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Caustics of developable surfaces*

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Abstract: While considering a mirror and light rays coming either from a point source or from infinity, the reflected light rays may have an envelope, called a caustic curve. In this paper, we study developable surfaces as mirrors. These caustic surfaces, described in a closed form, are also developable surfaces of the same type as the original mirror surface. We provide efficient, algorithmic computation to find the caustic surface of each of the three types of developable surfaces (cone, cylinder, and tangent surface of a spatial curve). We also provide a potential application of the results in contemporary free-form architecture design.

Key words: Caustics; Developable surface; Reflected light rays; Curve of regression
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1 Introduction

When light rays are reflected from a curved mirror, the following optical phenomenon may be observed: the reflected light rays may possess an envelope, called a caustic curve or surface. These caustics can not only appear in our everyday experience, such as on the surface of coffee in our coffee cup, but also play an important role in the sciences, from physics to computer graphics (Arnold et al., 1985; Lock and Andrews, 1992). Scientists, from the ancient Greeks through Huygens to contemporary opticians, engineers, and geometers, have studied reflected and refracted light rays from the theoretical point of view as well as through various applications.

In recent years, these surfaces have been usually described as parametric free-form surfaces, typically as Bézier or B-spline surfaces in various applications, to provide more freedom for users in the interactive design process (Tang et al., 2016). Optical studies frequently apply these free-form surfaces in lens design (Liu P et al., 2012; Ponce-Hernández et al., 2020) or light-emitting diode (LED) illumination research (Wu et al., 2013).

In engineering and architecture, this problem is especially relevant in terms of developable surfaces, i.e., curved surfaces that can be unfolded to (and therefore created from) a planar shape. There are basically three different types of developable surfaces. Two of them are the well-known cone and cylinder, while the third one is a more general type, namely, the tangent surface of spatial curves. From the computational point of view, this latter type is the most challenging one in engineering applications, but at the same time, this type provides much more freedom in engineering design than the classical cones and cylinders (Seguin et al., 2021). These surfaces are used, among other applications, for creating special developable mechanisms, e.g., for cylindrical (Greenwood et al., 2019) and for conical (Hyatt et al., 2020).

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Non-developable surfaces are also frequently approximated by developable ones, for instance, to ease fabrication from sheet metal (Liu XH et al., 2016). For an excellent overview of developable Bézier surfaces, readers can refer to Zhang and Wang (2006). In this paper, we also follow this construction.

One of the most spectacularly evolving application of developable surfaces can be found in architecture, precisely, in the so-called free-form architecture. Due to evident mechanical and material restrictions, special attention has been paid to developable surfaces in this field (Pottmann et al., 2015; Martín-Pastor, 2019).

The caustics of classical planar curves are well-known and widely studied (Yates, 1947; Lockwood, 1967). In the case of surfaces, theoretical results are also known. For a given surface, a somewhat similar notion is the focal surface, i.e., the surface formed by the centers of the curvature spheres. The relationship between the focal and caustic surfaces has been established (Pottmann and Wallner, 2000) and it has been proven that the focal surface of a developable surface will be developable of the same type as the original one (Pottmann and Wallner, 2000). This result, in theory, yields the same consequence in terms of caustic surfaces, but these theoretical outcomes do not provide exact, constructive, and algorithmic solutions to compute and display these surfaces in practical applications. To calculate the caustics of a given surface, numerical solutions have been provided (Schwartzburg et al., 2014).

Instead of numerical calculation, in this paper, we provide the exact computation and closed formulae for the caustics of developable surfaces. These caustic surfaces are of utmost importance in contemporary architecture (Pottmann et al., 2015), where caustics may appear, e.g., as an outcome of the reflected sunshine beams.

2 Developable surfaces as mirrors

As is well-known, considering the curve

$$\boldsymbol{r}(t)=\begin{bmatrix} r_x(t) \\ r_y(t) \\ r_z(t) \end{bmatrix} \tag{1}$$

and direction $\boldsymbol{g}(t),\ t\in[a,b]$, the surface

$$\boldsymbol{s}(t,u)=\boldsymbol{r}(t)+u\boldsymbol{g}(t)\,,\ u\in\mathbb{R} \tag{2}$$

is a ruled one. The given curve is the directrix of the surface, while for any fixed $t_0\in[a,b]$, the lines $\boldsymbol{r}(t_0)+u\boldsymbol{g}(t_0)$ are called the generators (or rulings). The surface is developable if the normals of the tangent planes along the generators are of constant direction; i.e., considering the partial derivatives

$$\frac{\partial}{\partial t}\boldsymbol{s}(t,u)=\dot{\boldsymbol{r}}(t)+u\dot{\boldsymbol{g}}(t)\,, \tag{3}$$

$$\frac{\partial}{\partial u}\boldsymbol{s}(t,u)=\boldsymbol{g}(t)\,, \tag{4}$$

the normal vector

$$\begin{aligned}\boldsymbol{n}(t,u)&=\frac{\partial}{\partial t}\boldsymbol{s}(t,u)\times\frac{\partial}{\partial s}\boldsymbol{s}(t,u)\\&=[\dot{\boldsymbol{r}}(t)+u\dot{\boldsymbol{g}}(t)]\times\boldsymbol{g}(t)\end{aligned} \tag{5}$$

does not depend on u for any t. In other words, the tangent planes of the surface along its generators coincide.

Now let us consider a developable surface as a mirror. Here, we assume that the light source is point-like, either being at infinity or not, and that none of the generators of the developable surface go through the light source. For any generator $\boldsymbol{r}(t_0)+u\boldsymbol{g}(t_0)$ of the surface, incoming light rays meeting the mirror surface along this generator are coplanar, and due to the fixed tangent plane along the generator, the reflected light rays are also coplanar. The plane of reflected light rays is to be computed first. This plane is fully determined by the selected generator and one single reflected light ray. In the following subsections, the computations are presented separately for the case when the light source is at infinity (yielding parallel light rays) and for the case when the source of light is a point of the affine space.

