Efficient Construction of a Substitution Box Based on a Mordell Elliptic Curve Over a Finite Field

Naveed Ahmed Azam^{*}, Umar Hayat, Ikram Ullah Department of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Japan Email: *azam@amp.i.kyoto-u.ac.jp *Corresponding author

Abstract: Elliptic curve cryptography (ECC) is used in many security systems due to its small key size and high security as compared to the other cryptosystems. In many well-known security systems substitution box (S-box) is the only non-linear component. Recently, it is shown that the security of a cryptosystem can be improved by using dynamic S-boxes instead of a static S-box. This fact necessitates the construction of new secure S-boxes. In this paper, we propose an efficient method for the generation of S-boxes based on a class of Mordell elliptic curves (MECs) over prime fields by defining different total orders. The proposed scheme is developed in such a way that for each input it outputs an S-box in linear time and constant space. Due to this property, our method takes less time and space as compared to all existing S-box construction methods over elliptic curve. Furthermore, it is shown by the computational results that the proposed method is capable of generating cryptographically strong S-boxes with comparable security to some of the existing S-boxes constructed over different mathematical structures.

Key words: Mordell elliptic curve; Finite field; Substitution box; Total order; Computational complexity

1 Introduction

Cryptography deals with the techniques to secure the private data. In these techniques, the data is transformed into an unreadable form by using some keys so that the adversaries cannot extract any useful information. S-box is the only non-linear component of many well-known cryptosystems including Advanced Encryption System (AES). It is therefore the security of such cryptosystems solely depends on the cryptographic properties of their S-boxes. Shannon [33] proved that a cryptosystem is secure, if it can create confusion and diffusion in the data up to a certain level. An S-box is cryptographically strong enough to create desire confusion and diffusion, if it satisfies certain tests including the test of non-linearity, approximation, strict avalanche, bit independence and algebraic complexity. Nowadays, AES is considered to be the most secured and widely used cryptosystem, and hence many cryptographers studied its S-box. The study in [35, 5, 26, 30] reveals that the AES S-box is vulnerable against algebraic attacks because of its sparse polynomial representation. It is also noticed that a cryptosystem based on a single S-box is unable to generate desirable security, if the data is highly correlated [2, 15]. Furthermore, it is shown that the security of a cryptosystem can be improved by using dynamic S-boxes instead of a static S-box, see for example [31, 20, 27, 1, 25, 19]. The two main reasons behind this are: (a) static S-box is vulnerable to data analysis attack and subkey attacks in which subkeys are obtained by using inverse subbyte, if inverse of the S-box is known [31]; and (b) it is shown in [20, 27, 1, 25, 19] that the algorithms using dynamic S-box are more complex and provide more overhead to the cryptanalysts when compared with static S-box. Different image encryption algorithms by using dynamic S-box are presented in [40, 37, 8, 24]. In these studies it turned out that the image cryptosystems based on a dynamic S-box provide better security when compared with the cryptosystems using a static S-box. Due to these reasons many researchers have proposed new S-box generation techniques based on different mathematical structures including algebraic, and differential equations.

For an S-box design technique, it is necessary that the resultant S-box: (a) inherits the properties of the underlying mathematical structure. This is an important requirement which leads to the efficient generation and better understanding of the cryptographic properties of the S-box; (b) is generated in low time and space complexity; and (c) satisfies the security tests. Of course, an S-box generation technique with high time complexity is not suitable for the cryptosystems using multiple, and dynamic S-boxes. Lui et al., [23] presented an improved AES S-box based on an algebraic method. Cui et al., [6] used an affine function to generate an S-box with 253 non-zero terms in its polynomial representation. Tran et al., [36] used composition of a Gray code instead of an affine mapping with the AES S-box to generate an S-box with high algebraic complexity. Khan and Azam [21] proposed different methods for the generation of cryptographically strong S-boxes based on a generalization of Gray S-box, and affine functions, see [22]. Azam [2] used the later S-boxes for the encryption of confidential images. Chaotic maps including Baker, logistic, and Chebyshev maps are used to generate new S-boxes in [11, 10, 29]. Similarly, elliptic curves (ECs) are also used in the field of cryptography for the development of highly secure cryptosystems. Miller [28] presented an EC based security system which has smaller key size and higher security as compared to RSA. Jung et al., [17] developed a link between the points on hyper-elliptic curves and non-linearity of an S-box. Havat et al., [12, 13] for the first time used EC over a prime field for the generation of dynamic S-boxes. In these schemes, an S-box is generated by using the x-coordinates of the points on an ordered EC over a prime p, where the ordering \prec on the points is performed with respect to their values i.e., for any two points (x_1, y_1) and (x_2, y_2) on the EC, $(x_1, y_1) \preceq (x_2, y_2)$, if $(y_1^2 \le y_2^2)$ (mod p). Actually, the scheme in [13] is a generalization of the scheme in [12]. Although these methods are capable of generating cryptographically strong Sboxes, but they have the next two weak points. Firstly, they need to compute and store the EC for their generation process. Due to this, the time and space complexity of these schemes are $\mathcal{O}(p^2)$ and $\mathcal{O}(p)$, respectively, where $p \geq 257$ is the prime of the underlying EC. Secondly, the output of these schemes is uncertain i.e., for each set of input parameters the algorithms do not necessarily output an S-box.

The purpose of this article is to develop such a novel and efficient S-box generation technique based on a finite Mordell elliptic curve (MEC) which generates secure S-box inheriting the properties of the underlying MEC for each set of input parameters. To achieve this, we defined some typical type of total orders on the points of the MEC, and then used y-coordinates instead of x-coordinates to obtain an S-box. The remaining paper is organized as follows: Some basic definitions and results related to EC are discussed in Section 2. The proposed algorithm is described in Section 3. Section 4 contains the security analysis, while a detailed comparison of the newly developed scheme with some of the existing methods is performed in Section 5. Finally, conclusions are drawn in Section 6.

