Geometry of forking in simple theories

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ABSTRACT. We investigate the geometry of forking for U-rank 2 elements in supersimple ω -categorical theories and prove stable forking and some structural properties for such elements. We extend this analysis to the case of U-rank 3 elements.

Simple theories were defined and initially investigated in 1980 by Shelah ([S2]). In the 90s Kim (and Pillay), inspired by the work of Hrushovski in the 80s on finite rank cases which showed the possibility of extending the machinery of forking from the stable case to a more general context, developed the basics of simple theories and showed that forking is well behaved in simple theories. Moreover, they showed that simple theories are exactly those theories where the notion of forking (as originally defined by Shelah) is well behaved (e.g. symmetric).

Early on in the investigation into simple theories, it seemed that the essential behavior of forking was the same as in stable theories and it was conjectured that every instance of forking in a simple theory is witnessed by a stable formula. This conjecture has been formalized in various forms as the stable forking conjecture. The truth of this conjecture would imply, among other things, the possibility of a certain lifting of techniques and even results from the deeply studied stable case to the simple one. Of the various formalizations which the stable forking conjecture has taken, counterexamples are known for many. Until now the only case where stable forking was known was for 1based simple theories with elimination of hyperimaginaries.

Simple theories differ from stable theories as they allow for the independence property while still not having the strict order property (every unstable theory has at least one of these two properties). The triangle-free random graph, a theory which has the independence property and not the strict order property, can be shown to be non-simple. Thus the property of simplicity forms a dividing line inside theories without the strict order property. One could conjecture that this line is defined by the fact that relations witnessing the independence property cannot be too intertwined with relations witnessing forking. A certain way to formalize this gives a second motivation for the stable forking conjecture, which could be described as stating that if two elements fork, then that forking can be witnessed by a relation without

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the independence property. The right form this formulation should take is not yet completely clear. In the ω -categorical case this formulation could state that the relation of forking (which is now first-order definable) does not have the independence property.

Here we will prove that the U-rank 2 elements of an ω -categorical supersimple theory satisfy stable forking. In fact, we will prove that the relation of forking itself cannot have the independence property and hence is stable. We will then show how a generalization of the given proof can be obtained to prove results for higher U-ranks. Specifically, we will investigate the situation for U-rank 3 elements and show results there.

Our method will be to look at the consequences which the independence property has for the geometry of forking. We will show that having the independence property has surprising consequences for the possible geometry of forking which will force, given simplicity, the relation of forking to be stable.

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Conventions and Notation. L is a possibly many sorted language. T is a complete first order theory in L. We work inside a monster model, M, which is some very saturated model of our theory. By an element a we will mean a (possibly imaginary) element of M. We will denote first order formulas in our language, of the form $\phi(x, y)$, as relations R(x, y) (this is to help flesh out the underlying combinatorial structure).

Definition. (1) An *IP*-sequence is an infinite indiscernible sequence \mathcal{I} which witnesses the *independence property* for a given formula R (fixed here throughout); i.e., for any finite disjoint subsets A and B of \mathcal{I} ,

$$M \models \exists x \left(\bigwedge_{a \in A} R(x, a) \land \bigwedge_{b \in B} \neg R(x, b) \right)$$

Recall that a theory T is said to have the independence property if it contains a formula having the independence property.

(2) Let \mathcal{I} be an IP-sequence. We say that c is generic for \mathcal{I} with respect to R if there exist infinite (disjoint) subsets A, B of \mathcal{I} such that $\bigwedge_{a \in A} R(c, a) \land \bigwedge_{b \in B} \neg R(c, b), c$ is not algebraic in \mathcal{I} , $\forall a \in I(a \notin acl(c, I - \{a\}))$ and $\mathcal{I} = I_1 \cup I_2$ such that I_1 and I_2 are indiscernible over c.

We say that c is generic for a for some R iff c is generic for some \mathcal{I} with respect to R such that $a \in \mathcal{I}$. We say c is generic in C for R, C a finite set, when c is generic for an IP containing all the elements of C. (Usually R will be obvious and so omitted.) Note that given a formula R with the independence property, we could always get such a configuration, i.e., find an a and csuch that c is generic for a with respect to R.

- (3) A formula $\phi(x, y)$ is *stable* if there do not exist $\langle a_i; i < \omega \rangle$, $\langle b_j; j < \omega \rangle$ such that $\phi(a_i, b_j)$ if and only if $i \leq j$. It is *unstable* otherwise.
- (4) We say an instance of forking is stable if it is witnessed by a stable formula. i.e., given a, b, if $a \not\downarrow_A b$ then there exists a stable formula $\psi(x, y) \in L(A)$ such that $\psi(a, b)$ and $\psi(x, b)$ forks over A.

