for obtaining the velocity $\hat{d}'(a)$,

$$d_0 = c_0 + (b-a)\left(\hat{d}'(a) - n\frac{c_1 - c_0}{u_n - u_{n-1}}\right), \quad (16)$$

Since we need to fix both $d_0$ and $d_L$ to obtain the developable patch $b(u,v)$, the construction from the previous section is required and hence such a patch must be of degree $n+1$. Since $c(u)$ is still of degree $n$, the calculation done in Eq. 16 is nonetheless valid whereas we keep the original vertices $c_0$ and $c_1$. Finally, shortening the surface patch as in Eq. 15 with $f(u)$ produces a triangular patch of degree $n+2$.

Summarising, the solution of this problem is reduced to the one of Problem 2:

1. Calculate the vertex $d_0$ and $\mathbf{v} = d_0 - c_0$ using Eq. 16.

2. Write the algebraic equation 10 with the B-spline polygon for $c(u)$, the list of knots and vectors for the rulings $\overrightarrow{c_0 d_0}$, $\overrightarrow{c_L d_L}$ and obtain a solution $M_0^*$.

3. Fix $\Lambda_0^*$ by choosing $d_0$ in Eq. 11 ($\sigma = 1$, but $\tau \neq 1$ in general).

4. Use the recursivity relation in Eq. 8 for computing the vertices of $d(u)$.

5. Increase by one the multiplicity of the inner knots of the boundary curves.

6. Formally raise the degree of $c(u)$ and compute the new B-spline vertices $\tilde{c}_i$ with Eq. 5.

7. Choose $f(u)$ so that $f(a) = 1$, $f(b) = 1/\tau$,

$$f(u) = \frac{b-u}{b-a} + \frac{1}{\tau}\frac{u-a}{b-a}.$$

8. Use this function to compute the B-spline vertices $\tilde{d}_i$ for the new boundary curve $\tilde{d}(u)$ with Eq. 13 and Eq. 12.

9. Increase by one the multiplicity of the inner knots of the boundary curves.

10. Formally raise the degree of $\tilde{c}(u)$ and compute the new B-spline vertices $\hat{c}_i$ with Eq. 5.

11. Use a function $\hat{f}(u) = u$ to shrink the first ruling to a point and compute the B-spline vertices $\hat{d}_i$ for the new boundary curve $\hat{d}(u)$ with Eq. 13 and Eq. 12.

12. The B-spline polygons $\{\hat{c}_0, \ldots, \hat{c}_{L'}\}$, $\{\hat{d}_0, \ldots, \hat{d}_{L'}\}$ form the B-spline net for the triangular developable patch complying with the prescription.

**Example 4** Consider a spline curve of degree three and three pieces with B-spline polygon

$$c_0 = (0,0,0),\ c_1 = (2,3,0),\ c_2 = (4,3,0),$$

$$c_3 = (5,0,0),\ c_4 = (7,2,1),\ c_5 = (9,-1,3),$$

and list of knots $\{0,0,0,0.3,0.7,1,1,1\}$. For the last ruling we choose direction $\mathbf{w} = (-1,0,1)$. Find a triangular developable surface patch bounded by $c(u)$, an unknown curve $\hat{d}(u)$ and the ruling defined by $\mathbf{w}$, such that $\hat{d}(0) = c_0$, $\hat{d}'(0) = (20, 30.5, 2)$, $\hat{d}(1) = c_5 + \mathbf{w} = (8,-1,4)$.

First of all, we calculate the first ruling of the developable surface. According to Eq. 16 we need

$$\mathbf{v} = d_0 - c_0 = \hat{d}'(0) + \frac{3}{0.3}(c_0 - c_1) = (0, 0.5, 2),$$

and we calculate the determinant in Eq. 10,

$$\det(a(M^*) - c_L, \mathbf{v}, \mathbf{w})$$

$$= \frac{8M^{*4} + 2.6M^{*3} - 16M^{*2} + 14.5M^* - 3.5}{M^{*3}(M^* - 0.3)(M^* - 0.7)},$$

so that developability is granted by choosing parameter $M^*$ as a real solution of

$$8M^{*4} + 2.6M^{*3} - 16M^{*2} + 14.5M^* - 3.5 = 0,$$

that is $M^* = -1.92,\ 0.38$. The other two solutions are complex.