2 Preliminaries

For a prime p, and two non-negative integers $a, b \leq p-1$, the EC $E_{p,a,b}$ over the prime field \mathbf{F}_p is defined to be the collection of infinity point O, and all ordered pairs $(x, y) \in \mathbf{F}_p \times \mathbf{F}_p$ satisfying the equation

$$y^2 \equiv x^3 + ax + b \pmod{p}.$$

We call p, a, and b the parameters of the EC $E_{p,a,b}$. An approximation for the number of points $\#E_{p,a,b}$ on $E_{p,a,b}$ can be obtained by using Hasse's formula [39, 3]

$$|\#E_{p,a,b} - p - 1| \le 2\sqrt{p}.$$

Mordell elliptic curve (MEC) is a special kind of elliptic curve with a = 0. The significance of some MECs $E_{p,0,b}$ is that they have exactly p + 1 points. The following Theorem [39] gives the information of such MECs.

Theorem 1 Let p > 3 be a prime such that $p \equiv 2 \pmod{3}$. Then for each $b \in \mathbf{F}_p$, the MEC $E_{p,0,b}$ has exactly p + 1 distinct points, and has each integer in [0, p - 1] exactly once as a y-coordinate.

Henceforth, a MEC $E_{p,0,b}$, where $p \equiv 2 \pmod{3}$, is simply denoted by $E_{p \equiv 2,b}$.

3 Description of the Proposed S-box Designing Technique

In this section, we give an informal intuition of our proposed method. Our aim is to develop such an S-box generation technique based on a MEC which outputs an S-box: (a) in linear time and constant space for each set of input parameters; (b) that inherits the properties of the underlying MEC; and (c) having high security against cryptanalysis. Note that the S-box design techniques proposed by [12, 13] do not satisfy conditions (a) and (b). One of the possible ways of designing such a technique is to input such an EC which contains all integer values from [0, 255] without repetition. It is, therefore, the proposed algorithm takes a MEC $E_{p\equiv 2,b}$ as an input, and uses y-coordinates to generate an S-box instead of x-coordinates. The next task is to use these y-coordinates in such a way that the resultant S-box inherits the properties of the underlying MEC. Of course, the usage of some arithmetic operations such as modulo operation for this purpose S-box will destroy the structure of the underlying MEC. Thus, we used the concept of total order on the MEC to get an S-box. Order theory is intensively used in formal methods, programming languages, logic, and statistic analysis. Now the natural question is how to define different orderings on the MEC. Note that for each x value of MEC, there are two y values. Thus, we can divide the orderings on MEC into two categories: (1) one is that in which the two y values of each x appear consecutively; and (2) the other one contains those orderings in which the two y values of each x do not appear consecutively. Based on this fact, we defined three different type of orderings on a given MEC $E_{p\equiv 2,b}$ to generate three different S-boxes.

3.1 The proposed orderings on a MEC $E_{p=2,b}$

The orderings used in the proposed method are discussed below.

(1) A natural ordering on a MEC: We define a natural ordering \prec_N on $E_{p\equiv 2,b}$ based on x-coordinates as follows

$$(x_1, y_1) \prec_N (x_2, y_2) \Leftrightarrow \begin{cases} \text{ either if } x_1 < x_2; \text{ or} \\ \text{ if } x_1 = x_2, \text{ and } y_1 < y_2, \end{cases}$$
(1)

where $(x_1, y_1), (x_2, y_2) \in E_{p \equiv 2, b}$.

The aim of this ordering is to sort the points on the MEC in such a way that the x-coordinates are in non-decreasing order, and the two y values corresponding to each x appear consecutively.

The next two orderings are introduced based on the following observation deduced from Theorem 1 to diffuse the y-coordinates on a MEC.

Observation: For any two distinct points (x_1, y_1) and (x_2, y_2) on the MEC $E_{p\equiv 2,b}$, and either $x_1 + y_1 = x_2 + y_2$ or $x_1 + y_1 \equiv x_2 + y_2 \pmod{p}$, it holds that $x_1 \neq x_2$.

(2) A diffusion ordering on a MEC: An ordering is defined on $E_{p\equiv 2,b}$ to diffuse the two y values of each x. Let (x_1, y_1) and (x_2, y_2) be any two points on $E_{p\equiv 2,b}$, the diffusion ordering \prec_D is defined to be

$$(x_1, y_1) \prec_D (x_2, y_2) \Leftrightarrow \begin{cases} \text{ either if } x_1 + y_1 < x_2 + y_2; \text{ or} \\ \text{ if } x_1 + y_1 = x_2 + y_2, \text{ and } x_1 < x_2. \end{cases}$$
(2)

Lemma 2 The relation \prec_D is a total order on the MEC $E_{p\equiv 2,b}$.

