On relatively analytic and Borel subsets

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Abstract

Define $\mathfrak{z}$ to be the smallest cardinality of a function $f : X \to Y$ with $X, Y \subseteq 2^\omega$ such that there is no Borel function $g \supseteq f$. In this paper we prove that it is relatively consistent with ZFC to have $\mathfrak{b} < \mathfrak{z}$ where $\mathfrak{b}$ is, as usual, smallest cardinality of an unbounded family in ω^ω. This answers a question raised by Zapletal.

We also show that it is relatively consistent with ZFC that there exists $X \subseteq 2^\omega$ such that the Borel order of X is bounded but there exists a relatively analytic subset of X which is not relatively coanalytic. This answers a question of Mauldin.

The following is an equivalent definition of $\mathfrak{z}$:

$$\mathfrak{z} = \min\{|X| : X \subseteq 2^\omega, \exists Y \subseteq X \quad Y \text{ is not Borel in } X\}$$

For one direction we can use for each $Y \subseteq X$ its characteristic function $f : X \to 2$. For the other direction use that a function is Borel iff the inverse image of each basic open set is Borel.

The following answers a question raised by Zapletal [5] see appendix A.

Theorem 1 *It is relatively consistent with ZFC that* $\mathfrak{b} < \mathfrak{z}$.

Define $p \in \mathbb{P}(A)$ for $A \subseteq 2^\omega$ iff p is a finite set of consistent sentences of the form:

1. "$x \in \cap_{m<\omega} U_{nm}$" where $x \in A$, $n \in \omega$, or

2. "$x \notin U_{nm}$" where $x \in 2^\omega$, $n, m \in \omega$, or

3. "$[s] \subseteq U_{nm}$" where $s \in 2^{<\omega}$, $n, m \in \omega$.

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By consistent we simply mean the following:

- p cannot contain both "$x \in \cap_{m<\omega} U_{nm}$" and "$x \notin U_{nk}$" for some x, n, k, and
- p cannot contain both "$x \notin U_{nm}$" and "$[x \restriction k] \subseteq U_{nm}$" for some $x.n, m, k$.

The ordering on $\mathbb{P}(A)$ is given by inclusion: $p \leq q$ iff $p \supseteq q$. Note that the set A enters into the picture only in sentence of type (1).

This partial order is from Miller [2] where there are versions for all countable Borel orders (this is for $\mathbf{\Sigma^0_3}$). It can be looked on as a generalization of almost disjoint forcing of Jensen and Solovay. I learned about describing almost disjoint forcing as sets of sentences from Jack Silver.

Now suppose that G is $\mathbb{P}(A)$-generic over V. Define

$$U^G_{nm} = \cup\{[s] : \text{“}[s] \subseteq U_{nm}\text{”} \in G\} \text{ and } W^G_n = \cap_{m<\omega} U^G_{nm}$$

Lemma 2 *For any $x \in V \cap 2^\omega$*

1. *$x \notin U^G_{nm}$ iff "$x \notin U_{nm}$" $\in G$*
2. *$x \in W^G_n$ iff "$x \in \cap_{m<\omega} U_{nm}$" $\in G$*
3. *$x \in A$ iff $x \in \cup_{n<\omega} W^G_n$*

Proof
To prove (1) working in V, fix $x \in 2^\omega$ and $n, m < \omega$. The following set is dense:

$$D_{x,n,m} = \{p \in \mathbb{P}(A) : \exists k \;\; \text{“}[x \restriction k] \subseteq U_{nm}\text{”} \in p \text{ or } \text{“}x \notin U_{nm}\text{”} \in p\}$$

To see this note that if "$x \notin U_{nm}$" is not in p we can always find k large enough so that $p \cup \{$"$[x \restriction k] \subseteq U_{nm}$"$\}$ is a consistent set of sentences. Now suppose $x \in U^G_{nm}$, then for some k we have that "$[x \restriction k] \subseteq U_{nm}$" $\in G$ and hence by consistency, "$x \notin U_{nm}$" $\notin G$. On the otherhand, if "$x \notin U_{nm}$" $\notin G$, then since $D_{x,n,m}$ is dense for some k we have that "$[x \restriction k] \subseteq U_{nm}$" $\in G$ and hence $x \in U^G_{nm}$.