For having $d_0 = (0, 0.5, 2)$ on the first ruling, we need to take $\sigma = 1$ in Eq. 11. The respective values of parameter $\Lambda^*$ are $-1.16$, $0.59$. We choose the first pair of parameters for our calculations, $\Lambda_0^* = -1.16$, $M_0^* = 0.59$. We calculate next the B-spline

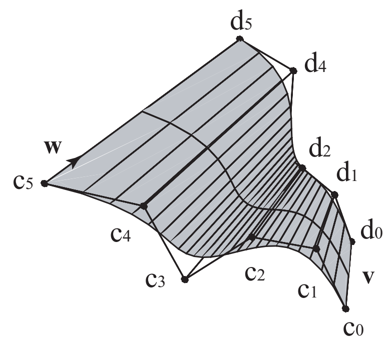


**Fig. 8 Developable surface of degree 3 and 3 pieces**

polygon for the second boundary curve according to Corollary 1,

$$d_{i+1} = \frac{(u_{i+n} - \Lambda^*)c_i + (\Lambda^* - u_i)c_{i+1} + (M^* - u_{i+n})d_i}{M^* - u_i}$$

for $i = 0 \ldots L-1$.

$$\begin{aligned}
d_0 &= (0, 0.5, 2)\\
d_1 &= \frac{(u_3 - \Lambda^*)c_0 + (\Lambda^* - u_0)c_1 + (M^* - u_3)d_0}{M^* - u_0}\\
&= (1.21, 2.39, 2.31)\\
d_2 &= \frac{(u_4 - \Lambda^*)c_1 + (\Lambda^* - u_1)c_2 + (M^* - u_4)d_1}{M^* - u_1}\\
&= (2.13, 2.17, 3.16)\\
d_3 &= \frac{(u_5 - \Lambda^*)c_2 + (\Lambda^* - u_2)c_3 + (M^* - u_5)d_2}{M^* - u_2}\\
&= (1.77, -0.07, 4.80)\\
d_4 &= \frac{(u_6 - \Lambda^*)c_3 + (\Lambda^* - u_3)c_4 + (M^* - u_6)d_3}{M^* - u_3}\\
&= (2.07, 1.22, 6.97)\\
d_5 &= \frac{(u_7 - \Lambda^*)c_4 + (\Lambda^* - u_4)c_5 + (M^* - u_7)d_4}{M^* - u_4}\\
&= (2.92, -1.00, 9.08).
\end{aligned}$$

Hence, $d_5 - c_5 = \tau \mathbf{w}$, with $\tau = 6.08$. We show the surface patch in Figure 8.

Next we shorten the surface patch so that the new boundary curve $\hat{d}(u)$ ends up at $(8, -1, 4)$. From the previous example we know that we are to increase the multiplicity of the inner knots by one,

$$\{0, 0, 0, 0, 0.3, 0.3, 0.7, 0.7, 1, 1, 1, 1\},$$

and formally raise the degree of $c(u)$ to four,

$$\tilde{c}_0 = (0, 0, 0),\ \tilde{c}_1 = (1.5, 2.25, 0),\ \tilde{c}_2 = (2.43, 3, 0),$$

$$\tilde{c}_3 = (3.79, 2.78, 0),\ \tilde{c}_4 = (4.5, 1.5, 0),$$

$$\tilde{c}_5 = (5.21, 0.51, 0.14),\ \tilde{c}_6 = (6.57, 1.57, 0.79),$$

$$\tilde{c}_7 = (7.5, 1.25, 1.5),\ \tilde{c}_8 = (9, -1, 3),$$

and shorten the director vector $\mathbf{w}(u)$ by a factor $f(u)$ as in Eq. 14,

$$f(u) = (1 - u) + \frac{u}{6.08},$$

so that the new boundary curve $\tilde{d}(u)$ has degree four and B-spline polygon using Eq. 12, given by