Proof. For each $(x_1, y_1) \in E_{p\equiv 2,b}$, we have $x_1 + y_1 = x_1 + y_1$, and therefore $(x_1, y_1) \prec_D (x_1, y_1)$. This implies that \prec_D is reflexive. Next, we need to show that \prec_D satisfies the antisymmetric property. Thus, for $(x_1, y_1), (x_2, y_2) \in$
$$\begin{split} E_{p\equiv2,b}, & \text{suppose that } (x_1,y_1) \prec_D (x_2,y_2), \text{ and } (x_2,y_2) \prec_D (x_1,y_1) \text{ hold. This} \\ & \text{implies that } x_1+y_1=x_2+y_2. & \text{This is because of the fact that } x_1+y_1 < x_2+y_2, \text{ and } x_2+y_2 < x_1+y_1 \text{ are the only cases for which the supposition} \\ & \text{and } x_1+y_1 \neq x_2+y_2 \text{ are true, which eventually imply that } x_1+y_1=x_2+y_2. \\ & \text{Now if } x_1 \neq x_2, \text{ then by the supposition and the fact } x_1+y_1=x_2+y_2. \\ & \text{we have } x_1 < x_2 \text{ and } x_2 < x_1, \text{ which lead to the contradiction } x_1=x_2. \\ & \text{Thus } x_1+y_1=x_2+y_2 \text{ and } x_1=x_2 \text{ hold, which ultimately imply that} \\ & y_1=y_2, \text{ and therefore } (x_1,y_1)=(x_2,y_2). \\ & \text{Now if } x_1+y_1=x_2+y_2 \text{ and } x_1=x_2 \text{ hold, which ultimately imply that} \\ & y_1=y_2, \text{ and therefore } (x_1,y_1) \prec_D (x_2,y_2), \text{ and } (x_2,y_2) \prec_D (x_3,y_3) \text{ hold, where} \\ & (x_1,y_1), (x_2,y_2), (x_3,y_3) \in E_{p\equiv2,b}. \\ & \text{Now if } x_1+y_1 < x_2+y_2 \text{ and } x_2+y_2 < x_3+y_3, \text{ then } x_1+y_1 < x_3+y_3, \text{ and therefore} \\ & (x_1,y_1) \prec_D (x_3,y_3). \\ & \text{Similarly, if } x_1+y_1=x_2+y_2=x_3+y_3, \text{ then } x_1 < x_2 \text{ and} \\ & x_2 < x_3, \text{ and hence } x_1+y_1=x_3+y_3 \text{ and } x_1 < x_3. \\ & \text{This completes the proof.} \\ & \blacksquare \\ \end{bmatrix}$$

(3) A modulo diffusion ordering on a MEC: The order \prec_M defined below produces diffusion in both *x*-coordinates and *y*-coordinates of the points on $E_{p\equiv 2,b}$. Let $(x_1, y_1), (x_2, y_2) \in E_{p\equiv 2,b}$, then

$$(x_1, y_1) \prec_M (x_2, y_2) \Leftrightarrow \begin{cases} \text{ either if } (x_1 + y_1 < x_2 + y_2) (\text{mod } p); \text{ or} \\ \text{if } x_1 + y_1 \equiv x_2 + y_2 (\text{mod } p), \text{ and } x_1 < x_2. \end{cases} (3)$$

Lemma 3 The relation \prec_M is a total order on the MEC $E_{p\equiv 2,b}$.

Lemma 3 can be proved by using the similar arguments as used in the proof of Lemma 2.

The effect of these orderings \prec_N, \prec_D and \prec_M on *y*-coordinates of the MEC $E_{101\equiv 2,1}$ is shown in Figure 1 by plotting them in a non-decreasing order of their points on the MEC w.r.t \prec_N, \prec_D and \prec_M , respectively.



Figure 1: The arrangements of *y*-coordinates of $E_{101\equiv2,1}$ w.r.t. the proposed orderings

Similarly, a relation among the sets of all y-coordinates of the MEC $E_{p\equiv 2,b}$ obtained by different proposed orderings \prec_H and \prec_K , where $H, K \in \{N, D, M\}$, is quantified by computing their correlation coefficient ρ_{HK} . The correlation results for different MECs are shown in Table 1. It is evident from the results that each ordering has different effect on the y-coordinates of the underlying MEC.

Table 1: Results of the correlation test

p	b	ρ_{ND}	ρ_{ND}	ρ_{DM}
101	1	-0.0588	0.0550	-0.0497
827	87	-0.0044	0.0008	0.0027
1013	118	0.0028	-0.0059	0.0003
2027	8	0.0007	-0.0068	-0.0002

3.2 The proposed S-box construction method

Let $E_{p\equiv2,b}$ be a Mordell elliptic curve (MEC), where $p \geq 257$. The lower bound on the prime p is 257 for the proposed method so that MEC has at least 256 points. An S-box $S_{p,b}^H$, where $H \in \{N, D, M\}$, is generated by selecting the ycoordinates on $E_{p\equiv2,b}$ which are in the interval [0, 255] as $S_{p,b}^H$: $\{0, 1, \ldots, 255\} \rightarrow$ $\{0, 1, \ldots, 255\}$ defined as $S_{p,b}^H(i) = y_i$, such that $(x_i, y_i) \in E_{p\equiv2,b}$, and $(x_{i-1}, y_{i-1}) \prec_H (x_i, y_i)$.

It is clear from Theorem 1 that $S_{p,b}^H$ is a bijection, which further implies that the proposed method generates an S-box for each set of input parameters.

Lemma 4 For any prime $p \ge 257$ such that $p \equiv 2 \pmod{3}$, integer $b \in [0, p-1]$, and $H \in \{N, D, M\}$, the S-box $S_{p,b}^H$ can be generated in time complexity $\mathcal{O}(p)$ and constant space.

Proof. The generation of $S_{p,b}^{H}$ requires calculation of 256 points on the MEC with y-coordinates in [0, 255], and then their sorting. The calculation of 256 points on the MEC can be done in $\mathcal{O}(p)$, since for each $y \in [0, 255]$, a for loop of size p is required to find integer x such that (x, y) is a point on the MEC. However, the sorting of these 256 points can be done in a constant time with respect to the ordering H. Thus, $S_{p,b}^{H}$ can be generated in $\mathcal{O}(p)$ time. Furthermore, the generation process store only 256 points on the MEC for sorting purpose, and therefore it takes constant space.