To prove (2) note that the following set is dense:

$$D_{x,n} = \{p \in \mathbb{P}(A) \; : \; \exists k \;\; \text{“}x \notin U_{nk}\text{”} \in p \text{ or } \text{“}x \in \cap_{m<\omega} U_{nm}\text{”} \in p\}$$

To see this note that if "$x \in \cap_{m<\omega} U_{nm}$" $\notin p$, then for large k (so that U_{nk} is not mentioned in p), the sentences $p \cup \{$"$x \notin U_{nk}$"$\}$ are consistent.

To prove (3) note that if $x \in A$ then the following is dense:

$$D_x = \{p \in \mathbb{P}(A) \ : \ \exists n \text{ "} x \in \cap_{m<\omega} U_{nm} \text{" } \in p\}$$

and we can only assert "$x \in \cap_{m<\omega} U_{nm}$" for $x \in A$.
QED

Note that it follows from the Lemma that $A \cap V = (\cup_{n<\omega} W_n^G) \cap V$ and so that A is a $\mathbf{\Sigma_3^0}$ relative to the ground model reals.

Lemma 3 $\mathbb{P}(A)$ *is ccc.*

Proof
This is a standard Δ-systems argument. Suppose two conditions p and q agree on all sentences of the form:

$$\text{"}[s] \subseteq U_{nm}\text{"}$$

and also they agree on all sentences of the form:

$$\text{"}x \in \cap_{m<\omega} U_{nm}\text{" or "}x \notin U_{nm}\text{"}$$

whenever x is mentioned in both p and q. Then $p \cup q$ is consistent.
QED

Next we must prove that $\mathbb{P}(A)$ does not add a dominating real.

Working in V, for $Y \subseteq 2^\omega$ countable define $p \in \mathbb{P}(A)_Y$ iff $p \in \mathbb{P}(A)$ and

$$\forall x, n, k \ \ (\text{"}x \notin U_{nk}\text{"} \in p \text{ or "}x \in \cap_{m<\omega} U_{nm}\text{"} \in p) \to x \in Y\}.$$

Or in otherwords, $\mathbb{P}(A)_Y$ are the conditions in $\mathbb{P}(A)$ which only mention elements of Y.

Lemma 4 *Suppose $p \in \mathbb{P}(A)$ and $q \in \mathbb{P}(A)_Y$. Then p and q are compatible iff r and q are compatible where*

$$r = p \setminus \{\text{"}x \in \cap_{m<\omega} U_{nm}\text{"} \ : \ x \notin Y, n < \omega\}$$

Proof
Incompatibility cannot arise between sentences of type (1) and (3). That is, any pair of the form:

$$\text{“}[s] \subseteq U_{nm}\text{”}, \quad \text{“}x \in \cap_{m<\omega} U_{nm}\text{”}$$

is consistent. It follows that the “$x \in \cap_{m<\omega} U_{nm}$” $\in p$ for which $x \notin Y$ cannot conflict with the sentences of q since by definition q cannot mention any x which is not in Y.
QED

Define. $T = (p, (t_i, n_i, m_i : i < N))$ is a Y-template iff

1. $p \in \mathbb{P}(A)_Y$, $t_i \in 2^{<\omega}$, $n_i, m_i, N \in \omega$,
2. if “$y \in \cap_{m<\omega} U_{n_i m}$” $\in p$, then $y \notin [t_i]$, and
3. if “$[s] \subseteq U_{n_i m_i}$” $\in p$, then $[s] \cap [t_i] = \emptyset$.

Define. For $\vec{x} = (x_i : i < N) \in \prod_{i<N} [t_i]$

$$p(\vec{x}) = p \cup \{\text{“}x_i \notin U_{n_i m_i}\text{”} \ : \ i < N\}$$

Note that by the definition of Y-template that $p(\vec{x}) \in \mathbb{P}(A)$, i.e., is consistent, for every $\vec{x} \in \prod_{i<N} [t_i]$.