$$\begin{aligned}
\tilde{d}_0 &= \tilde{d}[0, 0, 0, 0] = f(0)d[0, 0, 0] + (1 - f(0))c[0, 0, 0] \\
&= d_0 = (0, 0.5, 2) \\
\tilde{d}_1 &= \tilde{d}[0, 0, 0, 0.3] = \frac{f(0.3)d[0, 0, 0] + 3f(0)d[0, 0, 0.3]}{4} \\
&+ \frac{(1 - f(0.3))c[0, 0, 0] + 3(1 - f(0)c[0, 0, 0.3]}{4} \\
&= (0.91, 1.89, 2.11) \\
\tilde{d}_2 &= \tilde{d}[0, 0, 0.3, 0.3] = \frac{f(0.3)d[0, 0, 0.3] + f(0)d[0, 0.3, 0.3]}{2} \\
&+ \frac{(1 - f(0.3))c[0, 0, 0.3] + (1 - f(0)c[0, 0.3, 0.3]}{2} \\
&= (1.51, 2.42, 2.20) \\
\tilde{d}_3 &= \tilde{d}[0, 0.3, 0.3, 0.7] = \frac{f(0.7)d[0, 0.3, 0.3]}{4} \\
&+ \frac{2f(0.3)d[0, 0.3, 0.7] + f(0)d[0.3, 0.3, 0.7]}{4} \\
&+ \frac{(1 - f(0.7))c[0, 0.3, 0.3] + 2(1 - f(0.3))c[0, 0.3, 0.7]}{4} \\
&+ \frac{(1 - f(0))c[0.3, 0.3, 0.7]}{4} = (2.39, 2.24, 2.37) \\
\tilde{d}_4 &= \tilde{d}[0.3, 0.3, 0.7, 0.7] = \frac{f(0.3)d[0.3, 0.7, 0.7]}{2} \\
&+ \frac{f(0.7)d[0.3, 0.3, 0.7] + (1 - f(0.3))c[0.3, 0.7, 0.7]}{2} \\
&= \frac{(1 - f(0.7))c[0.3, 0.3, 0.7]}{2} = (2.97, 1.26, 2.37) \\
\tilde{d}_5 &= \tilde{d}[0.3, 0.7, 0.7, 1] = \frac{f(1)d[0.3, 0.7, 0.7]}{4} \\
&+ \frac{2f(0.7)d[0.3, 0.7, 1] + f(0.3)d[0.7, 0.7, 1]}{4} \\
&+ \frac{(1 - f(1))c[0.3, 0.7, 0.7]}{4} \\
&+ \frac{2(1 - f(0.7))c[0.3, 0.7, 1] + (1 - f(0.3))c[0.7, 0.7, 1]}{4} \\
&= (3.64, 0.39, 2.34) \\
\tilde{d}_6 &= \tilde{d}[0.7, 0.7, 1, 1] = \frac{f(1)d[0.7, 0.7, 1] + f(0.7)d[0.7, 1, 1]}{2} \\
&+ \frac{(1 - f(1))c[0.7, 0.7, 1] + (1 - f(0.7))c[0.7, 1, 1]}{2} \\
&= (5.20, 1.37, 2.48) \\
\tilde{d}_7 &= \tilde{d}[0.7, 1, 1, 1] = \frac{3f(1)d[0.7, 1, 1] + f(0.7)d[1, 1, 1]}{4} \\
&+ \frac{3(1 - f(1))c[0.7, 1, 1] + (1 - f(0.7))c[1, 1, 1]}{4} \\
&= (6.26, 1.15, 2.87) \\
\tilde{d}_8 &= \tilde{d}[1, 1, 1, 1] = f(1)d[1, 1, 1] + (1 - f(1))c[1, 1, 1] \\
&= (8, -1, 4),
\end{aligned}$$

where the auxiliary points are computed with blossoms in Appendix B. The result of this restriction of the surface patch is shown in Figure 9.

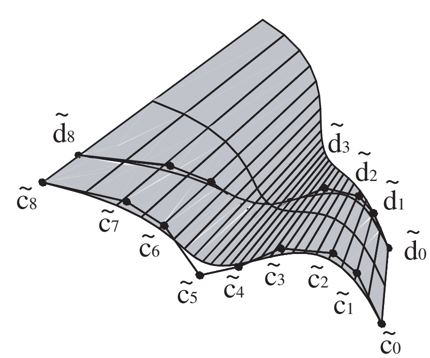


**Fig. 9 Restriction of the developable surface patch in Figure 8**

Finally, following Eq. 15, we further trim the surface patch bounded by $c(u)$ and $\tilde{d}(u)$ to shrink the first ruling to the vertex $c_0$.

Since we are raising the degree of the curves from four to five, we have to increase the multiplicity of the inner knots by one,

$$\{0,0,0,0,0,0.3,0.3,0.3,0.7,0.7,0.7,1,1,1,1,1\}.$$

The curve $c(u)$ becomes formally of degree five using Eq. 5 with B-spline polygon,