We say that a theory T satisfies *stable forking* if every instance of forking is stable. (For other definitions see references.)

We will note that though the definition of genericity seems complicated it is actually a trivial use of the independence property; compactness and the definition of the independence property allows us, given an R with the inependence property, to create such a situation (in the appropriate types in our case), thus allowing us to prepare immediately the generic situation which saves a lot of writing in the long run. We will constantly use the generic situation, which always exists given the independence property, to reach contradictions.

We will only use a few basic results of forking calculus in simple theories:

- 0. Forking is i) Symmetric: $a \downarrow_A b$ iff $b \downarrow_A a$, and ii) Transitive: Suppose $A \subseteq B \subseteq C$, then $a \downarrow_A C$ iff $(a \downarrow_B A \text{ and } a \downarrow_B C)$.
- 1. Let a fork with b_i for i < n and $\{b_i\}_{i < n}$ be independent. Then $U(a/\{b_i\}_{i < n}) \leq U(a) n$. (This is clear using symmetry and transitivity).
- 2. The Independence Theorem over Lascar strong types. ([W] 2.5.20)) If $B \downarrow_A C$, tp(b/AB) and tp(c/AC) do not fork over A, and Lstp(b/A) = Lstp(c/A), then there is an $a \models Lstp(b/A) \cup tp(b/AB) \cup tp(c/AC)$, with $a \downarrow_A BC$.
- 3. Type-definability of Lascar strong type. ([W] 2.7.9)
- 4. If $a \downarrow_A b$ and Lstp(a/A) = Lstp(b/A) then a and b begin a Morley sequence over A. ([W] 2.7.7).
- 5. The Lascar inequality (or equality in the finite U-rank case). ([W] 5.1.6) $U(a/bA) + U(b/A) \leq U(ab/A) \leq U(a/bA) \oplus U(b/A)$.
- 6. For every a and A, $a \downarrow_B A$ for some $B \subseteq A$, $|B| \leq |T|$.

1. U-RANK 2 STABLE FORKING

In this section we will prove stable forking for U-rank 2 elements, by analyzing the consequences which the independence property has for forking in simple theories. Simplifications of the proof can be achieved, but the following proof provides a better background for generalization; also, Theorem 1 is an interesting theorem on its own right, as it gives information about theories having the independence property but not the strict order property.

Theorem 1. Let T be a simple theory. Suppose e is generic for a with respect to R, R(x, a) forks over A, U(a/A) = U(e/A) = 2, and Lstp(b/Ae) = Lstp(a/Ae). Then $a \not\downarrow_A b$.

Proof. We will omit A in the proof, but all our calculations and analysis are over A. As e is generic for a with respect to R, we have by definition of genericity (i.e. the method of getting generic sequences) an IP sequence I with respect to R such that e is generic for I, and generic for each $c, c \in I$.

Claim 1.1. $\forall c \in I(a \not\downarrow c)$

Proof. $a, c \in I$, so there are unboundedly many d's which are generic for them satisfying $R(d, a) \wedge R(d, c)$ and U(d)=2 (by the indiscernibility properties in the definition of genericity). But then $d \not\downarrow a$ and $d \not\downarrow c$, so if $a \downarrow c$ then by Fact 1 above we get U(d/ac) = 0, hence d is algebraic over ac in contradiction to the way we chose d. (We use here our definition of genericity). So $a \not\downarrow c$, as desired. \Box

Now suppose for a contradiction that $a \downarrow b$. We look at the two possible cases:

1) Suppose $e \downarrow a$. As e is generic for I, and as R witness forking, there are unboundadly many c's in I such that $e \not\downarrow c$; but as $a \not\downarrow c$ we get again by fact 1 that c is algebraic in ea which is a contradiction.

2) Suppose $e \not \downarrow a$. Now by the Lascar inequalities (fact 5), U(eab) = U(ab) + U(e/ab). As we are assuming $a \downarrow b$, we have that U(ab) = 4, so U(b/ae) = U(aeb) - U(ae) = 4 + U(e/ab) - U(ae). As we assumed $a \not \downarrow e$, $U(ae) = U(a) + U(e/a) \le 2 + 1 = 3$, so $U(b/ae) \ge 1 = U(b/e)$ hence $b \downarrow_e a$.

As Lstp(b/e) = Lstp(a/e), and as we obtained $b \downarrow_e a$, by fact 4 above we can continue ab to a Morley sequence over $e, \{a_i\}_{i \le w}$.