It is evident from Lemma 4 that the time and space complexity of the proposed S-box generation method is independent of the parameter b and the ordering on the underlying MEC. An algorithmic description of the proposed generation method is given in Algorithm 1.

Algorithm 1 The proposed S-box generation method

Input: A Mordell elliptic curve $E_{p,b}$, where $p \equiv 2 \pmod{3}$, with a total order $H \in \{N, D, M\}.$ **Output:** The proposed S-box $S_{p,b}^H$. 1: $A := \emptyset$; /* The set of 256 points of the MEC with y-coordinates in $[0, 255]^*$ / 2: for each $y = 0, 1, \dots, 255$ do while $x \in [0, p-1]$ do 3: if $x^3 + b \equiv y^2 \pmod{p}$ then 4: $A := A \cup \{(x, y)\}$ 5: 6: end if end while 7:8: end for 9: Sort A with respect to the ordering H; 10: Output all y-coordinates of the points in A preserving their order as the S-box $S_{p,b}^H$.

The S-boxes $S_{1667,351}^N, S_{3299,1451}^D$ and $S_{4229,2422}^M$ generated by the proposed technique are presented in Tables (11)-(13), respectively.

4 Security Analysis

Several standard tests are applied on the S-boxes obtained by the proposed method to test their cryptographic strength. A brief introduction to these security tests, and their results for some of the newly generated S-boxes $S_{1667,351}^N$, $S_{1949,544}^N$, $S_{3023,626}^N$, $S_{3299,1451}^D$, $S_{3041,1298}^D$, $S_{3347,2937}^D$, $S_{4229,2422}^M$, $S_{4217,1156}^M$ and $S_{3299,1400}^M$ are discussed in this section.

4.1 Non-Linearity (NL)

It is important for an S-box to create confusion in the data up to a certain level to keep the data secure from the adversaries. The confusion creation capability of an S-box S over the Galois Field $GF(2^8)$ is measured by its non-linearity $\mathcal{N}(S)$, which is defined below

$$\mathcal{N}(S) = \min_{\alpha,\beta,\gamma} \{ x \in GF(2^8) : \alpha \cdot S(x) \neq \beta \cdot x \oplus \gamma \},\$$

where $\alpha \in GF(2^8)$, $\gamma \in GF(2)$, $\beta \in GF(2^8) \setminus \{0\}$ and "·" represents dot product over GF(2).

An S-box with high NL is capable of generating high confusion in the data. However, it is also shown in [38] that an S-box with high NL may not satisfy other cryptographic properties. The NL of some of the newly constructed Sboxes is listed in Table 2. Note that each listed S-box has NL 106, which is large enough to create high confusion.

Table 2: Non-linearity of the newly generated S-boxes

S-boxes	$S^{N}_{1667,351}$	$S^{N}_{1949,544}$	$S^{N}_{3023,626}$	$S^{D}_{3299,1451}$	$S^{D}_{3041,1298}$	$S^{D}_{3347,2937}$	$S^{M}_{4229,2422}$	$S^{M}_{4217,1156}$	$S^{M}_{3299,1400}$
NL	106	106	106	106	106	106	106	106	106

4.2 Approximation Attacks

A cryptographically strong S-box must have high resistance against approximation attacks. The approximation attacks can be divided into two categories namely linear approximation attacks, and differential approximation attacks which are explained below.

The resistance of an S-box S against linear approximation attacks is measured by calculating its maximum number $\mathcal{L}(S)$ of coincident input bits with the output bits. The mathematical expression of $\mathcal{L}(S)$ is as follows

$$\mathcal{L}(S) = \frac{1}{2^8} \left\{ \max_{\alpha,\beta} \left\{ \left| \# \left\{ x \in GF(2^8) : \alpha \cdot x = \beta \cdot S(x) \right\} - 2^7 \right| \right\} \right\},\$$

where $\alpha \in GF(2^8)$ and $\beta \in GF(2^8) \setminus \{0\}$.

An S-box S is said to be highly resistive against linear approximation attacks if it has low value of $\mathcal{L}(S)$. The LAP of the newly generated S-boxes is listed in Table 3. The average LAP of all of the listed S-boxes is 0.1371 which is very



low, and hence the proposed scheme is capable of generating S-boxes with high resistance against linear approximation attacks.

4.2.1 Differential Approximation Probability (DAP)

The strength of an S-box against differential approximation attacks is measured by calculating its DAP. For an S-box S, the DAP $\mathcal{D}(S)$ is the maximum probability of a specific change Δy in the output bits S(x) when the input bits x are changed to $x \oplus \Delta x$ i.e.,

$$\mathcal{D}(S) = \frac{1}{2^8} \left\{ \max_{\Delta x, \Delta y} \left\{ \# \left\{ x \in GF(2^8) : S(x \oplus \Delta x) = S(x) \oplus \Delta y \right\} \right\} \right\},\$$

where Δx , $\Delta y \in GF(2^8)$, and " \oplus " is bit-wise addition over GF(2).

The smaller is the value of DAP, the higher is the security of the S-box against differential approximation attacks. The experimental results of DAP on the newly generated S-boxes are presented in Table 4. It is evident from Table 4 that the newly generated S-boxes have high resistance against differential attacks.