$$\begin{aligned}
\hat{c}_0 &= \hat{c}[0,0,0,0,0] = \tilde{c}[0,0,0,0] = (0,0,0)\\
\hat{c}_1 &= \hat{c}[0,0,0,0,0.3] = \frac{\tilde{c}[0,0,0,0]+4\tilde{c}[0,0,0,0.3]}{5}\\
&= (1.20, 1.80, 0.0)\\
\hat{c}_2 &= \hat{c}[0,0,0,0.3,0.3] = \frac{2\tilde{c}[0,0,0,0.3]}{5}\\
&+ \frac{3\tilde{c}[0,0,0.3,0.3]}{5} = (2.06, 2.70, 0.0)\\
\hat{c}_3 &= \hat{c}[0,0,0.3,0.3,0.3] = \frac{3\tilde{c}[0,0,0.3,0.3]}{5}\\
&+ \frac{2\tilde{c}[0,0.3,0.3,0.3]}{5} = (2.66, 2.96, 0.0)\\
\hat{c}_4 &= \hat{c}[0,0.3,0.3,0.3,0.7] = \frac{\tilde{c}[0,0.3,0.3,0.3]}{5}\\
&+ \frac{3\tilde{c}[0,0.3,0.3,0.7]+\tilde{c}[0.3,0.3,0.3,0.7]}{5}\\
&= (3.69, 2.69, 0.0)\\
\hat{c}_5 &= \hat{c}[0.3,0.3,0.3,0.7,0.7] = \frac{2\tilde{c}[0.3,0.3,0.3,0.7]}{5}\\
&+ \frac{3\tilde{c}[0.3,0.3,0.7,0.7]}{5} = (4.34, 1.79, 0.0)\\
\hat{c}_6 &= \hat{c}[0.3,0.3,0.7,0.7,0.7] = \frac{3\tilde{c}[0.3,0.3,0.7,0.7]}{5}\\
&+ \frac{2\tilde{c}[0.3,0.7,0.7,0.7]}{5} = (4.66, 1.27, 0.03)\\
\hat{c}_7 &= \hat{c}[0.3,0.7,0.7,0.7,1] = \frac{\tilde{c}[0.3,0.7,0.7,0.7]}{5}\\
&+ \frac{3\tilde{c}[0.3,0.7,0.7,1]+\tilde{c}[0.7,0.7,0.7,1]}{5}\\
&= (5.31, 0.72, 0.20)\\
\hat{c}_8 &= \hat{c}[0.7,0.7,0.7,1,1] = \frac{2\tilde{c}[0.7,0.7,0.7,1]}{5}\\
&+ \frac{3\tilde{c}[0.7,0.7,1,1]}{5} = (6.34, 1.39, 0.68)\\
\hat{c}_9 &= \hat{c}[0.7,0.7,1,1,1] = \frac{3\tilde{c}[0.7,0.7,1,1]}{5}\\
&+ \frac{2\tilde{c}[0.7,1,1,1]}{5} = (6.94, 1.44, 1.07)\\
\hat{c}_{10} &= \hat{c}[0.7,1,1,1,1] = \frac{4\tilde{c}[0.7,1,1,1]+\tilde{c}[1,1,1,1]}{5}\\
&= (7.80, 0.80, 1.80)\\
\hat{c}_{11} &= \hat{c}[1,1,1,1,1] = \tilde{c}[1,1,1,1] = (9,-1,3),
\end{aligned}$$

and following Eq. 15, we shrink the rulings with a factor $\hat{f}(u)=u$. The auxiliary points are computed using the multiaffinity property of blossoms in Appendix C.

Making use of Eq. 12, we obtain the B-spline polygon of the final boundary curve $\hat{d}(u)$ of degree five,

$$\begin{aligned}
\hat{d}_0 &= \hat{d}[0,0,0,0,0] = \hat{f}(0)\tilde{d}[0,0,0,0]\\
&+ (1-\hat{f}(0))\tilde{c}[0,0,0,0] = \tilde{c}_0 = (0,0,0)\\
\hat{d}_1 &= \hat{d}[0,0,0,0,0.3] = \frac{\hat{f}(0.3)\tilde{d}[0,0,0,0]}{5}\\
&+ \frac{4\hat{f}(0)\tilde{d}[0,0,0,0.3]+(1-\hat{f}(0.3))\tilde{c}[0,0,0,0]}{5}\\
&+ \frac{4(1-\hat{f}(0)\tilde{c}[0,0,0,0.3]}{5}\\
&= (1.20, 1.83, 0.12)\\
\hat{d}_2 &= \hat{d}[0,0,0,0.3,0.3] = \frac{2\hat{f}(0.3)\tilde{d}[0,0,0,0.3]}{5}\\
&+ \frac{3\hat{f}(0)\tilde{d}[0,0,0.3,0.3]+2(1-\hat{f}(0.3))\tilde{c}[0,0,0,0.3]}{5}\\
&+ \frac{3(1-\hat{f}(0)\tilde{c}[0,0,0.3,0.3]}{5} = (1.99, 2.66, 0.25)\\
\hat{d}_3 &= \hat{d}[0,0,0.3,0.3,0.3] = \frac{3\hat{f}(0.3)\tilde{d}[0,0,0.3,0.3]}{5}\\
&+ \frac{2\hat{f}(0)\tilde{d}[0,0.3,0.3,0.3]+3(1-\hat{f}(0.3))\tilde{c}[0,0,0.3,0.3]}{5}\\
&+ \frac{2(1-\hat{f}(0)\tilde{c}[0,0.3,0.3,0.3]}{5} = (2.50, 2.86, 0.40)\\
\hat{d}_4 &= \hat{d}[0,0.3,0.3,0.3,0.7] = \frac{\hat{f}(0.7)\tilde{d}[0,0.3,0.3,0.3]}{5}\\
&+ \frac{3\hat{f}(0.3)\tilde{d}[0,0.3,0.3,0.7]+\hat{f}(0)\tilde{d}[0.3,0.3,0.3,0.7]}{5}\\
&+ \frac{(1-\hat{f}(0.7))\tilde{c}[0,0.3,0.3,0.3]}{5}\\
&+ \frac{3(1-\hat{f}(0.3))\tilde{c}[0,0.3,0.3,0.7]}{5}\\
&+ \frac{(1-\hat{f}(0))\tilde{c}[0.3,0.3,0.3,0.7]}{5} = (3.29, 2.52, 0.75)\\
\hat{d}_5 &= \hat{d}[0.3,0.3,0.3,0.7,0.7] = \frac{2\hat{f}(0.7)\tilde{d}[0.3,0.3,0.3,0.7]}{5}\\
&+ \frac{3\hat{f}(0.3)\tilde{d}[0.3,0.3,0.7,0.7]}{5}\\
&+ \frac{2(1-\hat{f}(0.7))\tilde{c}[0.3,0.3,0.3,0.7]}{5}\\
&+ \frac{3(1-\hat{f}(0.3))\tilde{c}[0.3,0.3,0.7,0.7]}{5}\\
&+ (3.65, 1.64, 1.09)
\end{aligned}$$