If $c \downarrow_e a$ then by definition of forking there exists some g such that $tp(gea_i) = tp(cea)$, for all i. By fact 6 c cannot fork with every pair of the Morley sequence, so by indiscernibility, and perhaps changing g we get an f such that $f \downarrow_e ab$ and $f \equiv_{ea_i} g$ for each i. But now $f \not\downarrow a$

and so $f \not\downarrow b$. If $a \downarrow b$ then U(f/ab) = 0, but U(f/e) = 1 as being nonalgebraic is type definable and hence is in tp(c/e). So $f \not\downarrow_e ab$ which is a contradiction to our assumption, and so we get a contradiction and $c \not\downarrow_e a$.

Now c was just one of infinitely many elements of I, and in particular, one of the infinitely many elements in I which fork with e. But $c \not\downarrow e$ and $c \not\downarrow_e a$ means $U(c/ae) \leq U(c) - 2 = 2 - 2 = 0$ which entails that c is algebraic over ae for infinitely many c's which is a contradiction. And so $a \not\downarrow b$ and we are done.

Theorem 2. Let T be a supersimple, ω -categorical theory. Then the Urank 2 elements satisfy stable forking. Moreover, the formula $\phi(x, y) = x \bigvee y \wedge U(x) = 2 \wedge U(y) = 2$ is stable.

Proof. Let a fork with d over A. There is a finite $B \subset A$ such that $a \not\downarrow_B d$; also as T is supersimple, we can get U(a/A) = U(a/B) and U(d/A) = U(d/B). So we can assume without loss of generality that A is finite. This is the only place we use supersimplicity rather than simplicity.

A will be omitted in the proof. Let R be a relation such that aRd, R witnesses forking, and xRy proves the types of x and y.

We note that as we will be dealing with IP sequences and generic elements, elements could be chosen to not be algebraic over other elements. We will obtain a contradiction to a POSSIBLE configuration, and hence get a contradiction to any configuration. We will constantly choose elements so as to be non-algebraic. (This will be much more noticeable in higher U-ranks).

Observation 1. We may assume R has the independence property (IP).

Proof. Every unstable formula either has the independence property or some conjunction of instances of the formula has the strict order property (the construction is given explicitly in [S1] Ch. II, §4). Now as T is simple, no formula of T has the strict order property, and so if R is unstable, it must be because R has the IP. Otherwise we are done as then R is stable and witnesses forking.

We assume R is such that aRb proves a is generic for b.

Definition. We define S to be the following relation: xSb iff tp(x) = tp(a), tp(b) = tp(d), and b forks with some c such that (xRc and Lstp(c) = Lstp(b)), and for all d such that (xRd and Lstp(c/x) = Lstp(d/x)) then b forks with d.

By ω -categoricity (and fact 3), this is definable.

Claim 2.1. If xRb then xSb.

This follows from theorem 1. We get that b forks with each member of the Lascar strong type over x to which b itself belongs.

Claim 2.2. S witnesses forking.

Proof. Let $\{b_i\}_{i < w}$ be Morley, and suppose there exists an x such that $xS(b_i)$ for all i < w. Then, by definition of S, there exists some c such that xRc, $Lstp(b_i) = Lstp(c)$ and b_i forks with c. Now notice that by definition we can take the same c for all b_i . (As the b_i s are Morley, they are all of the same Lstp). (There are only a bounded number of Lascar strong types over x, and so some Lascar strong type repeats for infinitely many c's and so b_i s, and thus we can get an infinite subset of our original Morley sequence, which is then still a Morley sequence for which it is the same c). But this is a contradiction to simplicity (fact 1).

We note that as we got that forking satisfies the definition of S, and that S witnesses forking, S is essentially forking.

Claim 2.3. S does not have the IP.

Proof. Let $\{b_j\}$ be a set witnessing the IP for S. We can assume the sequence $\{b_j\}$ is indiscernible, and as tp(b) is proved by S, that $b_0 = b$. Notice that as the b_j s are indiscernible, all of them are of the same Lascar strong type. There exists an $x, xSb_0 \wedge \neg xSb_1$. As $x \downarrow b$, there is some c, $Lstp(c/x) = Lstp(b_0/x)$ and xRc such that b_1 does not fork with c.

Remark. Given an IP set $\{b_i\}_{i < w}$, all of them indiscernible (we can achieve this by compactness, and as S proves type of b, can assume $b_0 = b$), then $\forall i, j(b_i \not \downarrow b_j)$.

Proof. Otherwise we have infinitely many x's such that xSb_0 and xSb_1 and $b_0 \downarrow b_1$. So by fact $4 U(x/b_0b_1) = 0$, so x has to be algebraic over b_0 and b_1 . Contradiction.