Table 4: DAP of the newly generated S-boxes

S-boxes	$S^{N}_{1667,351}$	$S^{N}_{1949,544}$	$S^{N}_{3023,626}$	$S^{D}_{3299,1451}$	$S^{D}_{3041,1298}$	$S^{D}_{3347,2937}$	$S^{M}_{4229,2422}$	$S^{M}_{4217,1156}$	$S^{M}_{3299,1400}$
DAP	0.0391	0.0391	0.0391	0.0391	0.0391	0.0391	0.0391	0.0391	0.0391

4.3 Strict Avalanche Criterion (SAC)

The diffusion creation capability of an S-box is calculated by SAC. The SAC of an S-box S is the measure of change in output bits when a single input bit is changed. The SAC of an S-box S with boolean functions S_i , where $1 \le i \le 8$, is computed by calculating an eight dimensional square matrix $M(S) = [m_{ij}]$ by using each of the eight elements $\alpha_i \in GF(2^8)$ with only one non-zero bit as

$$m_{ij} = \frac{1}{2^8} \left(\sum_{x \in GF(2^8)} w \left(S_i(x \oplus \alpha_j) \oplus S_i(x) \right) \right)$$

where w(v) denotes the number of non-zero bits in the vector v.

SAC test is fulfilled, if all entries of M(S) are close to 0.5. The entries of SAC matrix corresponding to each newly generated S-boxes $S_{1667,351}^N$, $S_{3299,1451}^D$ and $S_{4229,2422}^M$ are plotted in a linear order in Figure 2. The average of minimum, and maximum values of M(S) corresponding to each of the newly generated S-boxes are 0.4115 and 0.6094, respectively. Table 5 clearly shows that the S-boxes

generated by the proposed method based on a MEC is capable of generating high diffusion in the data.

Table 5: SAC of the newly generated S-boxes

S-boxes	$S^{N}_{1667,351}$	$S^{N}_{1949,544}$	$S^{N}_{3023,626}$	$S^{D}_{3299,1451}$	$S^{D}_{3041,1298}$	$S^{D}_{3347,2937}$	$S^{M}_{4229,2422}$	$S^{M}_{4217,1156}$	$S^{M}_{3299,1400}$
SAC(max)	0.5938	0.625	0.6563	0.6406	0.6094	0.6094	0.5938	0.6094	0.625
SAC(min)	0.4531	0.4219	0.4219	0.4063	0.4219	0.4063	0.375	0.3906	0.3594



Figure 2: SAC matrix plot for $S^N_{1667,351},\,S^D_{3299,1451}$ and $S^M_{4229,2422}$

4.4 Bit Independence Criterion (BIC)

BIC is also an important test to measure the diffusion creation strength of an S-box. The main idea of this test is to investigate the dependence of a pair of output bits when an input bit is reversed. The BIC of an S-box S over $GF(2^8)$ with S_i boolean functions is also calculated by computing a square matrix $N(S) = [n_{ij}]$ of dimension eight as follows

$$n_{ij} = \frac{1}{2^8} \left(\sum_{\substack{x \in GF(2^8) \\ 1 \le k \le 8}} w \left(S_i(x \oplus \alpha_j) \oplus S_i(x) \oplus S_k(x + \alpha_j) \oplus S_k(x) \right) \right)$$

Of course $n_{ii} = 0$. An S-box is said to be good if all off-diagonal values of its BIC matrix are near to 0.5. The experimental results of this test on the newly generated S-boxes $S_{1667,351}^N, S_{3299,1451}^D$ and $S_{4229,2422}^M$, excluding the value 0, are shown in a linear order in Figure 3. The minimum, and maximum values of BIC matrix N(S) of each of the newly generated S-boxes are listed in Table 6. It is evident from Figure 3 and Table 6 that the S-boxes generated by the proposed methods are strong enough to generate high diffusion in the data.

Table 6: BIC of the newly generated S-boxes

S-boxes	$S^{N}_{1667,351}$	$S^{N}_{1949,544}$	$S^{N}_{3023,626}$	$S^{D}_{3299,1451}$	$S^{D}_{3041,1298}$	$S^{D}_{3347,2937}$	$S^{M}_{4229,2422}$	$S^{M}_{4217,1156}$	$S^{M}_{3299,1400}$
BIC(max)	0.5273	0.5293	0.5313	0.5371	0.5273	0.5254	0.5254	0.5313	0.5449
BIC(min)	0.4648	0.4629	0.4707	0.4707	0.4844	0.4746	0.4688	0.4766	0.4727

Figure 3: BIC matrix plot for $S^{N}_{1667,351}, S^{D}_{3299,1451}$ and $S^{M}_{4229,2422}$

4.5 Algebraic Complexity(AC)

The resistance of an S-box against algebraic attacks is measured by computing its linear polynomial. The AC of an S-box is the number of non-zero terms in its linear polynomial. The greater is the AC, the greater is the security of the S-box against algebraic attacks. The AC of the newly generated S-boxes is computed, and is presented in Table 7. The minimum, and maximum values of AC of the newly generated S-boxes are 253, and 255, respectively, which are very close to the optimal value 255. Thus, the proposed method is able to generate S-boxes with good AC based on a MEC.

Table 7: The AC of the newly generated S-boxes

S-boxes	$S^{N}_{1667,351}$	$S^{N}_{1949,544}$	$S^{N}_{3023,626}$	$S^{D}_{3299,1451}$	$S^{D}_{3041,1298}$	$S^{D}_{3347,2937}$	$S^{M}_{4229,2422}$	$S^{M}_{4217,1156}$	$S^{M}_{3299,1400}$
AC	254	254	255	255	254	255	253	253	255

5 Comparison and Discussion

A detailed comparison of the proposed S-box construction method is performed in this section.