Lemma 5 *Suppose that* $|\!\vdash \tau \in \omega$*, there exists* $\Sigma \subseteq \mathbb{P}(A)_Y$ *a maximal antichain deciding* τ*, and* $(p, (t_i, n_i, m_i : i < N))$ *is a* Y*-template. Then there exists* $k < \omega$ *so that for every* $\vec{x} \in \prod_{i<N} [t_i]$ *there exists* $q \in \mathbb{P}(A)_Y$ *such that* $p(\vec{x}) \cup q \in \mathbb{P}(A)$ *and* $q |\!\vdash \tau < k$*.*

Proof
For $q \in \mathbb{P}(A)_Y$ define

$$U_q = \{\vec{x} \in \prod_{i<N} [t_i] \ : \ p(\vec{x}) \cup q \in \mathbb{P}(A)\}$$

Note that U_q is open. To see this, suppose $\vec{x} \in U_q$ so that $p(\vec{x}) \cup q \in \mathbb{P}(A)$. Note that although some x_i might be in Y it can't be that “$x_i \notin U_{n_i m_i}$” $\in p(\vec{x})$ and “$x_i \in \cap_{m<\omega} U_{n_i m}$” $\in q$, because they are compatible. Hence, there must be a sufficiently small neighborhood of x_i say $t'_i = x_i \restriction k_i \supseteq t_i$ with the properties that

1. if "$z \in \cap_{m<\omega} U_{n_i m}$" $\in p \cup q$, then $z \notin [t'_i]$, and
2. if "$[s] \subseteq U_{n_i m_i}$" $\in p \cup q$, then $[s] \cap [t'_i] = \emptyset$.

Hence, $\vec{x} \in \prod_{i<N} [t'_i] \subseteq U_q$.

Now since $\Sigma \subseteq \mathbb{P}(A)_Y$ is a maximal antichain we know that

$$\cup \{U_q : q \in \Sigma\} = \prod_{i<N} [t_i]$$

So by compactness since each U_q is open, there exists a finite $F \subseteq \Sigma$ such that

$$\cup \{U_q : q \in F\} = \prod_{i<N} [t_i]$$

and since each $q \in \Sigma$ decides τ, the Lemma follows.
QED

In order to prove the full result we must show that the iteration does not add a dominating real. To do this we prove the following stronger property (see Bartoszynski and Judah [1] definition 6.4.4):

Lemma 6 *The poset $\mathbb{P}(A)$ is really $\sqsubseteq^{\text{bounded}}$-good, i.e., for every name τ for an element of ω^ω there exists $g \in \omega^\omega$ such that for any $x \in \omega^\omega$ if there exists $p \in \mathbb{P}(A)$ such that $p | \vdash \forall^\infty n \; x(n) < \tau(n)$, then $\forall^\infty n \; x(n) < g(n)$.*

Proof

Suppose that $| \vdash \tau \in \omega^\omega$. Let $Y \subseteq 2^\omega$ be countable so that for every $n < \omega$ there exists a maximal antichain $\Sigma \subseteq \mathbb{P}(A)_Y$ which decides $\tau(n)$. List all Y-templates as $(T_n : n < \omega)$. By Lemma 5 there exists $g \in \omega^\omega$ with the property that for every $l < \omega$ and $n < l$ if

$$T_n = (p, (t_i, n_i, m_i : i < N))$$

then for every $\vec{x} \in \prod_{i<N} [t_i]$ there exists $q \in \mathbb{P}(A)_Y$ such that $p(\vec{x}) \cup q \in \mathbb{P}(A)$ and $q | \vdash \tau(l) < g(l)$. (To get $g(l)$ apply Lemma 5 to $\tau = \tau(l)$ and each of the templates $(T_n : n < l)$ and then take $g(l)$ to be the maximum of all the $k's$.)

Now suppose that $p_0 | \vdash \forall l > l_0 \;\; x(l) < \tau(l)$ and

$$p_0 = p \cup \{z_i \in \cap_{m<\omega} U_{n'_i, m} : i < N'\} \cup \{x_i \notin U_{n_i m_i} \; : \; i < N\}$$

where $p \in \mathbb{P}(A)_Y$ and $z_i, x_i \notin Y$.

Take t_i sufficiently long so that $t_i \subseteq x_i$ and

$$T = (p, (t_i, n_i, m_i : i < N))$$

is a Y-template. Assume that l_0 is sufficiently large so that $T = T_k$ for some $k < l_0$. By our construction for each $l > l_0$, there exists $q \in \mathbb{P}(A)_Y$ such that $p(\vec{x}) \cup q \in \mathbb{P}(A)$ and $q | \vdash \tau(l) < g(l)$. But by Lemma 4 this means that $p_0 \cup q \in \mathbb{P}(A)$ and hence $x(l) < g(l)$.
QED

The above proof is similar to that of Lemma 6.5.8 [1].