Now, by the IP and the assumption regarding ω -categoricity it is easy to see that there must be an x and a y such that $Lstp(x/b_0) = Lstp(y/b_0)$ and xSb_0 and not xSb_1 and ySb_0 and ySb_1 . (e.g. look at the ω many subsets of ω which are $\omega - j$ for each j. then there is x_j which witnesses x_jSb_k iff not k = j. Now at least 2 of them have the same Lstp over b_0 , and then can assume j = 1 without loss of generality). For this x there is a c as before. x could also be chosen to have infinitely

 $\mathbf{6}$

many b's from the IP of the same Lstp over x as b_0 and hence will also fork with c. Now, by our previous remark, $b_0 \not\downarrow b_1$; by our choice of c, $b_0 \not\downarrow c$, and also $c \downarrow b_1$, hence by fact 1 we get $U(b_0/cb_1) \leq 2-2 = 0$ which is a contradiction.

And so we are done proving our Theorem. Notice that if we take $R(x,y) := x \not\downarrow y \land U(x) = U(y) = 2$ (which is definable in an ω -categorical theory), then if the relation R has the independence property, an extension of R which also states the types of x and y for some x and y has the IP, while still witnessing forking and the ranks; but as we proved that cannot happen, the relation of forking, plus the ranks, cannot have the independence property.

The fact that we got a precise formula (moreover that it is forking) is very useful when one comes to extend this result to higher U-ranks (e.g., for a proof by induction), and allows the result itself to be used and not just the methodology of the proof.

Remark. (1) I will mention that we could prove claim 2.3 using the independence theorem over Lascar strong types. Such a proof, though longer in this case, allows for certain generalizations to higher U-ranks by changing our elements to tuples.

(2) Using regular type machinery one could take our U-rank 2 elements to be pairs of U-rank 1 elements.

2. The U-rank 3 case of stable forking

In this section we will prove that forking between elements of U-rank 3 is either stable, or is witnessed by an IP sequence with a particular U-rank configuration. Our elements in this section are of U-rank 3. In this case we have several possible independence property sequences. Let x be generic to an IP-sequence I with respect to R which witnesses forking, and a, b, c distinct elements in I. We will divide the possible IP's into 4 cases:

- (1) U(b/a)=1. We name it 3-1.
- (2) U(b/a)=2. We name it 3-2.
- (3) U(b/a)=3 and U(d/ab)=2. We name it 3-3-2.
- (4) U(b/a)=3 and U(d/ab)=1. We name it 3-3-1.

In our proof R will be taken to be the forking relation itself (and stating R(x, y) implies U(x) = U(y) = 3). We will show that in this case, the only possible IP-sequences is of the form 3-3-1. We note that the 4 cases defined above are the only possible IP sequences, as

if a sequence were of the form 3-3-3 then there would be only a finite number of elements of U-rank 3, forking with any 3 elements from the IP, in contradiction of it being an IP. Likewise we cannot have U-rank 0 in IP sequences due to indiscernibility.

If we are interested in having stable forking for elements of U-rank ≤ 3 then it is sufficient to prove it for U-rank 3 elements, as if U(a) < 3 we look at the following structure. We partition the universe into 2 disjoint unary predicates P and Q. P we take as our original structure, while Q is an infinite set with no relations. There are no relations between the universe of the 2 predicates. We now add to a the needed number of elements from Q to make it U-rank 3 (it is actually clear that in this case U(a) = 2 and we will add an element, q, from Q to a. As the only relation with regard Q is equality, we get U(aq) = 3. We do this with different q's for each member of the IP-sequence. Now if we prove stable forking in this case, it would translate immediately to stable forking in P. We will later show that in the case of both elements having U-rank ≤ 3 but one of the elements having U-rank 2, we have stable forking.

Theorem 3. T supersimple, ω -categorical. Let U(x) = U(a) = 3 and x forks with a. Then either the forking is stable or, with respect to the relation $R(z, y) := z \not\downarrow_A y \land U(z) = U(y) = 3$, x is generic for a and the only IP-sequences which witness this genericity are 2-independent. Furthermore, the sequences are of the form 3-3-1.

Proof. Similarly to the Rank 2 case, without loss of generality we can drop the base set A.

We first look at the case where the IP sequences are not 2-independent (i.e. of form 3-1 and 3-2).

Claim 3.1. Suppose that there is an IP-sequence \mathcal{I} whose elements are not 2-independent. Then whenever $x \not \downarrow a, x \not \downarrow b$, all generic, and Lstp(a/x) = Lstp(b/x) then $a \not \downarrow b$.

Proof. We assume towards contradiction that $a \downarrow b$. There are two subcases.

Subcase 1. Let \mathcal{I} be of the form 3-1.

Let x be generic for \mathcal{I} . Recall that \mathcal{I} is IP with regard to the relation of forking. Let x fork with a and not fork with d. So U(xda) = U(xd) + U(a/xd) = 7 (as a is not algebraic over xd) = U(ad) + U(x/ad) = 4 + U(x/ad) < 7 (as x forks with a), contradiction. So no such \mathcal{I} exists.