2.1 Light source at infinity

Without loss of generality of the forthcoming computation, we can assume that the direction of parallel light rays is $\boldsymbol{d}=[1\ 0\ 0]^{\mathrm{T}}$. We further assume that none of the generators are parallel to the given direction. Let the selected generator of the surface be $\boldsymbol{r}(t_0)+u\boldsymbol{g}(t_0)$.

The direction $\overline{\boldsymbol{d}}(t_0)$ of the reflected rays can be computed by reflecting $\boldsymbol{d}$ with respect to the tangent plane along the selected generator with normal vector $\boldsymbol{n}(t_0)$. Since the endpoint of $\boldsymbol{d}$ is on the unit sphere,

$$x^2+y^2+z^2=1, \tag{6}$$

the endpoint of the reflected ray $\overline{\boldsymbol{d}}(t_0)$ will also be on this sphere. Thus, we have to compute the intersection point of the unit sphere and the line:

$$\begin{aligned}\boldsymbol{d}+\lambda\boldsymbol{n}(t_0)&=\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix}+\lambda\begin{bmatrix}n_x(t_0)\\n_y(t_0)\\n_z(t_0)\end{bmatrix}\\&=\begin{bmatrix}1+\lambda n_x(t_0)\\\lambda n_y(t_0)\\\lambda n_z(t_0)\end{bmatrix},\end{aligned}\tag{7}$$

where $\lambda\in\mathbb{R}$. Solving the equation

$$[1+\lambda n_x(t_0)]^2+[\lambda n_y(t_0)]^2+[\lambda n_z(t_0)]^2=1,\tag{8}$$

one can find

$$\begin{aligned}\lambda&=\frac{-2n_x(t_0)}{n_x^2(t_0)+n_y^2(t_0)+n_z^2(t_0)}\\&=-\frac{2n_x(t_0)}{\|\boldsymbol{n}(t_0)\|^2},\ \text{ if }\|\boldsymbol{n}(t_0)\|\neq 0.\end{aligned}\tag{9}$$

Therefore, the direction of the reflected rays along this generator is

$$\begin{aligned}\overline{\boldsymbol{d}}(t_0)&=\begin{bmatrix}1-\frac{2n_x(t_0)}{\|\boldsymbol{n}(t_0)\|^2}n_x(t_0)\\-\frac{2n_x(t_0)}{\|\boldsymbol{n}(t_0)\|^2}n_y(t_0)\\-\frac{2n_x(t_0)}{\|\boldsymbol{n}(t_0)\|^2}n_z(t_0)\end{bmatrix}\\&=\boldsymbol{d}-\frac{2n_x(t_0)}{\|\boldsymbol{n}(t_0)\|^2}\boldsymbol{n}(t_0).\end{aligned}\tag{10}$$

2.2 Light source at an affine point

Without loss of generality of the foregoing computation, we can assume that the source of light is at the origin of the coordinate system. We further assume that none of the generators pass through this point. Again, let the selected generator of the surface be $\boldsymbol{r}(t_0)+u\boldsymbol{g}(t_0)$.

As we have observed, light rays intersecting this generator form a plane, and the reflected light rays will also form a plane passing through this generator. To determine this plane, it is enough to reflect one single light ray, e.g., the one intersecting the directrix curve $\boldsymbol{r}(t)$ at this generator. In other words, we have to reflect the vector $\boldsymbol{r}(t_0)$ (this is the direction of a light ray coming from the origin, i.e., $\boldsymbol{r}(t_0)=\boldsymbol{d}(t_0)$) with respect to the normal vector $\boldsymbol{n}(t_0)$ of the tangent plane.

The endpoint of $\boldsymbol{r}(t_0)=\boldsymbol{d}(t_0)$ is on the sphere:

$$x^2+y^2+z^2=\|\boldsymbol{r}(t_0)\|^2,\tag{11}$$

and therefore the endpoint of the reflected vector $\overline{\boldsymbol{d}}(t_0)$ also has to be on this sphere.

Thus, we have to compute the intersection point of the line passing through $\boldsymbol{r}(t_0)$ with the direction vector $\boldsymbol{n}(t_0)$, i.e., of the line

$$\begin{aligned}\boldsymbol{r}(t_0)+\lambda\boldsymbol{n}(t_0)&=\begin{bmatrix}r_x(t_0)\\r_y(t_0)\\r_z(t_0)\end{bmatrix}+\lambda\begin{bmatrix}n_x(t_0)\\n_y(t_0)\\n_z(t_0)\end{bmatrix}\\&=\begin{bmatrix}r_x(t_0)+\lambda n_x(t_0)\\r_y(t_0)+\lambda n_y(t_0)\\r_z(t_0)+\lambda n_z(t_0)\end{bmatrix}\end{aligned}\tag{12}$$

and sphere (11). Solving the equation

$$\|\boldsymbol{r}(t_0)\|^2=\|\boldsymbol{r}(t_0)+\lambda\boldsymbol{n}(t_0)\|^2,\tag{13}$$

one obtains

$$\begin{aligned}\lambda=&-\frac{2n_x(t_0)r_x(t_0)+2n_y(t_0)r_y(t_0)}{n_x^2(t_0)+n_y^2(t_0)+n_z^2(t_0)}\\&-\frac{2n_z(t_0)r_z(t_0)}{n_x^2(t_0)+n_y^2(t_0)+n_z^2(t_0)}\\=&-\frac{2\boldsymbol{r}(t_0)\cdot\boldsymbol{n}(t_0)}{\|\boldsymbol{n}(t_0)\|^2},\ \text{ if }\|\boldsymbol{n}(t_0)\|\neq 0.\end{aligned}\tag{14}$$

The reflected vector is

$$\overline{\boldsymbol{d}}(t_0)=\boldsymbol{r}(t_0)-\frac{2\boldsymbol{r}(t_0)\cdot\boldsymbol{n}(t_0)}{\|\boldsymbol{n}(t_0)\|^2}\boldsymbol{n}(t_0),\tag{15}$$

which, together with the selected generator, determines the plane of the reflected rays.