Now we prove Theorem 1. Starting with a model of CH we iterate with finite support ω_2 times

$$\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \overset{\circ}{\mathbb{P}} (\overset{\circ}{A}_\alpha)$$

where we dovetail to list all $A \subseteq 2^\omega$ of size ω_1 in the final model. Since the finite support iteration of really $\sqsubseteq^{\text{bounded}}$-good ccc forcing adds no dominating real (see Bartoszynski and Judah [1] Theorem 6.5.4), we have that in the resulting model that $\mathfrak{b} = \omega_1$. On the other hand by Lemma 2 we have that $\mathfrak{z} = \omega_2$.
QED

Define (see Zapletal [5] Appendix A)

$$\mathfrak{sn} = \min\{|X| : X \subseteq \mathcal{T}, \forall A \ \ \mathbf{\Sigma^1_1} \quad X \cap A \neq X \cap WF\}$$

where $\mathcal{T}$ is the set of ω-trees and WF is the set of well-founded trees. An equivalent definition is:

$$\mathfrak{sn} = \min\{|X| \ : \ X \subseteq 2^\omega \ \exists A \ \ \mathbf{\Sigma^1_1} \ \ \forall B \ \ \mathbf{\Pi^1_1} \ \ X \cap A \neq X \cap B\}$$

The equivalence is easy to show because the set of well-founded trees is a universal $\mathbf{\Pi^1_1}$ set. It is not hard to see that $\mathfrak{z} \leq \mathfrak{sn}$. So we have the relative consistency of $\mathfrak{b} < \mathfrak{sn}$.

The following proposition is mostly due to Rothberger [4]. It implies that we must go up to at least the third level of the Borel hierarchy to get the consistency of $\mathfrak{b} < \mathfrak{sn}$.

Proposition 7 *For κ an infinite cardinal the following are equivalent:*

1. $\mathfrak{b} > \kappa$

2. *For all $X \subseteq 2^\omega$ with $|X| \leq \kappa$ and for all $\mathbf{\Sigma^1_1}$ sets $A \subseteq 2^\omega$ there exists a $\mathbf{\Sigma^0_2}$ set $B \subseteq 2^\omega$ such that $X \cap A = X \cap B$.*

3. *For all $X \subseteq 2^\omega$ with $|X| \leq \kappa$ and for all $\mathbf{\Sigma^0_2}$ sets $A \subseteq 2^\omega$ there exists a $\mathbf{\Pi^0_2}$ set $B \subseteq 2^\omega$ such that $X \cap A = X \cap B$.*

4. *For all $X \subseteq 2^\omega$ with $|X| \leq \kappa$ and for all countable $A \subseteq X$ there exists a $\mathbf{\Pi^0_2}$ set $B \subseteq 2^\omega$ such that $A = X \cap B$.*

Proof

(2) $\rightarrow$ (3) and (3) $\rightarrow$ (4) are trivial.

To see (1) $\rightarrow$ (2) let

$$A = \{x \in 2^\omega \ : \ \exists y \in \omega^\omega \ \ (x, y) \in C\}$$

where $C \subseteq 2^\omega \times \omega^\omega$ is closed. Suppose that $A \cap X = \{x_\alpha : \alpha < \kappa\}$. Choose $y_\alpha \in \omega^\omega$ so that $(x_\alpha, y_\alpha) \in C$ for each $\alpha < \kappa$. Since $\mathfrak{b} > \kappa$ we can choose $z_n \in \omega^\omega$ for $n < \omega$ so that for all $\alpha < \kappa$ there exists $n < \omega$ with $y_\alpha \leq z_n$ (pointwise). Define

$$C_n = \{(x, y) \in C : y \leq z_n\}$$

C_n is compact and therefore so is its projection:

$$A_n = \{x \in 2^\omega \ : \ \exists y \ (x, y) \in C_n\}$$

But $A \cap X = \cup_{n<\omega} A_n \cap X$.