Subcase 2. Let \mathcal{I} be of the form 3-2.

We have xRa, xRb where $a \downarrow b, Lstp(a/x) = Lstp(b/x)$, as well as an IP \mathcal{I} of form 3-2 for which x is generic. In \mathcal{I} , let $xRa \land xRe \land xRg \land xRh \land$

 $\neg xRd \land \neg xRf$. Now U(a/x) = U(x/a) = 2 as otherwise by genericity (and Lstp(b/x)=Lstp(a/x)) $U(xab) = U(a) + U(x/a) + U(b/xa) \leq 3 + 1 + 1 = 5$ while U(ab) = 6 contradicting the Lascar inequality.

Suppose towards contradiction that U(x/ae) = 2.

By U-rank calculations this implies U(e/ax) = 2. So e does not fork with a over x. But then we can extend the type of e over x to both a and b, and by the Independence theorem over Lascar strong types (U-rank calculations prove $a \downarrow_x b$) we get an l, $tp(l/xa) = tp(e/xa) \land tp(l/xb) =$ tp(e/xb), which does not fork with ab over x, but which forks with aand with b (which are independent). But now l forks with a and with b, and a is free from b, so $U(l/ab) \leq 1 \land U(l/x) = 2$ contradiction.

So U(x/ae) = 1 and U(e/ax) = 1. Notice that we are not using here the fact that a, e are part of an IP-sequence for which x is generic, but only that Lstp(a/x) = Lstp(b/x). (The case where a, e are not necessarily part of such an \mathcal{I} allows also for U(x/ae) = 0.) So $e \not\downarrow_a x$, U(e/a) = 2 and U(e/ax) = 1.

Now, U(xaeg) = 6 + U(g/xae) = 7 (for g's not algebraic over xae, which means for almost all in the IP).

U(xaeg) = 5 + U(g/ae) + U(x/aeg) = 6 + U(g/ae) so U(g/ae) = 1. (for x not algebraic over *aeg*, hence for all generic x for the IP, but in particular could always find such an x, and as the conclusion does not mention x, the conclusion follows). This is not related to whether x is connected or not to a or e or g. So for every a, e, g in \mathcal{I} , U(e/a) = 2 and U(g/ae) = 1.

Now U(g/xdf) = 1, so x does not fork with g (or h) over df, also g does not fork with h over df (otherwise h would be algebraic over gdf), so by the independence theorem over Lascar strong types, we get an $z, z \downarrow_{df} gh$, so U(z/df) = 1 as U(z/gh) = 1 (recall this is the case whether z is connected to an IP-sequence containing them or not, as long as z forks with both of them). Also notice that x or df could be chosen so that x is not algebraic over df. (We constantly talk about the generic situation).

(Notice this is the same computation for e instead of f).

So U(zdf) = 5 + U(z/df) = 6 but U(zdf) = 6 + U(f/zd) and so we obtain a contradiction, as f is not algebraic over zd.

Claim 3.2. The relation R cannot have the IP with respect to an IP sequence which is not 2 free.

Proof. We prove this similarly to the U-rank 2 case (or rather, the remark after the proof where an alternative, a bit longer, proof is mentioned).

Lemma. If x does not fork with b, then there exists a c in each Lascar strong type over x such that b does not fork with xc (even infinitely many such c's).

Proof. Suppose not. Then b is free from x but forks with xc for all c of some Lascar strong type over x. So b forks with c over x for all such c's. Take a Morley sequence in b over x. As x is free from b, then that sequence is actually Morley over the empty set (or whatever base set we're working over). But now as there are only boundedly many Lascar strong types and the series can be taken to be arbitrarily long, we can assume there is some Lascar strong type over x, b_i forks with xc, for each b_i in the Morley sequence. But then $xc \bigvee_{b_{ii} < \alpha} b_{\alpha}$ contradicting simplicity. \Box

Let \mathcal{I} be an IP, x generic for \mathcal{I} and a, b be in I. By our previous claim we get U(b/a) = 2 for each a, b in \mathcal{I} . Let $xRa \wedge x(\neg R)b$. Then as x does not fork with b, by our previous lemma there exists some cLstp(c/x) = Lstp(a/x) such that b does not fork with xc. By our first claim $a \not\downarrow c$.

We will now use the Independence Theorem over Lascar strong types to get a contradiction. We first prove the requirements:

0) As before, we can assume U(x/a) = 2 as otherwise $U(xab) = U(a) + U(x/a) + U(b/xa) \le 6$ while U(xb) = 6 so U(a/xb) = 0 but this is true for infinitely many *a*'s so we get a contradiction, as we can assume *x* is connected to infinitely many *a*'s (or at least could always choose such an *x*).