$$\begin{aligned}
\hat{d}_6 &= \hat{d}[0.3,0.3,0.7,0.7,0.7] = \frac{3\hat{f}(0.7)\tilde{d}[0.3,0.3,0.7,0.7]}{5}\\
&+ \frac{2\hat{f}(0.3)\tilde{d}[0.3,0.7,0.7,0.7]}{5}\\
&+ \frac{3(1-\hat{f}(0.7))\tilde{c}[0.3,0.3,0.7,0.7]}{5}\\
&+ \frac{2(1-\hat{f}(0.3))\tilde{c}[0.3,0.7,0.7,0.7]}{5}\\
&+ (3.83,1.15,1.30)\\
\hat{d}_7 &= \hat{d}[0.3,0.7,0.7,0.7,1] = \frac{\hat{f}(1)\tilde{d}[0.3,0.7,0.7,0.7]}{5}\\
&+ \frac{3\hat{f}(0.7)\tilde{d}[0.3,0.7,0.7,1]+\hat{f}(0.3)\tilde{d}[0.7,0.7,0.7,1]}{5}\\
&+ \frac{(1-\hat{f}(1))\tilde{c}[0.3,0.7,0.7,0.7]}{5}\\
&+ \frac{3(1-\hat{f}(0.7))\tilde{c}[0.3,0.7,0.7,1]}{5}\\
&+ \frac{(1-\hat{f}(0.3))\tilde{c}[0.7,0.7,0.7,1]}{5} = (4.25,0.62,1.70)\\
\hat{d}_8 &= \hat{d}[0.7,0.7,0.7,1,1] = \frac{2\hat{f}(1)\tilde{d}[0.7,0.7,0.7,1]}{5}\\
&+ \frac{3\hat{f}(0.7)\tilde{d}[0.7,0.7,1,1]+2(1-\hat{f}(1))\tilde{c}[0.7,0.7,0.7,1]}{5}\\
&+ \frac{3(1-\hat{f}(0.7))\tilde{c}[0.7,0.7,1,1]}{5} = (5.18,1.24,2.15)\\
\hat{d}_9 &= \hat{d}[0.7,0.7,1,1,1] = \frac{3\hat{f}(1)\tilde{d}[0.7,0.7,1,1]}{5}\\
&+ \frac{2\hat{f}(0.7)\tilde{d}[0.7,1,1,1]+3(1-\hat{f}(1))\tilde{c}[0.7,0.7,1,1]}{5}\\
&+ \frac{2(1-\hat{f}(0.7))\tilde{c}[0.7,1,1,1]}{5} = (5.77,1.30,2.47)\\
\hat{d}_{10} &= \hat{d}[0.7,1,1,1,1] = \frac{4\hat{f}(1)\tilde{d}[0.7,1,1,1]}{5}\\
&+ \frac{\hat{f}(0.7)\tilde{d}[1,1,1,1]+4(1-\hat{f}(1))\tilde{c}[0.7,1,1,1]}{5}\\
&+ \frac{(1-\hat{f}(0.7))\tilde{c}[1,1,1,1]}{5}\\
&= (6.67,0.72,3.03)\\
\hat{d}_{11} &= \hat{d}[1,1,1,1,1] = \hat{f}(1)\tilde{d}[1,1,1,1]\\
&+ (1-\hat{f}(1))\tilde{c}[1,1,1,1] = \tilde{d}_{11} = (8,-1,4).
\end{aligned}$$

The triangular B-spline net for the surface patch which satisfies the requirements of the example is shown in Figure 10. We check that in fact the velocity of the boundary curve $\hat{d}(u)$ of degree $n = 5$ is as