To see (4) $\rightarrow$ (1) let $X \subseteq \omega^\omega$ with $|X| = \kappa$. Now since ω^ω is homeomorphic to $[\omega]^\omega$ and $[\omega]^\omega \subseteq P(\omega) \simeq 2^\omega$ by applying (4) we can find a $\mathbf{\Pi^0_2}$ set $G \subseteq P(\omega)$ such that

$$G \cap (X \cup [\omega]^{<\omega}) = [\omega]^{<\omega}$$

But note that $F = P(\omega) \setminus G$ is a σ-compact set which is disjoint from $[\omega]^{<\omega}$, i.e. a subset of $[\omega]^\omega \simeq \omega^\omega$ and covers X. But is easy to show that for any σ-compact subset F of ω^ω there exists $f \in \omega^\omega$ such that $g \leq^* f$ for all $g \in F$. QED

Remark. One way to get the consistency of $\mathfrak{b} < \mathfrak{z} < \mathfrak{sn}$ is as follows: Start with a ground model of $2^\omega = \omega_1$, $2^{\omega_1} = \omega_2$, and $2^{\omega_2} = \omega_{17}$. Do a finite support iteration of $\mathbb{P}(A_\alpha)$ for $\alpha < \omega_3$, so that for each α either $A_\alpha = A$ the universal $\mathbf{\Sigma^1_1}$-set or $|A_\alpha| = \omega_1$ as in the above proof. In the final model we will have $\mathfrak{b} = \omega_1$ since it is an iteration of really $\sqsubseteq^{\text{bounded}}$-good ccc partial orders. Also we will have $\mathfrak{z} \leq \omega_2$ because $2^{\omega_2} = \omega_{17}$ and $2^{\omega} = \omega_3$. We also have $\mathfrak{z} \geq \omega_2$ because of dovetailing over all $|A| = \omega_1$. And we will have $\mathfrak{sn} = \omega_3 = \mathfrak{c}$ because we have cofinally used the universal $\mathbf{\Sigma^1_1}$-set.

The following Theorem answers a question of Dan Mauldin (see [3] problem 7.8).

Theorem 8 *It is relatively consistent with ZFC that there exist a separable metric space X such that the Borel order of X is bounded, but not every relatively analytic subset of X is Borel in X.*

Proof
We use almost exactly the same partial order but with one crucial difference. Instead of using arbitrary subsets $A \subseteq 2^{\omega}$ we let $B \subseteq 2^{\omega}$ be a fixed universal $\mathbf{\Pi^0_3}$ set. The partial order $\mathbb{P}(B)$ is Borel, ccc, and adds a generic $\mathbf{\Sigma^0_3}$ set whose intersection with the ground model is the same as B's with the ground model.

Define. A partially ordered set $\mathbb{P}$ is very Souslin iff

1. $\mathbb{P}$ is ccc,
2. $\mathbb{P}$, $\leq$, $\{(p,q) \in \mathbb{P}^2 : p, q \text{ incompatible } \}$ are $\mathbf{\Sigma^1_1}$, and
3. $\{\Sigma \in \mathbb{P}^{\omega} \ : \ \Sigma \text{ enumerates a maximal antichain } \}$ is $\mathbf{\Sigma^1_1}$.

We will need the following Lemma:

Lemma 9 *(Zapletal [5] see Appendix C, Lemmas C.0.14 and C.0.17) Suppose $\mathbb{P}$ is a very Souslin real partial order and $\mathbb{P}^{\omega_2}$ the countable support iteration of $\mathbb{P}$. Then*

$$V^{\mathbb{P}^{\omega_2}} \models \mathfrak{sn} = \omega_1$$

Clearly this means that partial order $\mathbb{P}(A)$ is not very Souslin even when A is taken to be analytic (so it is Souslin). However if we change A to make it Borel, then it is very Souslin.

Lemma 10 *The partial order $\mathbb{P}(B)$ is very Souslin.*

Proof
The following sets are Borel:

1. $\mathbb{P}(B)$
2. $\{(p,q) \in \mathbb{P}(B) \times \mathbb{P}(B) \ : \ p \subseteq q\}$
3. $\{(p,q) \in \mathbb{P}(B) \times \mathbb{P}(B) \ : \ p$ and q are incompatible $\}$
4. $\{(p,Y) \ : \ Y \in [2^\omega]^\omega$ and $p \in \mathbb{P}(B)_Y\}$
5. $\{((T_n : n < \omega), Y) \ : \ Y \in [2^\omega]^\omega$ and $\{T_n : n < \omega\} =$ all Y-templates $\}$

Next we verify that being a maximal antichain in $\mathbb{P}(B)$ is $\mathbf{\Sigma^1_1}$.