1) We can get a y such that y does not fork with b over a, and Lstp(y/a) = Lstp(x/a).

We take a generic y for \mathcal{I} which forks with a and b. Suppose y forks with b over a. As $\forall d \in I(U(d/a) = 2)$ and as y is of U-rank 2 over a we get by our result on U-rank 2 elements that the forking formula has to be stable. In particular it cannot divide the (indiscernible over a) IP into 2 infinite parts ([S1] Ch. II Theorem 2.20 proves that if $\phi(z, t)$ is stable and J, some indiscernible sequence, then the set of j's in J such that $\phi(a, j)$ is either finite, or cofinite (in J) for all a's). So there are infinitely many d's which y does not fork with, but which it forks with over a (or we can find a different b that x forks with and y does not fork with over a). But now U(xda) = U(xd) + U(a/xd) = 6 + 1 = 7(the 1 is as a forks with both x and d which are independent), while U(xda) = U(da) + U(x/da) = 5 + 1 (the 1 is by our assumption of forking over a) Contradiction. The fact that we can get Lstp(y/a) =Lstp(x/a) can now be easily seen by counting (same argument as in the U-rank 2 case). 2) x does not fork with c over a: Suppose it does fork. Then $U(xabc) = U(bxc) + U(a/bxc) \ge 8 + 1 = 9$ as a cannot be algebraic (or at least chosen as not to be algebraic from having the IP). But also $U(xabc) = U(ac) + U(x/ac) + U(b/xac) \le 5 + 1 + 2 = 8$ ($U(ac) \le 5$ by our previous claim that x cannot fork with 2-independent elements of the same Lstp over it). contradiction.

3) b does not fork with c over a: If it does, then U(abc) < 7 but U(bc) = 6 so U(a/bc) = 0 hence a is algebraic over bc which is a contradiction, as again, can choose a configuration where that does not happen.

So we can now use the independence theorem over Lstp and get a z such that z forks with b and with c and such that z does not fork with bc over a. But as U(z/a) = 2 and z forks both with b and with c and b and c are independent, then U(z/bc) < 2 hence z forks with bc over a and we get a contradiction.

And so if there is an IP sequence which is not 2-independent, then the forking relation is stable.

We now look at the case where the IP sequence is 2-independent. In the 3-3-2 case we can use our U-rank 2 result and not just its methods.

Suppose there exists an IP sequence which is 2-independent and such that for a, b, d in \mathcal{I} U(d/ab) = 2 (i.e. of form 3-3-2). Then we look at the new sequence \mathcal{I} over a. Over a any 2 elements in \mathcal{I} fork with each other. Now this is still an IP sequence as if x forks with b then it forks with b over a as a and b are independent. If x is free from b we show it is free from b over a. Let x fork with a and d but not with b. Then U(xabd) = U(abd) + U(x/abd) = 8 + 1 = 9 so 9 = U(xabd) = U(xb) + U(a/xb) + U(d/xab). Now as U(xb) = we get U(a/xb) = 2 so U(xab) = U(a/xb) = 8 hence U(b/xa) = 8 - U(xa) = 3 and we get U(b/xa) = U(b/a) so x does not fork with b over a as we wanted to show.

So we got that if there exists such an IP sequence with respect to forking, then there exists one which is not 2-independent, and as we know no such sequence exists, then also the independent sequence cannot exist.

Hence we obtain that the only possible IP-sequence, with respect to forking, is of the form 3-3-1 as desired. \Box

For a discussion of the 3-3-1 case, see [P].

We finish this section with the following remark (T is as usual):

Remark. Suppose x forks with a and $U(x) = 2 \wedge U(a) = 3$, then this is an instance of stable forking.

Let $R(z,t) := z \not \downarrow t \land U(z) \land tp(t)$. As explained before theorem 6, we add an element q to x such that U(xq) = 3. Then x still forks with aq. If R is not stable, we have an xq and an \mathcal{I} for which xq is generic for, such that the elements of \mathcal{I} are of the type of a. As we know the only possible IP is of form 3-3-1, and that xq is generic for some a, band a, c and $\{a, b, c\}$ is independent and xq forks with all 3 elements. But now as the forking cannot occur with respect to the q's as the only way to fork with them is with equality, x forks with a, b and c which are independent, while x is U-rank 2, which is a contradiction. And so we are done.