2.3 The family of planes of reflected rays and their envelope surface

In the preceding subsections, we have computed the vector $\overline{\boldsymbol{d}}(t_0)$, and this computation is necessary for determining the plane of reflected rays at each generator $\boldsymbol{r}(t_0)+u\boldsymbol{g}(t_0)$ of the developable surface. Forthcoming computations are independent of the actual position of the light source.

The reflected rays form a one-parameter family of planes:

$$\boldsymbol{P}(u,v,t)=\boldsymbol{r}(t)+u\boldsymbol{g}(t)+v\overline{\boldsymbol{d}}(t),\tag{16}$$

where $t\in[a,b]$ is the family parameter and $u,v\in\mathbb{R}$. It is easy to see that the envelope surface of this one-parameter family of reflected light planes is also developable (Do Carmo, 2016). This envelope

surface is the caustic surface of the original mirror surface.

In what follows, we will determine this caustic surface in the standard form of developable surfaces analogous to Eq. (2), i.e., in the following form:

$$\boldsymbol{e}(t,\lambda)=\boldsymbol{q}(t)+\lambda\boldsymbol{f}(t),\ t\in[a,b],\ \lambda\in\mathbb{R}, \tag{17}$$

where $\boldsymbol{q}(t)$ is the curve and $\boldsymbol{f}(t)$ is the direction of the generators passing through the points of this curve. We apply the computations as

$$\frac{\partial}{\partial u}\boldsymbol{P}(u,v,t)=\boldsymbol{g}(t), \tag{18}$$

$$\frac{\partial}{\partial v}\boldsymbol{P}(u,v,t)=\overline{\boldsymbol{d}}(t), \tag{19}$$

$$\frac{\partial}{\partial t}\boldsymbol{P}(u,v,t)=\dot{\boldsymbol{r}}(t)+u\dot{\boldsymbol{g}}(t)+v\dot{\overline{\boldsymbol{d}}}(t), \tag{20}$$

and therefore we have

$$\begin{aligned}&\det\left[\ \tfrac{\partial}{\partial u}\boldsymbol{P}(u,v,t)\quad \tfrac{\partial}{\partial v}\boldsymbol{P}(u,v,t)\quad \tfrac{\partial}{\partial t}\boldsymbol{P}(u,v,t)\ \right]\\&=\left[\boldsymbol{g}(t)\times\overline{\boldsymbol{d}}(t)\right]\cdot\left[\dot{\boldsymbol{r}}(t)+u\dot{\boldsymbol{g}}(t)+v\dot{\overline{\boldsymbol{d}}}(t)\right]=0,\end{aligned} \tag{21}$$

from which we can obtain

$$\begin{aligned}&\left[\boldsymbol{g}(t)\times\overline{\boldsymbol{d}}(t)\right]\cdot\dot{\boldsymbol{r}}(t)+\left[\boldsymbol{g}(t)\times\overline{\boldsymbol{d}}(t)\right]\cdot u\dot{\boldsymbol{g}}(t)\\&+\left[\boldsymbol{g}(t)\times\overline{\boldsymbol{d}}(t)\right]\cdot v\dot{\overline{\boldsymbol{d}}}(t)=0.\end{aligned} \tag{22}$$

The normal vector of the plane $\boldsymbol{P}(u,v,t)$ is

$$\boldsymbol{g}(t)\times\overline{\boldsymbol{d}}(t), \tag{23}$$

the derivative of which is

$$\dot{\boldsymbol{g}}(t)\times\overline{\boldsymbol{d}}(t)+\boldsymbol{g}(t)\times\dot{\overline{\boldsymbol{d}}}(t). \tag{24}$$

The cross product of the normal vector (23) and its derivative (24) is the direction of the generators of the envelope surface passing through the points of $\boldsymbol{r}(t)$. This direction can be computed as

$$\begin{aligned}\boldsymbol{f}(t)&=\left[\boldsymbol{g}(t)\times\overline{\boldsymbol{d}}(t)\right]\\&\quad\times\left[\dot{\boldsymbol{g}}(t)\times\overline{\boldsymbol{d}}(t)+\boldsymbol{g}(t)\times\dot{\overline{\boldsymbol{d}}}(t)\right]\\&=\left[\boldsymbol{g}(t)\times\overline{\boldsymbol{d}}(t)\right]\times\left[\dot{\boldsymbol{g}}(t)\times\overline{\boldsymbol{d}}(t)\right]\\&\quad+\left[\boldsymbol{g}(t)\times\overline{\boldsymbol{d}}(t)\right]\times\left[\boldsymbol{g}(t)\times\dot{\overline{\boldsymbol{d}}}(t)\right].\end{aligned} \tag{25}$$

To simplify this expression, we apply the formula of

$$\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})=(\boldsymbol{a}\cdot\boldsymbol{c})\,\boldsymbol{b}-(\boldsymbol{a}\cdot\boldsymbol{b})\,\boldsymbol{c}. \tag{26}$$

Accordingly, the two terms on the right-hand side of Eq. (25) can be written as

$$\begin{aligned}&\left[\boldsymbol{g}(t)\times\overline{\boldsymbol{d}}(t)\right]\times\left[\dot{\boldsymbol{g}}(t)\times\overline{\boldsymbol{d}}(t)\right]\\&=\left[\left(\boldsymbol{g}(t)\times\overline{\boldsymbol{d}}(t)\right)\cdot\overline{\boldsymbol{d}}(t)\right]\dot{\boldsymbol{g}}(t)\\&\quad-\left[\left(\boldsymbol{g}(t)\times\overline{\boldsymbol{d}}(t)\right)\cdot\dot{\boldsymbol{g}}(t)\right]\overline{\boldsymbol{d}}(t)\\&=-\left[\left(\boldsymbol{g}(t)\times\overline{\boldsymbol{d}}(t)\right)\cdot\dot{\boldsymbol{g}}(t)\right]\overline{\boldsymbol{d}}(t),\end{aligned} \tag{27}$$

and

$$\begin{aligned}&\left[\boldsymbol{g}(t)\times\overline{\boldsymbol{d}}(t)\right]\times\left[\boldsymbol{g}(t)\times\dot{\overline{\boldsymbol{d}}}(t)\right]\\&=\left[\left(\boldsymbol{g}(t)\times\overline{\boldsymbol{d}}(t)\right)\cdot\dot{\overline{\boldsymbol{d}}}(t)\right]\boldsymbol{g}(t)\\&\quad-\left[\left(\boldsymbol{g}(t)\times\overline{\boldsymbol{d}}(t)\right)\cdot\boldsymbol{g}(t)\right]\dot{\overline{\boldsymbol{d}}}(t)\\&=\left[\left(\boldsymbol{g}(t)\times\overline{\boldsymbol{d}}(t)\right)\cdot\dot{\overline{\boldsymbol{d}}}(t)\right]\boldsymbol{g}(t).\end{aligned} \tag{28}$$

