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On the consistency strength of the Inner Model Hypothesis

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The *Inner Model Hypothesis (IMH)* and the *Strong Inner Model Hypothesis (SIMH)* were introduced in [5]. In this article we establish some upper and lower bounds for their consistency strength.

We repeat the statement of the IMH, as presented in [5]. A sentence in the language of set theory is *internally consistent* iff it holds in some (not necessarily proper) inner model. The meaning of internal consistency depends on what inner models exist: If we enlarge the universe, it is possible that more statements become internally consistent. The *Inner Model Hypothesis* asserts that the universe has been maximised with respect to internal consistency:

The Inner Model Hypothesis (IMH): If a statement φ without parameters holds in an inner model of some outer model of V (i.e., in some model compatible with V), then it already holds in some inner model of V.

Equivalently: If φ is internally consistent in some outer model of V then it is already internally consistent in V. This is formalised as follows. Regard V as a countable model of Gödel-Bernays class theory, endowed with countably many sets and classes. Suppose that V^* is another such model, with the same ordinals as V. Then V^* is an *outer model of* V (V is an *inner model of* V^*) iff the sets of V^* include the sets of V and the classes of V^* include the classes of V. V^* is *compatible with* V iff V and V^* have a common outer model.

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Remark. The Inner Model Hypothesis, like Lévy-Shoenfield absoluteness, is a form of absoluteness between V and arbitrary outer models of V, which need not be generic extensions of V. Formally speaking, the notion of "arbitrary outer model" does depend on the background universe in which V is situated as a countable model. However, a typical model of the IMH is minimal in the sense that for some real R, it is the smallest transitive model of Gödel-Bernays containing R (see Theorem 8 below). For minimal models, the choice of background universe is irrelevant, and if there is a model of the IMH then there is a minimal one. Thus we may in fact regard the IMH as an intrinsic hypothesis about V, independent of any background universe. An alternative way to "internalise" the IMH is to restrict the notion of outer model to class-generic extensions which preserve the axioms of Gödel-Bernays. This weakened form of the IMH is expressible in the language of class theory, and the results of this paper would not be affected by this change. However, this is not in the spirit of [5], where the IMH is introduced as a form of absoluteness which is no way limited by the notion of forcing.

Theorem 1 *([5]) The Inner Model Hypothesis implies that for some real R, ZFC fails in $L_\alpha[R]$ for all ordinals α. In particular, there are no inaccessible cardinals, the reals are not closed under # and the singular cardinal hypothesis holds.*

Theorem 2 *The IMH implies that there is an inner model with measurable cardinals of arbitrarily large Mitchell order.*

Proof. Assume not and let K denote Mitchell's core model for sequences of measures (see [6]). Let δ be the maximum of ω_1 and the strict supremum of the Mitchell orders of measurable cardinals in K. By Mitchell's Covering Theorem for K we have:

$(*)$ *cof* $(\alpha) \geq \delta$, α *regular in* $K \to$ *cof* $(\alpha) =$ *card* (α).

Now iterate K by applying each measure of order 0 exactly once, i.e., if K_i is the i-th iterate of K, K_{i+1} is formed by applying the measure of order 0 in K_i at κ_i, where κ_i is the least measurable of K_i not of the form $\pi_{i_0 i}(\kappa_{i_0})$ for any $i_0 < i$. It is easy to see that $i < j$ implies $\kappa_i < \kappa_j$ and hence this iteration is normal. Let $\sigma_{ij} : K_i \to K_j$ be the resulting iteration map from K_i to K_j for $i \leq j \leq \infty$, let K' denote K_∞ and let $\sigma : K \to K'$ denote $\sigma_{0\infty}$.