3. Generalizing Theorem 1

In this section we will prove a generalization of Theorem 1. This theorem will show again a consequence of a formula both having the independence property and witnessing forking in a simple theory. We will prove the theorem for U-rank 3 elements as well as remark on a possibility for generalization to arbitrary finite U-rank. One of the reasons for an interest in this theorem is that it, again, gives us information on the area of non-simple theories without the strict order property. Beside that, this theorem gives us structure information inside simple theories, as even given stable forking, still forking can exist with respect to a formula with the independence property (stable forking only states that a stable formula witnessing the forking exists).

Definition. We say an IP sequence I is sound if for $a \in I : U(a) = n$ for some $n < \omega$ and every size n-1 subset, J, of I is independent, and U(a/J) = n - 1 for $a \notin J$.

For U-rank 3, an IP sequence is sound if for $a, b, d \in \mathcal{I}$, $U(a) = U(b/a) = 3 \wedge U(c/ab) = 2$.

We will prove that, as in the U-rank 2 case, if a formula xRa both witnesses forking and has the IP with respect to a sound IP sequence, then x cannot fork with 3 independent elements of U-rank 3 of the same Lascar strong type over it. This can be generalized to the general finite U-rank situation by adding a base set, C, and then demanding that a, that b and that c continue the base set to a generic IP for x. This is essentially the same proof, only with trivial alteration of the U-rank calculations.

Theorem 4. Let T be a simple theory. Let $x \not\downarrow a$, where U(x) = U(a) = 3; let the formula R witness the forking and have the independence

property where x generic in a with respect to a sound IP sequence. Then, if $xRa \wedge xRb \wedge xRc$, a, b and c of the same Lascar strong type over x, then a \downarrow bc.

Proof. We divide the proof to 3 subclaims, from which the theorem follows.

Claim 4.1. Suppose a and b are elements of a sound IP-sequence \mathcal{I} for which x is generic. suppose a and c be elements of another such sequence, that Lstp(b/ax) = Lstp(c/ax) and that a, b, c are pairwise independent. Then $\{a, b, c\}$ is not independent.

Proof. Suppose for a contradiction that $\{a, b, c\}$ are independent in the sense of forking. Let d continue a, b to a sound IP-sequence, I. We remember that for any $d \in \mathcal{I}$, U(d/ab) = 2 and U(b/a) = 3.

Suppose that $d \downarrow_a xb$. Now as Lstp(b/ax) = Lstp(c/ax), we can find a Morley sequence $\{b_j\}$ over ax whose first two elements are b, c. By indiscernibility, for each member of this Morley sequence we get a corresponding d; in particular we can find such a d_i, d_j for some b_i, b_j so that $Lstp(d_i/ax) = Lstp(d_j/ax)$. By boundedness of the number of distinct Lascar strong types and type-definability of the independence property, we may assume i = 0, j = 1, i.e., these are the corresponding d's for b and c respectively.

By assumption, $d_0 \downarrow_a xb$ and so also $d_1 \downarrow_a xc$, as $Lstp(cd_0/ax) = Lstp(cd_1/ax)$. By U-rank calculations, $b \downarrow_a xc$ as: U(xabc) = U(abc) + U(x/abc) (by the Lascar inequality for the finite U-rank case) $= 3^2 + 0$ (x has to be algebraic over abc as they are independent and x forks with each of them) = U(x) + U(a/x) + U(c/xa) + U(b/xac); so as $U(x) = 3, U(a/x) \leq 2$ and the last 2 terms are each ≤ 2 , hence U(b/xac) has to equal 2 so U(b/xac) = 2 = U(b/xa) hence $b \downarrow_{xa} c$.

By the independence theorem over Lascar strong types, we can find an *e* such that $e \downarrow_{ax} bc$ and $Lstp(e/xa) = Lstp(d_0/xa)$ and $(e/xab) = (d_0/xab)$ and $(e/xac) = (d_1/xac)$. As $bc \downarrow_x a$ we have $e \downarrow_x abc$. But now in particular $e \not\downarrow ab$ and $e \not\downarrow ac$, and as $b \downarrow_a c$ we get $e \not\downarrow_a bc$ (for otherwise $e \downarrow_a bc \land c \downarrow ab \rightarrow e \downarrow_a b \land a \downarrow b \rightarrow e \downarrow ab$, a contradiction). So $U(e/abc) \leq 1$, while U(e/x) = 2, which gives $e \not\downarrow_x abc$, a contradiction.

So it must be the case that $d \not\downarrow_a xb$. Now, U(xabd) = U(x) + U(a/x) + U(b/ax) + U(d/xab) = 3 + 2 + 2 + 1 = 8 = U(abd) + U(x/abd) = U(ab) + U(d/ab) + U(x/abd) = 6 + 2 + U(x/abd), so U(x/abd) = 0, meaning that x is algebraic over abd. But a, b, d are part of an IP-sequence generic for x (a here refers to the set of a_i 's which form part of this sequence), which means that x is not algebraic with the IP sequence, so we get a contradiction.