5.1 Time and Space Complexity

It is always desirable to have algorithms with low time and space complexity from implementation point of view. The time and space complexity of the proposed method and other S-box generation methods [12, 13] based on ECs are compared in Table 8. Note that each method in [12, 13] has quadratic time complexity, while the proposed method takes linear time in the underlying prime p for the generation of an S-box. However, the space complexity of the methods in [12, 13] is $\mathcal{O}(p)$, where p is the underlying prime, while it is constant for the proposed method. Hence, the newly developed method is more suitable for the implementation when compared to all existing S-box generation methods over EC.

Table 8: Comparison of time and space complexity of the proposed method with other methods over ECs

S-box	Ref. [12]	Ref. [13]	Proposed method
Time complexity	$\mathcal{O}(p^2)$	$\mathcal{O}(p^2)$	$\mathcal{O}(p)$
Space complexity	$\mathcal{O}(p)$	$\mathcal{O}(p)$	O(1)

5.2 Generation Efficiency

For a good dynamic S-box construction scheme, it is necessary to ensure the generation of S-box for each valid input parameters, and construct enough number of distinct S-boxes. It is evident from Theorem 1 that the proposed method always generate an S-box for each input, while the output of the methods in [12, 13] are uncertain i.e., they do not guarantee the construction of S-boxes for each input. This implies that the proposed method is better than the other existing schemes over ECs.

The proposed method can generate at most p-1 number of distinct Sboxes for a given prime p and ordering, since for each $b \in [1, p-1]$ it can generate exactly one S-box. We generated all S-boxes by the proposed method for different primes p = 257, 263, 269, 281, 293, 1013, 1019, 1031, 1049, 1061 and 1997 and each ordering developed in this paper. The number of distinct S-boxes for each ordering is same for all the primes and is listed in listed in Table 9. It is evident from Table 9 that the number of distinct S-boxes generated by proposed S-box design scheme attains the optimal value and increases with the increase in the size of the prime. Hence, one can generate the desired number of distinct S-boxes by using the proposed method on a appropriate prime.

Table 9: The number of distinct S-boxes constructed by the proposed scheme for some primes

p	257	263	269	281	293	1013	1019	1031	1049,	1061	1997
Distinct S-boxes	256	262	268	280	292	1012	1018	1030	1048	1060	1996

5.3 Cryptographic Properties

The cryptographic properties of some of the S-boxes constructed by the proposed method are compared with some of the well-known existing S-boxes due to [11, 10, 29, 32, 16, 18, 14, 7, 9, 4, 34] generated by different mathematical structures. The properties of the S-boxes used in this comparison are listed in Table 10. Note that the non-linearity (NL) of the S-boxes $S_{1667,351}^N$, $S_{3299,1451}^D$ and $S_{4229\ 2422}^M$ is greater than that of the S-boxes in [11, 10, 12, 18, 9, 4, 34], and hence the newly generated S-boxes create better confusion in the data when compared to the later S-boxes. This implies that the proposed technique is capable of generating S-boxes with good NL when compared to some of the other existing techniques. Moreover, the linear approximation probability (LAP) of the newly generated S-boxes is better than the LAP of the S-boxes in [11, 10, 29, 9, 4, 34], while their differential approximation probability (DAP) is at most the DAP of the S-boxes in [11, 10, 29, 12, 18, 9, 4, 34]. Thus, the Sboxes generated by the proposed technique have same or better security against approximation attacks as compared to the other S-boxes. Similarly, the SAC, BIC and AC test results of the newly generated S-boxes are comparable with the S-boxes listed in Table 10. Hence, the proposed S-box generation technique based on a MEC is capable of generating S-boxes with cryptographic properties comparable with some of the existing S-box construction techniques based on different mathematical structures.

Table 10: Comparison of the newly generated S-boxes with some of the existing S-boxes

S-boxes	NL	LAP	DAP	SAC(Max)	SAC(Min)	BIC(Max)	BIC(Min)	AC
Ref. [11]	103	0.1328	0.0391	0.5703	0.4414	0.5039	0.4961	255
Ref. [10]	102	0.1484	0.0391	0.6094	0.375	0.5215	0.4707	254
Ref. [29]	106	0.1406	0.0391	0.5938	0.4375	0.5313	0.4648	251
Ref. [12]	104	0.0391	0.0391	0.625	0.3906	0.53125	0.4707	255
Ref. [18]	104	0.109	0.0469	0.593	0.39	0.499	0.454	255
Ref. [7]	112	0.062	0.0156	0.562	0.453	0.504	0.480	9
Ref. [9]	74	0.2109	0.0547	0.6875	0.1094	0.5508	0.4023	253
Ref. [4]	100	0.1328	0.0547	0.6094	0.4219	0.5313	0.4746	255
Ref. [34]	103	0.1328	0.0391	0.5703	0.3984	0.5352	0.4727	255
$S^{N}_{1667,351}$	106	0.1328	0.0391	0.5938	0.4531	0.5273	0.4648	254
$S^{D}_{3299,1451}$	106	0.1484	0.0391	0.6406	0.4063	0.5371	0.4707	255
$S^{M}_{4229,2422}$	106	0.1328	0.0391	0.5938	0.375	0.5254	0.4688	253

6 Conclusion

In this article, we presented an S-box design scheme based on y-coordinates of a finite Mordell elliptic curve (MEC), where prime is congruent to 2 modulo 3. The technique uses some special type of total orders on the points of the MEC, and generates an S-box. The main advantages of the proposed method are that it has linear time complexity, constant space complexity and generate an S-box for each input parameter which are not possible in all existing S-box generation schemes over elliptic curves. Several standard security tests are performed on the S-boxes generated by the proposed method to analyze its cryptographic efficiency. Experimental results show that the proposed scheme can generate cryptographically strong S-boxes. Furthermore, it is shown by computational results that the cryptographic properties of the newly generated S-boxes are comparable with some of the well-known existing S-boxes generated by different mathematical structures.