Claim. $\Sigma \subseteq \mathbb{P}(B)$ is a maximal antichain iff

1. Σ is an antichain and
2. there exists $Y \subseteq 2^\omega$ countable and $(T_n : n < \omega)$ such that
   - $\Sigma \subseteq \mathbb{P}(B)_Y$ and
   - $(T_n : n < \omega)$ enumerates the set of all Y-templates

   and for all n if $T_n = (p, (t_i, n_i, m_i : i < N))$, then there exists K, $(t_i^j : j < K)$, and $(q_j : j < K)$ such that

   (a) $\prod_{i<N} [t_i] = \cup_{j<K} \prod_{i<N} [t_i^j]$
   (b) $q_j \in \Sigma$
   (c) $q_j \cup p \in \mathbb{P}(B)$
   (d) "$y \in \cap_{m<\omega} U_{n_i,m}$" $\in q_j \rightarrow y \notin [t_i^j]$
   (e) "$[s] \subseteq U_{n_i m_i}$" $\in q_j \rightarrow [t_i^j] \cap [s] = \emptyset$

Proof
Condition (2) is just a detailed restatement of Lemma 5 and its proof. It guarantees by Lemma 4 that every $p \in \mathbb{P}(B)$ is compatible with some $q \in \Sigma$.

This proves the claim and the lemma easily follows.
QED

Hence by Zapletal's Lemma 9 if we iterated $\mathbb{P}(B)$ with countable support ω_2 times then in the resulting model $\mathfrak{sn} = \omega_1$. Hence there is some $X \subseteq 2^\omega$ of size ω_1 with a relatively analytic set which is not relatively coanalytic. (Actually the proof of Lemma 9 shows that the ground model reals would do

for such an X). But note that every $\mathbf{\Pi^0_3}$ set occurs as a cross section of our universal $\mathbf{\Pi^0_3}$-set B and by Lemma 2 becomes $\mathbf{\Sigma^0_3}$ with respect to the ground model. Hence it is easy to see that for every $X \subseteq 2^\omega$ of size ω_1 for every $\mathbf{\Sigma^0_3}$ B there exists a $\mathbf{\Pi^0_3}$ C such that $X \cap B = X \cap C$. This proves Theorem 8.
QED

References

[1] Bartoszyński, Tomek; Judah, Haim; **Set theory. On the structure of the real line.** A K Peters, Ltd., Wellesley, MA, 1995. xii+546 pp.

[2] Miller, Arnold W.; On the length of Borel hierarchies. Ann. Math. Logic 16 (1979), no. 3, 233–267.

[3] Miller, Arnold W.; Some interesting problems, in Set Theory of the Reals, ed Haim Judah, Israel Mathematical Conference Proceedings, vol 6 (1993), 645-654, American Math Society, continuously updated on my home page.

[4] Rothberger, Fritz; Sur les familles indénombrables de suites de nombres naturels et les problèmes concernant la propriété C. (French) Proc. Cambridge Philos. Soc. 37, (1941). 109–126.

[5] Zapletal, J.; **Descriptive set theory and definable forcing**, to appear.

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Appendix

(Not intended for publication, electronic version only.)

Our first proof of $\mathfrak{b} < \mathfrak{sn}$ used large cardinals and the following Lemma:

Lemma 11 *(Zapletal [5] Thm 5.4.12) (LC) Suppose $\mathbb{P}$ is a real, proper, universally Baire forcing such that*

$$V^{\mathbb{P}} \models V \cap \omega^\omega \textit{is unbounded in } \leq^*$$

Then

$$V^{\mathbb{P}^{\omega_2}} \models V \cap \omega^\omega \textit{is unbounded in } \leq^*$$

where $\mathbb{P}^{\omega_2}$ stands for the ω_2 iteration with countable support of $\mathbb{P}$.

The hypothesis (LC) stands for large cardinals, for example, unboundedly many measurable Woodin cardinals would be enough. In otherwords for a nice enough forcing, not adding a dominating real is preserved by the iteration. It is easy to get a two step iteration so that neither step adds a dominating real but the two steps do. For example, force ω_1-Cohen reals followed by the Heckler partial order of the ground model.

Fix $A \subseteq 2^\omega$ a universal Σ^1_1 set, i.e., it is lightface Σ^1_1 and every boldface $\mathbf{\Sigma^1_1}$ occurs as a cross section via some effective homeomorphism of $2^\omega \times 2^\omega$ and 2^ω. In this case the partial order $\mathbb{P}(A)$ is $\mathbf{\Sigma^1_1}$, ccc, and determined by a real - so it satisfies the hypothesis of the Lemma.