Therefore, the direction of the generators of the envelope surface can simply be written in the following form:

$$\begin{aligned}\boldsymbol{f}(t)&=\left[\left(\boldsymbol{g}(t)\times\overline{\boldsymbol{d}}(t)\right)\cdot\dot{\overline{\boldsymbol{d}}}(t)\right]\boldsymbol{g}(t)\\&\quad-\left[\left(\boldsymbol{g}(t)\times\overline{\boldsymbol{d}}(t)\right)\cdot\dot{\boldsymbol{g}}(t)\right]\overline{\boldsymbol{d}}(t).\end{aligned} \tag{29}$$

Now, if one can fix u and v in a way that determinant (22) vanishes $\forall t\in[a,\ b]$, then by substituting these values of u and v into Eq. (16), we obtain a curve (or a constant vector) $\boldsymbol{q}(t)$, based on which the caustic envelope surface can be written in the classic form of developable surfaces in Eq. (17).

3 Caustics of developable surfaces

As is well-known, there are three types of developable surfaces: (generalized) cone, (generalized) cylinder, and the tangent surface of spatial curves. In this section, we specify the general computations in Section 2 to the three different types of developable surfaces.

3.1 Cones as mirrors

Considering a cone as a mirror, defined by curve $\boldsymbol{r}(t)$, apex $\boldsymbol{p}$, and light rays with direction $\boldsymbol{d}(t)$, the caustic surface, i.e., the envelope surface of the reflected light rays with direction $\overline{\boldsymbol{d}}(t)$, is also a cone (Pottmann and Wallner, 2000). Herein, we present the exact computation of the caustic cone surface and provide the solution in a closed form.

In the case of a conic mirror surface, the curve $\boldsymbol{r}(t)$ of Eq. (2) is an arbitrary planar or spatial curve,

while the generators can be written as

$$\boldsymbol{g}(t) = \boldsymbol{p} - \boldsymbol{r}(t),\ t \in [a, b], \tag{30}$$

where $\boldsymbol{p}$, the apex of the cone, is an arbitrary point out of curve $\boldsymbol{r}$. The family of planes in Eq. (16) will be of the specific form:

$$\boldsymbol{P}(u, v, t) = \boldsymbol{r}(t) + u(\boldsymbol{p} - \boldsymbol{r}(t)) + v\overline{\boldsymbol{d}}(t), \tag{31}$$

where $t \in [a,\ b]$ and $u, v \in \mathbb{R}$.

In this case, determinant (22) will be

$$\begin{aligned}&\left[(\boldsymbol{p} - \boldsymbol{r}(t)) \times \overline{\boldsymbol{d}}(t)\right] \cdot \dot{\boldsymbol{r}}(t)\\ &- \left[(\boldsymbol{p} - \boldsymbol{r}(t)) \times \overline{\boldsymbol{d}}(t)\right] \cdot u\dot{\boldsymbol{r}}(t)\\ &+ \left[(\boldsymbol{p} - \boldsymbol{r}(t)) \times \overline{\boldsymbol{d}}(t)\right] \cdot v\dot{\overline{\boldsymbol{d}}}(t)\\ =&\left[(\boldsymbol{p} - \boldsymbol{r}(t)) \times \overline{\boldsymbol{d}}(t)\right] \cdot (1-u)\dot{\boldsymbol{r}}(t)\\ &+ \left[(\boldsymbol{p} - \boldsymbol{r}(t)) \times \overline{\boldsymbol{d}}(t)\right] \cdot v\dot{\overline{\boldsymbol{d}}}(t),\end{aligned} \tag{32}$$

which vanishes $\forall t \in [a, b]$ if $u = 1$ and $v = 0$. Substituting these parameter values into Eq. (31), we obtain

$$\boldsymbol{P}(u, v, t) = \boldsymbol{r}(t) + \boldsymbol{p} - \boldsymbol{r}(t) = \boldsymbol{p}. \tag{33}$$

That is, the caustic envelope surface is also a cone with apex $\boldsymbol{p}$. The direction of the generators can be written as

$$\begin{aligned}\boldsymbol{f}(t) =&\left\{\left[(\boldsymbol{p} - \boldsymbol{r}(t)) \times \overline{\boldsymbol{d}}(t)\right] \cdot \dot{\overline{\boldsymbol{d}}}(t)\right\}(\boldsymbol{p} - \boldsymbol{r}(t))\\ &+ \left\{\left[(\boldsymbol{p} - \boldsymbol{r}(t)) \times \overline{\boldsymbol{d}}(t)\right] \cdot \dot{\boldsymbol{r}}(t)\right\}\overline{\boldsymbol{d}}(t).\end{aligned} \tag{34}$$

Based on the calculations above, the caustic envelope surface of the cone can be written as

$$\boldsymbol{e}(t, \lambda) = \boldsymbol{p} + \lambda \boldsymbol{f}(t),\ t \in [a, b],\ \lambda \in \mathbb{R}, \tag{35}$$

where $\boldsymbol{p}$ is the apex and $\boldsymbol{f}(t)$ denotes the direction of generators.

An example is shown in Fig. 1.