References

- Agarwal P, Singh A, Kilicman A, 2018. Development of Key–Dependent Dynamic S-Boxes with Dynamic Irreducible Polynomial and Affine Constant. Advances in Mechanical Engineering, 10(7):1-18.
- [2] Azam N A, 2017. A Novel Fuzzy Encryption Technique Based on Multiple Right Translated AES Gray S-Boxes and Phase Embedding. Security and Communication Networks, Volume 2017, 1-9.
- [3] Brown D R L, 2009. SEC 1: Elliptic curvecryptography. Mossossaiga: Certicom Corp.

- [4] Chen G, Chen Y, Liao X, 2007. An extended method for obtaining S-boxes based on three-dimensional chaotic baker maps. Chaos, Solitons and Fractals, 31(3):571-579.
- [5] Courtois N T, Josef P, 2002. Cryptanalysis of block ciphers with over defined systems of equations. ASIACRYPT 2002 LNCS, 2501, 267-287.
- [6] Cui L, Cao Y, 2007. A new S-box structure named affine power-affine. International Journal of Innovative Computing, Information and Control, 3:751-759.
- [7] Daemen J, Rijmen V, 2002. The Design of Rijndael-AES: The Advanced Encryption Standard, Springer, Berlin, Germany.
- [8] Devaraj P, Kavitha C, 2016. An Image Encryption Scheme Using Dynamic Sboxes. Nonlinear Dyn, 86:927-940.
- [9] Gautam A, Gaba G S, Miglani R et al., 2015. Application of Chaotic Functions for Construction of Strong Substitution Boxes. Indian Journal of Science and Technology, 8(28):1-5.
- [10] Guo C, 2008. A novel heuristic method for obtaining S-boxes. Chaos, Solitons and Fractals, 36:1028-1036.
- [11] Guoping T, Xiaofeng L, Yong C, 2005. A novel method for designing S-boxes based on chaotic maps. Solitons and Fractals, 23:413-419.
- [12] Hayat U, Azam N A, Asif M, 2018. A Method of Generating 8 × 8 Substitution Boxes Based on Elliptic Curves. Wireless Personal Communications, 101:439-451.
- [13] Hayat U, Azam N A, 2019. A Novel Image Encryption Scheme Based on Elliptic Curves. Signal Processing, 155:391-402.
- [14] Hussain I, Shah T, Gondal M A et al., 2013. A group theoretic approach to construct cryptographically strong substitution boxes. Neural Computing and Applications, 23:97-104.
- [15] Hussain I, Azam N A, Shah T, 2014. Stego optical encryption based on chaotic S-box transformation. Optics and Laser Technology, 61:50-56.
- [16] Jakimoski G, Kocarev L, 2001. Chaos and cryptography: block encryption ciphers. IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications, 48:163-170.
- [17] Jung H C, Seongtaek C, Choonsik P, 1999. S-boxes with controllable nonlinearity. EUROCRYPT '99. LNCS, 1592, 286:294.
- [18] Kim J, Phan R C W, 2009. Advanced differential-style cryptanalysis of the NSA's skipjack block cipher. Cryptologia, 33:246-270.
- [19] Kazlauskas K, Kazlauskas J, 2009. Key-Dependent S-Box Generation in AES Block Cipher System. INFORMATICA, 20(1):23-34.

- [20] Katiyar S, Jeyanthi N, 2016. Pure Dynamic S-box Construction. International Journal of Computers, 1:42-46.
- [21] Khan M, Azam N A, 2015. Right translated AES Gray S-box. Security and Network Communication, 8:1627-1635.
- [22] Khan M, Azam N A, 2015. S-boxes based on affine mapping and orbit of power function. 3D Research, 6(12):1-15.
- [23] Liu J, Wai B, Cheng, X, et al., 2005. An AES S-box to increase complexity and cryptographic analysis. In Proceedings of the 19th international conference on advanced information networking and applications, Taiwan, 724-728.
- [24] Liu Y, Wang J, Fan, J, et al., 2016). Image Encryption Algorithm Based on Chaotic System and Dynamic S-boxes Composed of DNA sequences. Multimed Tools Appl, 75:4363–4382.
- [25] Maram B, Gnanasekar J M, 2016. Evaluation of Key Dependent S-Box Based Data Security Algorithm using Hamming Distance and Balanced Output. TEM J, 5:67-75.
- [26] Murphy S, Robshaw M J, 2002. Essential algebraic structure within the AES. Proceedings of the 22th annual international cryptology. Berlin: Springer, 1-16.
- [27] Manjula G, Mohan H S, 2013. Constructing Key Dependent Dynamic S-Box for AES Block Cipher System. 2nd International Conference on Applied and Theoretical Computing and Communication Technology (iCATccT), 613-617.
- [28] Miller V, 1986. Uses of elliptic curves in cryptography. Advances in Cryptology, 85:417-426.
- [29] Neural Y W, Li Y, Min L et al., 2010. A method for designing S-box based on chaotic neuralnetwork. In 2010 Sixth international conference on natural computation (ICNC).
- [30] Rosenthal J, 1949. A polynomial description of the Rijndael advanced encryption standard. Journal of Algebra and its Applications, 2:223-236.
- [31] Rahnama B, Kran Y, Dara R, 2013. Countering AES Static S-Box Attack, SIN 13. Proceedings of the 6th International Conference on Security of Information and Networks, 256-260.
- [32] Shi X Y, Xiao H, You X C et al., 1997. A method for obtaining cryptographically strong 8× 8 S-boxes. Conference on Information Network and Application, 2:689-693.
- [33] Shannon C E, 1949. Communications theory of secrecy systems. Bell Labs Technical Journal, 12:656-715.
- [34] Tang G, Liao X, Chen Y, 2005. A novel method for designing S-boxes based on chaotic maps. Chaos, Solitons and Fractals, 23(2):413-419.