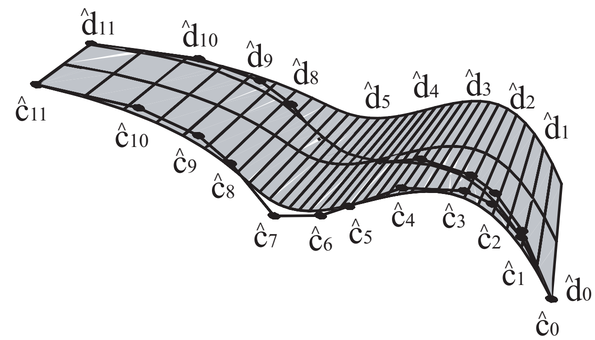


**Fig. 10 Restriction to a triangular patch of the developable surface patch in Figure 9**

prescribed,

$$\begin{aligned}
\hat{d}'(0) &= n\frac{\hat{d}_1-\hat{d}_0}{\hat{u}_n-\hat{u}_{n-1}} = \frac{5}{0.3}(1.20,1.83,0.12)\\
&= (20.00,30.50,2.00).
\end{aligned}$$

## 8 Conclusions

We have made use of a procedure of degree elevation for obtaining spline developable surfaces from which we know the segments of the first and last rulings and one of the curves of the boundary. It consists of first solving the problem with free endpoints of the rulings and then moving the resulting boundary curve along the rulings to match the endpoints and increase the degree of the curves by one. This solution is also used to solve the problem of finding a triangular spline developable patch from which we know the last ruling, one of the curves of the boundary and the initial velocity of the other curve.

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## A Auxiliary points for $c(u)$

We perform here calculations of auxiliary points for the curve $c(u)$ over the list of knots $\{0,0,0,0.3,0.7,1,1,1\}$ which are needed for Example 3, taking into account that blossoms are multi-affine Eq. (5):

$$\begin{aligned}
c[0,0,0] &= c_0 = C_0 = (0,0,0)\\
c[0,0,0.3] &= c_1 = C_1 = (2,3,0)\\
c[0,0.3,0.3] &= C_2 = \frac{0.7-0.3}{0.7-0}c[0,0,0.3]\\
&\quad + \frac{0.3-0}{0.7-0}c[0,0.7,0.3] = \frac{0.4c_1+0.3c_2}{0.7}\\
&= (2.86,3,0)\\
c[0,0.3,0.7] &= c_2 = (4,3,0)\\
c[0.3,0.3,0.3] &= C_3 = \frac{0.7-0.3}{0.7-0}c[0,0.3,0.3]\\
&\quad + \frac{0.3-0}{0.7-0}c[0.3,0.3,0.7] = \frac{0.4C_2+0.3C_4}{0.7}\\
&= (3.48,2.61,0)\\
c[0.3,0.3,0.7] &= C_4 = \frac{1-0.3}{1-0}c[0,0.3,0.7]\\
&\quad + \frac{0.3-0}{1-0}c[1,0.3,0.7]\\
&= 0.7c_2+0.3c_3 = (4.3,2.1,0)\\
c[0.3,0.7,0.7] &= C_5 = \frac{1-0.7}{1-0}c[0,0.3,0.7]\\
&\quad + \frac{0.7-0}{1-0}c[1,0.3,0.7] = 0.3c_2+0.7c_3\\
&= (4.7,0.9,0)\\
c[0.3,0.7,1] &= c_3 = (5,0,0)\\
c[0.7,0.7,0.7] &= C_6 = \frac{1-0.7}{1-0.3}c[0.3,0.7,0.7]\\
&\quad + \frac{0.7-0.3}{1-0.3}c[0.7,0.7,1] = \frac{0.3C_5+0.4C_7}{0.7}\\
&= (5.52,1.04,0.33)\\
c[0.7,0.7,1] &= C_7 = \frac{1-0.7}{1-0.3}c[0.3,0.7,1]\\
&\quad + \frac{0.7-0.3}{1-0.3}c[1,0.7,1] = \frac{0.3c_3+0.4c_4}{0.7}\\
&= (6.14,1.14,0.57)\\
c[0.7,1,1] &= c_4 = C_8 = (7,2,1)\\
c[1,1,1] &= c_5 = C_9 = (9,-1,3).
\end{aligned}$$