3.2 Cylinders as mirrors

Considering a cylinder as a mirror, defined by curve $\boldsymbol{r}(t)$, direction $\boldsymbol{a}$, and light rays with direction $\boldsymbol{d}(t)$, it is known that the caustic surface, i.e., the envelope surface of the reflected light rays with direction $\overline{\boldsymbol{d}}(t)$, is also a cylinder (Pottmann and Wallner, 2000). In this subsection, we present the computation of this caustic surface and provide a closed form of it.

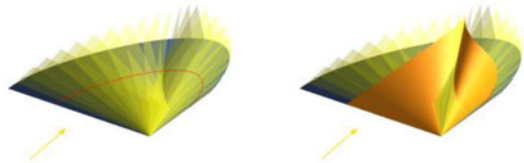

Fig. 1 The family of planes of reflected rays (in yellow) and their envelope caustic surface (in orange) in the case of a conic mirror (in blue) defined by a cubic Bézier curve (in red)

The direction of the incoming light rays (in yellow) is shown. References to color refer to the online version of this figure

If the mirror surface in Eq. (2) is a cylinder, then curve $\boldsymbol{r}(t)$ is an arbitrary planar or spatial curve, and the direction of the generators $\boldsymbol{g}(t)$ is constant, i.e.,

$$\boldsymbol{g}(t) = \boldsymbol{a},\ t \in [a, b], \tag{36}$$

where $\boldsymbol{a}$ is a vector (not parallel to the plane of $\boldsymbol{r}$ when $\boldsymbol{r}(t)$ is planar). The family of planes in Eq. (16) of the reflected rays can be written as

$$\boldsymbol{P}(u, v, t) = \boldsymbol{r}(t) + u\boldsymbol{a} + v\overline{\boldsymbol{d}}(t),\ t \in [a, b],\ u, v \in \mathbb{R}. \tag{37}$$

The direction of generators of the caustic envelope surface will be

$$\begin{aligned}\boldsymbol{f}(t) =&\boldsymbol{a} \times \overline{\boldsymbol{d}}(t) \cdot \dot{\overline{\boldsymbol{d}}}(t)\,\boldsymbol{a} - \left[\left(\boldsymbol{a} \times \overline{\boldsymbol{d}}(t)\right) \cdot \boldsymbol{0}\right]\overline{\boldsymbol{d}}(t)\\ =&\left(\boldsymbol{a} \times \overline{\boldsymbol{d}}(t)\right) \cdot \dot{\overline{\boldsymbol{d}}}(t)\,\boldsymbol{a}.\end{aligned} \tag{38}$$

That is, $\boldsymbol{f}(t)$ is parallel to $\boldsymbol{a}$, $\forall t \in [a, b]$. This means that the caustic surface of a cylindrical mirror is also a cylinder, and that the generators of these two cylinders are parallel.

Determinant (22) becomes

$$\begin{aligned}0 =&\left(\boldsymbol{a} \times \overline{\boldsymbol{d}}(t)\right) \cdot \dot{\boldsymbol{r}}(t) + \left(\boldsymbol{a} \times \overline{\boldsymbol{d}}(t)\right) \cdot u\boldsymbol{0}\\ &+ \left(\boldsymbol{a} \times \overline{\boldsymbol{d}}(t)\right) \cdot v\dot{\overline{\boldsymbol{d}}}(t)\\ =&\left(\boldsymbol{a} \times \overline{\boldsymbol{d}}(t)\right) \cdot \dot{\boldsymbol{r}}(t) + \left(\boldsymbol{a} \times \overline{\boldsymbol{d}}(t)\right) \cdot v\dot{\overline{\boldsymbol{d}}}(t),\end{aligned} \tag{39}$$

from which one can express v as

$$-\left(\boldsymbol{a} \times \overline{\boldsymbol{d}}(t)\right) \cdot \dot{\boldsymbol{r}}(t) = \left(\boldsymbol{a} \times \overline{\boldsymbol{d}}(t)\right) \cdot v\dot{\overline{\boldsymbol{d}}}(t), \tag{40}$$

$$v = -\frac{\left(\boldsymbol{a} \times \overline{\boldsymbol{d}}(t)\right) \cdot \dot{\boldsymbol{r}}(t)}{\left(\boldsymbol{a} \times \overline{\boldsymbol{d}}(t)\right) \cdot \dot{\overline{\boldsymbol{d}}}(t)}, \tag{41}$$

if $\left(\boldsymbol{a} \times \overline{\boldsymbol{d}}(t)\right) \cdot \dot{\overline{\boldsymbol{d}}}(t) \neq 0$.

Substituting the calculated v and $u = 0$ into Eq. (22), it vanishes $\forall t \in [a, b]$. Substituting the

same v and $u = 0$ into the family of planes in Eq. (37), the following curve is obtained:

$$\boldsymbol{h}(t) = \boldsymbol{r}(t) - \frac{\left(\boldsymbol{a} \times \overline{\boldsymbol{d}}(t)\right) \cdot \dot{\boldsymbol{r}}(t)}{\left(\boldsymbol{a} \times \overline{\boldsymbol{d}}(t)\right) \cdot \dot{\overline{\boldsymbol{d}}}(t)} \overline{\boldsymbol{d}}(t). \tag{42}$$

Based on the calculations presented above, the caustic surface of the cylinder can be written as

$$\boldsymbol{e}(t, \lambda) = \boldsymbol{h}(t) + \lambda \boldsymbol{a},\ t \in [a, b],\ \lambda \in \mathbb{R}. \tag{43}$$

A cylindrical mirror and its caustic surface are shown in Fig. 2.

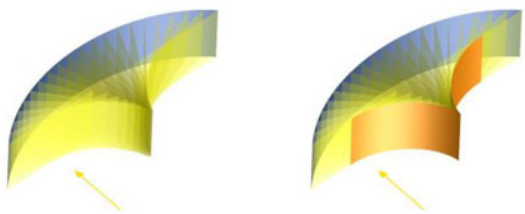

Fig. 2 The family of planes of reflected rays (in yellow) and their envelope caustic surface (in orange) in the case of a cylindric mirror (in blue)

The direction of the incoming light rays (in yellow) is shown. References to color refer to the online version of this figure

3.3 Tangent surface of a spatial curve as a mirror

Considering a general tangent surface of a spatial curve $\boldsymbol{r}(t)$ as a mirror and light rays with direction $\boldsymbol{d}(t)$, the caustic surface, i.e., the envelope surface of the reflected light rays with direction $\overline{\boldsymbol{d}}(t)$, has been proven to be a tangent surface (Pottmann and Wallner, 2000), but the exact construction of this surface is not known. Now, we present the computation of the caustic surface and provide the solution in a closed form.