- [35] Thomas J, Knudsen L R, 1997. The interpolation attack on block ciphers. International workshop on fast software encryption (FSE). Fast Software Encryption, 28-40.
- [36] Tran M T, Bui D K, Doung A D, 2008. Gray S-box for advanced encryption standard. International Conference on Computational Intelligence and Security, 1:253-258.
- [37] Wang X, Wang Q, 2014. A Novel Image Encryption Algorithm Based on Dynamic S-boxes Constructed by Chaos. Nonlinear Dyn, 5:567-576.
- [38] Willi M, Othmar S 1990. Nonlinearity criteria for cryptographic functions. Advances in Cryptology-EUROCRYPT '89 LNCS. 434:549-562.
- [39] Washington L C, 2008. Number Theory: Elliptic Curves and Cryptography. vol. 50 of Discrete Mathematics and Its Applications. Chapman and Hall/CRC, 2nd ed.
- [40] Zaibi G, Kachouri A, Peyrard F, et al., 2009. On dynamic chaotic S-Box. In Information Infrastructure Symposium, 2009. GIIS'09. Global. IEEE, 1-5.

7 Appendix: S-boxes generated by proposed method

Table 11: The S-box $S^{N}_{\rm 1667,351}$ generated by the proposed method based on the natural ordering

154	217	227	110	85	29	199	37	68	21	91	78	208	3	148	40
198	52	54	2	73	7	168	201	229	184	146	6	172	28	44	67
195	53	106	10	204	131	157	185	187	156	206	161	81	103	211	33
96	159	72	134	164	143	140	193	145	231	237	12	221	188	197	116
47	19	129	104	51	236	56	133	55	220	87	1	203	117	210	24
4	174	175	113	34	213	171	255	30	43	130	191	57	137	76	234
247	244	173	223	63	60	230	166	8	190	139	99	49	200	23	245
58	102	226	83	122	70	241	94	127	41	194	233	97	251	107	26
109	61	248	90	192	167	147	82	158	225	36	50	84	92	88	38
74	136	138	232	62	176	128	189	124	118	169	14	228	0	243	181
123	254	20	202	75	149	219	120	160	9	253	39	180	207	114	142
183	93	101	15	238	177	132	212	35	250	239	249	179	17	65	186
11	125	178	45	170	141	121	126	119	64	144	182	112	22	165	222
100	69	252	216	13	27	152	235	80	5	196	59	25	151	79	155
240	77	115	71	31	105	95	86	209	150	98	89	163	246	66	18
162	214	218	42	242	46	111	48	215	224	135	108	153	32	16	205

Table 12: The S-box $S^D_{3299,1451}$ generated by the proposed method based on the diffusion ordering

- 33	151	65	207	12	103	96	123	190	126	82	155	21	1	229	186
61	224	42	179	63	178	73	153	138	168	146	41	46	9	109	184
124	243	236	57	19	6	100	94	69	48	116	216	54	228	90	81
47	13	88	197	247	129	206	198	221	5	78	80	150	200	145	55
60	105	212	18	210	43	137	250	135	166	52	115	91	208	25	199
77	170	121	122	11	254	27	157	175	34	104	201	95	222	133	176
36	3	141	218	30	162	220	193	28	110	223	161	74	182	226	113
0	112	234	144	241	20	156	62	49	23	26	35	148	101	233	56
181	130	118	149	70	173	71	45	50	204	10	87	232	93	177	67
4	120	8	40	72	125	92	114	68	83	225	246	158	143	53	196
249	242	136	195	160	213	131	107	66	29	230	188	38	111	205	253
171	251	102	235	31	127	217	17	183	117	37	211	164	97	119	219
167	134	24	16	255	2	32	215	227	154	187	75	231	240	172	142
244	89	14	98	76	85	147	79	64	180	214	139	152	238	51	185
22	44	194	99	39	169	203	189	108	86	132	237	163	239	209	245
59	202	15	58	248	128	174	140	192	191	106	165	159	84	7	252

Table 13: The S-box $S^M_{4229,2422}$ generated by using the proposed method based on the modulo diffusion ordering

15	13	247	249	167	183	179	173	101	204	105	210	214	205	199	19
164	38	85	72	98	90	113	12	239	217	165	228	123	195	26	216
207	30	182	219	14	215	232	135	241	145	17	244	223	114	29	70
104	81	71	99	191	128	227	86	172	185	5	75	197	184	109	248
162	250	25	110	125	230	129	35	102	234	54	171	194	16	33	73
155	246	154	84	149	134	238	18	240	67	200	253	61	31	170	180
55	20	224	187	10	147	92	133	196	242	146	27	34	140	28	192
63	127	143	203	137	2	74	193	65	4	124	51	107	24	42	122
103	22	41	226	235	252	116	212	77	49	48	201	148	221	251	80
229	115	93	139	181	52	97	119	189	166	21	45	53	100	32	131
112	94	59	142	117	36	153	254	66	158	79	121	8	130	132	60
245	231	126	152	151	89	0	39	160	136	37	78	236	56	206	157
222	174	82	69	6	83	220	3	57	111	208	47	141	87	168	176
11	118	169	58	243	120	150	91	190	23	178	44	7	43	177	76
161	144	163	68	88	138	218	108	159	186	40	237	175	46	198	96
202	9	62	50	64	233	255	209	188	1	106	225	95	213	156	211