Claim 4.2. Assume x is generic for a with respect to R with a sound IP sequence, xRa and that $Lstp(c/ax) = Lstp(b/ax) \wedge Lstp(b/x) = Lstp(a/x)$; furthermore that $\{a, b, c\}$ are pairwise independent. Then $a \not \downarrow bc$.

Proof. As Lstp(a/x) = Lstp(b/x), we can continue a, b to a Morley sequence \mathcal{J} over x. x is generic in a with respect to a sound IP sequence \mathcal{I} . Let $e \in \mathcal{I}$, then $e \downarrow a$. By rank calculations again, $e \downarrow_x a$ so we can find such an e simultaneously for all the elements of any Morley sequence over x, in particular for \mathcal{J} . Now as there are unboundedly many elements in the sequence, we can find 2 such that $Lstp(a_i/ex) = Lstp(a_j/ex)$, and since the a_k 's are Morley over x, ecannot fork with all such pairs over x; so there exist a pair a_i, a_j so that $e \downarrow_x a_i a_j$. But then by indiscernibility and type-definability of our assumptions, we can take a_i to be a and a_j to be b. But now e, a, b satisfy the hypotheses of Claim 4.1, playing the parts of a, b, crespectively. So by Claim 4.1, $e \not\downarrow ab$, and by our choice of $e, e \downarrow_x ab$.

As in Claim 4.1, suppose $e \downarrow_{ax} b$. Then the Independence Theorem over Lascar strong types gives a d such that $d \downarrow_{ax} bc$, Lstp(d/abx) = Lstp(e/abx) = Lstp(d/acx), in the obvious sense (here e, d play the parts of d, e in Claim 4.1, respectively). Now as d extends the type of $e, d \not\downarrow ab, d \not\downarrow ac, d \downarrow_{ax} bc$. $\{a, b, c\}$ is independent, $U(d/abc) \leq 1$ and as $d \downarrow_{ax} bc$ then $U(d/ax) \leq 1$ but U(e/xab) = U(e/x) = 2 so $e \not\downarrow_{ax} b$. So we have $e \not\downarrow_{ax} b$, hence $e \not\downarrow_x ab$. But this contradicts our choice of e. And so we are done.

Claim 4.3. Let $xRa \wedge xRb \wedge xRc$, Lstp(a/x) = Lstp(b/x) = Lstp(c/x), a, b, c 2-independent. Then $a \not\downarrow_c b$.

Proof. Suppose $\{a, b, c\}$ are independent. Then $b \downarrow_x c$ and Lstp(b/x) = Lstp(c/x), so we can continue b, c to a Morley sequence. Let d, e be in this Morley sequence such that Lstp(d/xa) = Lstp(e/xa), and also such that $a \downarrow_x de$. The first requirement is achieved by basic counting, and the second since $a \bigvee_x de \land a \bigvee_x fg \land a \bigvee_x hi$, so then as d, e, f, g, h, i are Morley over x and $a \bigvee_x x$, a would fork too much for its U-rank 3. Now Lstp(d/xb) = Lstp(e/xb), as d, e are part of a Morley sequence over x which starts with b, as well as Lstp(d/xc) = Lstp(e/xc). Also $d \downarrow c \land d \downarrow b \land d \downarrow e$ and $e \downarrow b \land e \downarrow c \land e \downarrow d$ as it is an indiscernible sequence. So by claim 2, $de \bigvee_a b$, $de \bigvee_c$, and $de \bigvee_a a$. As a, b, c are independent, $de \bigvee_a b$ and $de \bigvee_{ab} c$, so $U(de/abc) \leq 3$, but U(de/x) = 4 so $de \bigvee_x abc$. Notice that all we used to get that $de \bigvee_x abc$ was $tp(de/xa) \land tp(de/xbc)$.

We now show we can get such a d, e such that $de \downarrow_x abc$ and get a contradiction. $de \downarrow_x a$, since chosen as such. As bcde is Morley over $x, bc \downarrow_x de$. Also by rank calculations $bc \downarrow_x a$, and obviously Lstp(de/x) = Lstp(de/x), so by the independence theorem over Lascar strong types we get d'e' such that $d'e' \downarrow_x bc$ and tp(d'e'/xbc) =tp(de/xbc) and tp(d'e'/xa) = tp(de/xa). So d'e' is exactly as de with regard to xa and xbc, which is what we used in the beginning, and we got the contradiction.

With this, our Theorem is done. \Box

This theorem gives us structure information regarding IP sequences for a formula which witnesses forking in simple theories. i.e., consequences of the cohabitation of independence property and forking in simple theories.

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