If the developable mirror surface in Eq. (2) is a general tangent surface of a spatial curve, then $\boldsymbol{r}(t)$ is an arbitrary spatial curve, while the direction of the generators fulfills the relationship as

$$\boldsymbol{g}(t) = \dot{\boldsymbol{r}}(t),\ t \in [a, b]. \tag{44}$$

In this case, determinant (22) is of the form as

$$\left(\dot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t)\right) \cdot u\ddot{\boldsymbol{r}}(t) + \left(\dot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t)\right) \cdot v\dot{\overline{\boldsymbol{d}}}(t), \tag{45}$$

and it vanishes $\forall t \in [a, b]$ when $u = v = 0$. This immediately yields the somewhat surprising fact that the caustic envelope surface contains the original curve $\boldsymbol{r}(t)$.

The normal vector of plane $\boldsymbol{P}(u, v, t)$ is

$$\dot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t), \tag{46}$$

while its derivative is

$$\ddot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t) + \dot{\boldsymbol{r}}(t) \times \dot{\overline{\boldsymbol{d}}}(t). \tag{47}$$

In this case, the direction of the generators of the caustic envelope surface can be written as

$$\begin{aligned}\boldsymbol{f}(t) = &\left[\left(\dot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t)\right) \cdot \dot{\overline{\boldsymbol{d}}}(t)\right] \dot{\boldsymbol{r}}(t) \\ &- \left[\left(\dot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t)\right) \cdot \ddot{\boldsymbol{r}}(t)\right] \overline{\boldsymbol{d}}(t),\end{aligned} \tag{48}$$

and the caustic envelope surface is of the form as

$$\boldsymbol{e}(t, \lambda) = \boldsymbol{r}(t) + \lambda \boldsymbol{f}(t),\ t \in [a, b],\ \lambda \in \mathbb{R}. \tag{49}$$

It is clear from this expression that the caustic surface is a ruled one. However, our aim is to prove that it is a developable surface, specifically, that it is a tangent developable surface of a spatial curve.

To reach this aim, we have to find the curve of regression, i.e., a curve $\boldsymbol{c}(t),\ t \in [a, b]$, the tangents of which are parallel to the directions $\boldsymbol{f}(t)$ of the generators of the surface, $\forall t \in [a, b]$.

Since each curve on the ruled surface in Eq. (49) can be considered a functional translation of curve $\boldsymbol{r}(t)$ along the rulings, we search for the curve $\boldsymbol{c}(t)$ in the following form:

$$\boldsymbol{c}(t) = \boldsymbol{r}(t) + \lambda(t) \boldsymbol{f}(t), \tag{50}$$

where the function $\lambda(t),\ t \in [a,\ b]$, is to be determined. Moreover, the derivative of this curve with respect to t is

$$\dot{\boldsymbol{c}}(t) = \dot{\boldsymbol{r}}(t) + \dot{\lambda}(t) \boldsymbol{f}(t) + \lambda(t) \dot{\boldsymbol{f}}(t), \tag{51}$$

which must be parallel to $\boldsymbol{f}(t)$; i.e., $\dot{\boldsymbol{c}}(t)$ must be orthogonal to vectors (46) and (47). These conditions yield the following system of equations:

$$\dot{\boldsymbol{c}}(t) \cdot \left(\dot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t)\right) = 0, \tag{52}$$

$$\dot{\boldsymbol{c}}(t) \cdot \left(\ddot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t) + \dot{\boldsymbol{r}}(t) \times \dot{\overline{\boldsymbol{d}}}(t)\right) = 0. \tag{53}$$

From Eq. (52), we obtain the following equality:

$$\begin{aligned}0 &= \left(\dot{\boldsymbol{r}}(t) + \dot{\lambda}(t) \boldsymbol{f}(t) + \lambda(t) \dot{\boldsymbol{f}}(t)\right) \cdot \left(\dot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t)\right) \\ &= \lambda(t) \dot{\boldsymbol{f}}(t) \cdot \left(\dot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t)\right),\end{aligned} \tag{54}$$

and from Eq. (54), we obtain

$$\begin{aligned}0 &= \left(\dot{\boldsymbol{r}}(t) + \dot{\lambda}(t)\boldsymbol{f}(t) + \lambda(t)\dot{\boldsymbol{f}}(t)\right) \\ &\quad \cdot \left(\ddot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t) + \dot{\boldsymbol{r}}(t) \times \dot{\overline{\boldsymbol{d}}}(t)\right) \\ &= \dot{\boldsymbol{r}}(t) \cdot \left(\ddot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t) + \dot{\boldsymbol{r}}(t) \times \dot{\overline{\boldsymbol{d}}}(t)\right) \\ &\quad + \lambda(t)\dot{\boldsymbol{f}}(t) \cdot \left(\ddot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t) + \dot{\boldsymbol{r}}(t) \times \dot{\overline{\boldsymbol{d}}}(t)\right) \\ &= \dot{\boldsymbol{r}}(t) \cdot \left(\ddot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t)\right) + \lambda(t)\dot{\boldsymbol{f}}(t) \\ &\quad \cdot \left(\ddot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t) + \dot{\boldsymbol{r}}(t) \times \dot{\overline{\boldsymbol{d}}}(t)\right),\end{aligned} \tag{55}$$

which further yields

$$\begin{aligned}&-\dot{\boldsymbol{r}}(t) \cdot \left(\ddot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t)\right) \\ &= \lambda(t)\dot{\boldsymbol{f}}(t) \cdot \left(\ddot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t) + \dot{\boldsymbol{r}}(t) \times \dot{\overline{\boldsymbol{d}}}(t)\right).\end{aligned} \tag{56}$$

Therefore,

$$\lambda(t) = \frac{-\dot{\boldsymbol{r}}(t) \cdot \left(\ddot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t)\right)}{\dot{\boldsymbol{f}}(t) \cdot \left(\ddot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t) + \dot{\boldsymbol{r}}(t) \times \dot{\overline{\boldsymbol{d}}}(t)\right)}, \tag{57}$$

whenever

$$\dot{\boldsymbol{f}}(t) \cdot \left(\ddot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t) + \dot{\boldsymbol{r}}(t) \times \dot{\overline{\boldsymbol{d}}}(t)\right) \neq 0 \tag{58}$$

holds. Here, we prove that this scalar product cannot be equal to zero.