And similarly for $d(u)$,

$$\begin{aligned}
d[0,0,0] &= d_0 = D_0 = (0,0,2)\\
d[0,0,0.3] &= d_1 = D_1 = (1.56,2.34,2.08)\\
d[0,0.3,0.3] &= D_2 = \frac{0.7-0.3}{0.7-0}d[0,0,0.3]\\
&\quad + \frac{0.3-0}{0.7-0}d[0,0.7,0.3] = \frac{0.4d_1+0.3d_2}{0.7}\\
&= (2.21,2.32,2.15)\\
d[0.3,0.3,0.3] &= D_3 = \frac{0.7-0.3}{0.7-0}d[0,0.3,0.3]\\
&\quad + \frac{0.3-0}{0.7-0}d[0.3,0.3,0.7] = \frac{0.4D_2+0.3D_4}{0.7}\\
&= (2.67,1.99,2.24)\\
d[0,0.3,0.7] &= d_2 = (3.09,2.29,2.26)\\
d[0.3,0.3,0.7] &= D_4 = \frac{1-0.3}{1-0}d[0,0.3,0.7]\\
&\quad + \frac{0.3-0}{1-0}d[1,0.3,0.7] = 0.7d_2+0.3d_3\\
&= (3.29,1.56,2.35)\\
d[0.3,0.7,0.7] &= D_5 = \frac{1-0.7}{1-0}d[0,0.3,0.7]\\
&\quad + \frac{0.7-0}{1-0}d[1,0.3,0.7] = 0.3d_2+0.7d_3\\
&= (3.55,0.58,2.46)\\
d[0.3,0.7,1] &= d_3 = (3.75,-0.15,2.55)\\
d[0.7,0.7,0.7] &= D_6 = \frac{1-0.7}{1-0.3}d[0.3,0.7,0.7]\\
&\quad + \frac{0.7-0.3}{1-0.3}d[0.7,0.7,1] = \frac{0.3D_5+0.4D_7}{0.7}\\
&= (4.15,0.68,2.84)\\
d[0.7,0.7,1] &= D_7 = \frac{1-0.7}{1-0.3}d[0.3,0.7,1]\\
&\quad + \frac{0.7-0.3}{1-0.3}d[1,0.7,1] = \frac{0.3d_3+0.4d_4}{0.7}\\
&= (4.59,0.75,3.12)\\
d[0.7,1,1] &= d_4 = D_8 = (5.22,1.42,3.55)\\
d[1,1,1] &= d_5 = D_9 = (6.76,-1.00,5.24).
\end{aligned}$$

## B Auxiliary points for $d(u)$

We compute here auxiliary points for the curve $d(u)$ over the list of knots $\{0,0,0,0.3,0.7,1,1,1\}$ which are needed for Example 4, using the property of multiaffinity (Eq. 5) for blossoms:

$$\begin{aligned}
d[0,0,0] &= d_0 = (0,0.5,2)\\
d[0,0,0.3] &= d_1 = (1.21,2.39,2.31)
\end{aligned}$$

$$
\begin{aligned}
d[0,0.3,0.3] &= \frac{0.7-0.3}{0.7-0} d[0,0,0.3] \\
&\quad + \frac{0.3-0}{0.7-0} d[0,0.7,0.3] = \frac{0.4d_1+0.3d_2}{0.7} \\
&= (1.61, 2.30, 2.67) \\
d[0,0.3,0.7] &= d_2 = (2.13, 2.17, 3.16) \\
d[0.3,0.3,0.7] &= \frac{1-0.3}{1-0} d[0,0.3,0.7] \\
&\quad + \frac{0.3-0}{1-0} d[1,0.3,0.7] = 0.7d_2 + 0.3d_3 \\
&= (2.02, 1.50, 3.65) \\
d[0.3,0.7,0.7] &= \frac{1-0.7}{1-0} d[0,0.3,0.7] \\
&\quad + \frac{0.7-0}{1-0} d[1,0.3,0.7] = 0.3d_2 + 0.7d_3 \\
&= (1.88, 0.60, 4.31) \\
d[0.3,0.7,1] &= d_3 = (1.77, -0.07, 4.80) \\
d[0.7,0.7,1] &= \frac{1-0.7}{1-0.3} d[0.3,0.7,1] \\
&\quad + \frac{0.7-0.3}{1-0.3} d[1,0.7,1] = \frac{0.3d_3+0.4d_4}{0.7} \\
&= (1.94, 0.67, 6.04) \\
d[0.7,1,1] &= d_4 = (2.07, 1.22, 6.97) \\
d[1,1,1] &= d_5 = (2.92, -1.00, 9.08).
\end{aligned}
$$

## C Auxiliary points for $\tilde{c}(u)$

Finally we calculate the auxiliary points which are necessary to formally raise the degree of the curve $\tilde{c}(u)$ with list of knots $\{0,0,0,0,0.3,0.3,0.7,0.7,1,1,1,1\}$ from four to five using the property of multiaffinity (Eq. 5) for blossoms:

$$
\begin{aligned}
\tilde{c}[0,0,0,0] &= \tilde{c}_0 = (0,0,0) \\
\tilde{c}[0,0,0,0.3] &= \tilde{c}_1 = (1.50, 2.25, 0.00) \\
\tilde{c}[0,0,0.3,0.3] &= \tilde{c}_2 = (2.43, 3.00, 0.00) \\
\tilde{c}[0,0.3,0.3,0.3] &= \frac{0.7-0.3}{0.7-0} \tilde{c}[0,0,0.3,0.3] \\
&\quad + \frac{0.3-0}{0.7-0} \tilde{c}[0,0.7,0.3,0.3] \\
&= \frac{0.4\tilde{c}_2+0.3\tilde{c}_3}{0.7} = (3.01, 2.90, 0.00) \\
\tilde{c}[0,0.3,0.3,0.7] &= \tilde{c}_3 = (3.79, 2.78, 0.00) \\
\tilde{c}[0.3,0.3,0.3,0.7] &= \frac{0.7-0.3}{0.7-0} \tilde{c}[0,0.3,0.3,0.7] \\
&\quad + \frac{0.3-0}{0.7-0} \tilde{c}[0.7,0.3,0.3,0.7] \\
&= \frac{0.4\tilde{c}_3+0.3\tilde{c}_4}{0.7} = (4.09, 2.23, 0.00) \\
\tilde{c}[0.3,0.3,0.7,0.7] &= \tilde{c}_4 = (4.50, 1.50, 0.00) \\
\tilde{c}[0.3,0.7,0.7,0.7] &= \frac{1-0.7}{1-0.3} \tilde{c}[0.3,0.3,0.7,0.7] \\
&\quad + \frac{0.7-0.3}{1-0.3} \tilde{c}[1,0.3,0.7,0.7] \\
&= \frac{0.3\tilde{c}_4+0.4\tilde{c}_5}{0.7} = (4.91, 0.93, 0.08) \\
\tilde{c}[0.3,0.7,0.7,1] &= \tilde{c}_5 = (5.21, 0.51, 0.14) \\
\tilde{c}[0.7,0.7,0.7,1] &= \frac{1-0.7}{1-0.3} \tilde{c}[0.3,0.7,0.7,1] \\
&\quad + \frac{0.7-0.3}{1-0.3} \tilde{c}[1,0.7,0.7,1] \\
&= \frac{0.3\tilde{c}_5+0.4\tilde{c}_6}{0.7} = (5.99, 1.12, 0.51) \\
\tilde{c}[0.7,0.7,1,1] &= \tilde{c}_6 = (6.57, 1.57, 0.79) \\
\tilde{c}[0.7,1,1,1] &= \tilde{c}_7 = (7.50, 1.25, 1.50) \\
\tilde{c}[1,1,1,1] &= \tilde{c}_8 = (9, -1, 3).
\end{aligned}
$$

And similarly for $\tilde{d}(u)$,

$$
\begin{aligned}
\tilde{d}[0,0,0,0] &= \tilde{d}_0 = (0, 0.5, 2) \\
\tilde{d}[0,0,0,0.3] &= \tilde{d}_1 = (0.91, 1.89, 2.11) \\
\tilde{d}[0,0,0.3,0.3] &= \tilde{d}_2 = (1.51, 2.42, 2.20) \\
\tilde{d}[0,0.3,0.3,0.3] &= \frac{0.7-0.3}{0.7-0} \tilde{d}[0,0,0.3,0.3] \\
&\quad + \frac{0.3-0}{0.7-0} \tilde{d}[0,0.7,0.3,0.3] \\
&= \frac{0.4\tilde{d}_2+0.3\tilde{d}_3}{0.7} = (1.89, 2.35, 2.28) \\
\tilde{d}[0,0.3,0.3,0.7] &= \tilde{d}_3 = (2.39, 2.24, 2.37) \\
\tilde{d}[0.3,0.3,0.3,0.7] &= \frac{0.7-0.3}{0.7-0} \tilde{d}[0,0.3,0.3,0.7] \\
&\quad + \frac{0.3-0}{0.7-0} \tilde{d}[0.7,0.3,0.3,0.7] \\
&= \frac{0.4\tilde{d}_3+0.3\tilde{d}_4}{0.7} = (2.64, 1.82, 2.37) \\
\tilde{d}[0.3,0.3,0.7,0.7] &= \tilde{d}_4 = (2.97, 1.26, 2.37) \\
\tilde{d}[0.3,0.7,0.7,0.7] &= \frac{1-0.7}{1-0.3} \tilde{d}[0.3,0.3,0.7,0.7] \\
&\quad + \frac{0.7-0.3}{1-0.3} \tilde{d}[1,0.3,0.7,0.7] \\
&= \frac{0.3\tilde{d}_4+0.4\tilde{d}_5}{0.7} = (3.35, 0.77, 2.35) \\
\tilde{d}[0.3,0.7,0.7,1] &= \tilde{d}_5 = (3.64, 0.39, 2.34) \\
\tilde{d}[0.7,0.7,0.7,1] &= \frac{1-0.7}{1-0.3} \tilde{d}[0.3,0.7,0.7,1] \\
&\quad + \frac{0.7-0.3}{1-0.3} \tilde{d}[1,0.7,0.7,1] \\
&= \frac{0.3\tilde{d}_5+0.4\tilde{d}_6}{0.7} = (4.53, 0.95, 2.42)
\end{aligned}
$$

$$
\begin{aligned}
\tilde{d}[0.7, 0.7, 1, 1] &= \tilde{d}_6 = (5.20, 1.37, 2.48)\\
\tilde{d}[0.7, 1, 1, 1] &= \tilde{d}_7 = (6.26, 1.15, 2.87)\\
\tilde{d}[1, 1, 1, 1] &= \tilde{d}_8 = (8, -1, 4).
\end{aligned}
$$