The first factor of the scalar product in inequality (58) is not vanishing, i.e., $\dot{\boldsymbol{f}}(t) \neq \boldsymbol{0}$, since the generators of the surface are not parallel (it is not a cylinder). Thus, $\dot{\boldsymbol{f}}(t)$ can be written in the following form:

$$\begin{aligned}\dot{\boldsymbol{f}}(t) =& \frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\dot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t)\right) \cdot \dot{\overline{\boldsymbol{d}}}(t)\right]\dot{\boldsymbol{r}}(t) \\ &+ \left[\left(\dot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t)\right) \cdot \dot{\overline{\boldsymbol{d}}}(t)\right]\ddot{\boldsymbol{r}}(t) \\ &- \frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\dot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t)\right) \cdot \ddot{\boldsymbol{r}}(t)\right]\overline{\boldsymbol{d}}(t) \\ &- \left[\left(\dot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t)\right) \cdot \ddot{\boldsymbol{r}}(t)\right]\dot{\overline{\boldsymbol{d}}}(t) \\ =& \frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\dot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t)\right) \cdot \dot{\overline{\boldsymbol{d}}}(t)\right]\dot{\boldsymbol{r}}(t) \\ &- \frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\dot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t)\right) \cdot \ddot{\boldsymbol{r}}(t)\right]\overline{\boldsymbol{d}}(t) \qquad (59) \\ &+ \left[\left(\dot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t)\right) \cdot \dot{\overline{\boldsymbol{d}}}(t)\right]\ddot{\boldsymbol{r}}(t) \\ &- \left[\left(\dot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t)\right) \cdot \ddot{\boldsymbol{r}}(t)\right]\dot{\overline{\boldsymbol{d}}}(t). \qquad (60)\end{aligned}$$

Since expression (59) is the linear combination of $\dot{\boldsymbol{r}}(t)$ and $\overline{\boldsymbol{d}}(t)$, this term is parallel to the plane spanned by $\dot{\boldsymbol{r}}(t)$ and $\overline{\boldsymbol{d}}(t)$, which is actually the plane $\boldsymbol{P}(u, v, t)$.

Applying identity (26) to expression (60), one can find

$$\begin{aligned}&\left[\left(\dot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t)\right) \cdot \dot{\overline{\boldsymbol{d}}}(t)\right]\ddot{\boldsymbol{r}}(t) \\ &- \left[\left(\dot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t)\right) \cdot \ddot{\boldsymbol{r}}(t)\right]\dot{\overline{\boldsymbol{d}}}(t) \\ &= \left[\dot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t)\right] \times \left[\ddot{\boldsymbol{r}}(t) \times \dot{\overline{\boldsymbol{d}}}(t)\right].\end{aligned} \tag{61}$$

However, this cross product is evidently orthogonal to one of its factors, i.e.,

$$\left(\dot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t)\right) \times \left(\ddot{\boldsymbol{r}}(t) \times \dot{\overline{\boldsymbol{d}}}(t)\right) \perp \left(\dot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t)\right). \tag{62}$$

Therefore, expression (60) is also parallel to the plane (orthogonal to the normal vector of the plane) spanned by $\dot{\boldsymbol{r}}(t)$ and $\overline{\boldsymbol{d}}(t)$, i.e., $\boldsymbol{P}(u, v, t)$. We obtain that both terms of $\dot{\boldsymbol{f}}(t)$ (expressions (59) and (60)) and $\dot{\boldsymbol{f}}(t)$ are parallel to the plane $\boldsymbol{P}(u, v, t)$ and orthogonal to its normal vector, i.e., $\dot{\boldsymbol{f}}(t) \perp \left(\dot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t)\right)$.

For the second factor of the scalar product in inequality (58), the equality

$$\ddot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t) + \dot{\boldsymbol{r}}(t) \times \dot{\overline{\boldsymbol{d}}}(t) = \frac{\mathrm{d}}{\mathrm{d}t}(\dot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t)) \tag{63}$$

holds; therefore, this term describes the change of the normal vector $\dot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t)$ of the plane $\boldsymbol{P}(u, v, t)$.

As a consequence, the scalar product, i.e., the left side of inequality (58), could be equal to zero only if either the second factor, i.e., the derivative of the normal vector $\dot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t)$, is the null vector or the normal vector is parallel to its derivative. Both would yield that the planes $\boldsymbol{P}(u, v, t)$ coincide or that they are parallel $\forall t$; i.e., curve $\boldsymbol{r}(t)$ is a planar curve. However, this contradicts our assumption that we are studying the tangent surfaces of spatial curves.

Back to the system of Eqs. (52) and (53), we have expressed $\lambda(t)$ from Eq. (53), but this must fulfill Eq. (52) as well.

In other words,

$$0 = \lambda(t)\dot{\boldsymbol{f}}(t) \cdot \left(\dot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t)\right) \tag{64}$$

must hold. However, this equation indeed holds, since we have just seen that $\dot{\boldsymbol{f}}(t) \perp \left(\dot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t)\right)$; i.e., the scalar product above equals zero, $\forall t$.

Based on the computation presented above, the caustic surface of a tangent surface can be computed

as a tangent surface of the spatial curve $\boldsymbol{c}(t) = \boldsymbol{r}(t) + \lambda(t)\,\boldsymbol{f}(t)$, where

$$\begin{aligned}\boldsymbol{f}(t) = &\left[\left(\dot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t)\right) \cdot \dot{\overline{\boldsymbol{d}}}(t)\right] \dot{\boldsymbol{r}}(t) \\ &- \left[\left(\dot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t)\right) \cdot \ddot{\boldsymbol{r}}(t)\right] \overline{\boldsymbol{d}}(t)\end{aligned} \tag{65}$$

and

$$\lambda(t) = \frac{-\dot{\boldsymbol{r}}(t) \cdot \left(\ddot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t)\right)}{\dot{\boldsymbol{f}}(t) \cdot \left[\ddot{\boldsymbol{r}}(t) \times \overline{\boldsymbol{d}}(t) + \dot{\boldsymbol{r}}(t) \times \dot{\overline{\boldsymbol{d}}}(t)\right]}. \tag{66}$$

Examples of this type of mirror and its caustic surface are shown in Figs. 3 and 4, respectively.

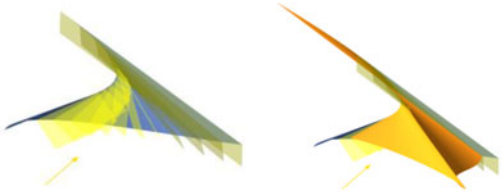

Fig. 3 The family of planes of reflected rays (in yellow) and their envelope caustic surface (in orange) in the case of a tangent surface of a spatial curve as a mirror (in blue)

The direction of the incoming light rays (in yellow) is shown. References to color refer to the online version of this figure

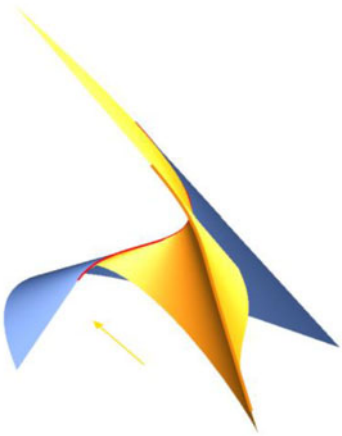

Fig. 4 The caustic surface (in orange) along with its regression curve in the case of a tangent surface of a spatial curve as a mirror (in blue) defined by a cubic Bézier curve (in red)

The direction of the incoming light rays (in yellow) is shown. References to color refer to the online version of this figure

4 An application: developable surfaces in architecture

Developable surfaces, especially the tangent surfaces of a curve, are commonly used in modern architecture (Glaeser and Gruber, 2007; Pottmann et al., 2015). If the surface of such a construction is made from a reflective material (e.g., metal sheets at the Guggenheim Museum in Bilbao), then the surface may behave like a mirror. When a free-form building or a sculpture reflects light rays, some places around the structure, where light rays are concentrated, can be of high temperature. This especially holds around the cusp of the intersection curve of the caustic surface of the building and the ground plane. Our aim is to find those "hot" points of the ground plane. Without loss of generality, we can assume that this plane is the $[x, y]$ plane.

The intersection of the caustic surface $\boldsymbol{e}(t, \lambda)$ and the $[x, y]$ plane is a curve, expressed as follows:

$$\boldsymbol{e}(t, \lambda) \cap [x, y] = \boldsymbol{e}_{\mathrm{c}}(t). \tag{67}$$

This curve can be described as

$$\boldsymbol{e}_{\mathrm{c}}(t) = \boldsymbol{e}(t, \lambda(t)), \tag{68}$$

where $\lambda(t)$ is the solution to the equation $\boldsymbol{e}(t, \lambda) = \boldsymbol{0}$ for parameter t. This curve usually has a cusp $\boldsymbol{e}(t_0, \lambda_0)$, which can be very hot due to physical reasons. Applying standard methods, in most cases, this cusp cannot be directly computed exactly. Based on our results, however, this cusp can simply be computed as the intersection of the curve of regression of the caustic surface and the $[x, y]$ plane (Fig. 5), as follows:

$$\boldsymbol{e}(t_0, \lambda_0) = \boldsymbol{c}(t) \cap [x, y]. \tag{69}$$

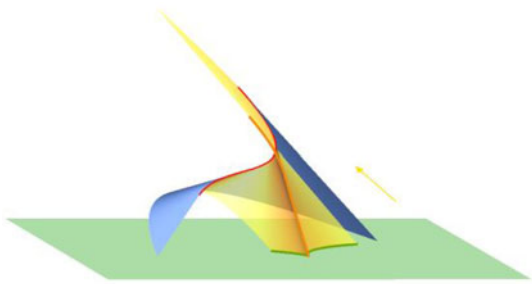

Fig. 5 The caustic surface (in orange) and its regression curve in the case of a tangent surface of a spatial curve as a mirror (in blue) defined by a cubic Bézier curve (in red)

The caustic surface intersects the $[x, y]$ plane in the curve $\boldsymbol{e}_{\mathrm{c}}(t)$ (in green). The curve of regression hits the $[x, y]$ plane in the cusp of the curve $\boldsymbol{e}_{\mathrm{c}}(t)$. The direction of the incoming light rays (in yellow) is shown. References to color refer to the online version of this figure

5 Conclusions

The caustics of developable surfaces have been studied in this paper. In the theory of developable surfaces, it is known that the caustics of each of the three different types of developable surfaces are developable surfaces of the same type. These caustic surfaces have been expressed in an exact, closed form. In the case of the tangent surface of a spatial curve, the curve of regression of the caustic surface has also been computed. This curve has been applied in a practical computation of finding the cusp of the intersection curve of a caustic surface and a plane, which may appear in applications such as free-form architecture.

Contributors

Miklós HOFFMANN, Imre JUHÁSZ, and Ede TROLL contributed equally, on a shared basis to the research, jointly developing the mathematical computations and creating the experimental figures.

Compliance with ethics guidelines

Miklós HOFFMANN, Imre JUHÁSZ, and Ede TROLL declare that they have no conflict of interest.

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