Lemma 3 *(a) For any α and any i, there is at most one $j \geq i$ such that $\sigma_{j,j+1}$ is discontinuous at $\sigma_{ij}(\alpha)$. If there is no such $j \geq i$, then $\sigma_{i\infty}$ is continuous at α.*
(b) For any α and any i, the K_i-cardinality of $\sigma_{i\infty}(\alpha)$ is at most α^+ of K_i. If $\sigma_{i\infty}$ is continuous at α and α is a K_i-cardinal, then the K_i-cardinality of $\sigma_{i\infty}(\alpha)$ equals α.
(c) If α is a limit cardinal of K_i then α is a closure point of $\sigma_{i\infty}$, i.e., $\sigma_{i\infty}[\alpha] \subseteq \alpha$. If in addition the K_i-cofinality of α is not measurable in K_i, then α is a fixed point of $\sigma_{i\infty}$.

Proof. (a) The map $\sigma_{j,j+1}$ is discontinuous at β iff β has K_j-cofinality κ_j. Thus if the K_j-cofinality of $\sigma_{ij}(\alpha)$ is not κ_j for any $j \geq i$, it follows that $\sigma_{j,j+1}$ is continuous at $\sigma_{ij}(\alpha)$ for all $j \geq i$. Otherwise let j be least so that $\sigma_{ij}(\alpha)$ has K_j-cofinality κ_j; then for k greater than j, $\sigma_{ik}(\alpha)$ has K_k-cofinality $\sigma_{jk}(\kappa_j)$, which by definition of the κ_k's does not equal κ_k. It follows that $\sigma_{k,k+1}$ is continuous at $\sigma_{ik}(\alpha)$ for k greater than j, as desired. The last statement is immediate.
(b) Define the ordering $\prec$ as follows: $(j,\alpha) \prec (k,\beta)$ iff $j \geq k$ and $\alpha < \sigma_{kj}(\beta)$. The relation $\prec$ is a well-founded partial ordering. We prove the desired property of (i,α) by induction on $\prec$.

We may assume that α is a cardinal of K_i. If $\sigma_{ij}(\alpha)$ does not have K_j-cofinality κ_j for any $j \geq i$, then $\sigma_{i\infty}$ is continuous at α and by induction applied to pairs $(i,\bar{\alpha})$, $\bar{\alpha} < \alpha$, we have that the K_i-cardinality of $\sigma_{i\infty}(\alpha) = \sup \sigma_{i\infty}[\alpha]$ is equal to α. Otherwise, choose the unique $j \geq i$ so that $\sigma_{ij}(\alpha)$ has K_j-cofinality κ_j; then σ_{ij} is continuous at α and therefore by induction applied to pairs $(i,\bar{\alpha})$, $\bar{\alpha} < \alpha$, $\sigma_{ij}(\alpha)$ has K_i-cardinality α. Let α^* denote $\sigma_{ij}(\alpha)$. Now $\sigma_{i,j+1}(\alpha) = \sigma_{j,j+1}(\alpha^*)$ has K_j-cardinality $(\alpha^*)^+$ of K_j. And by induction applied to pairs $(j+1,\beta)$, $\beta < \sigma_{j,j+1}(\alpha^*)$, we have that $\sigma_{j+1,\infty}(\beta)$ has K_{j+1}-cardinality at most β^+ of K_{j+1} for each $\beta < \sigma_{j,j+1}(\alpha^*)$. So as $\sigma_{j+1,\infty}$ is continuous at $\sigma_{j,j+1}(\alpha^*)$ and $\sigma_{j,j+1}(\alpha^*)$ is a cardinal of K_{j+1}, it follows that the K_{j+1}-cardinality of $\sigma_{j+1,\infty}(\sigma_{j,j+1}(\alpha^*)) = \sigma_{j\infty}(\alpha^*)$ is $\sigma_{j,j+1}(\alpha^*)$, and therefore the K_j-cardinality of $\sigma_{j\infty}(\alpha^*)$ is $(\alpha^*)^+$ of K_j. As α^* has K_i-cardinality α, it follows that the K_i-cardinality of $\sigma_{i\infty}(\alpha) = \sigma_{j\infty}(\alpha^*)$ is at most α^+ of K_i, as desired.
(c) The first statement follows immediately from (b). For the second statement, suppose that α is a closure point of $\sigma_{i\infty}$ and the K_i-cofinality of α is not measurable in K_i. We show by induction on $j \geq i$ that σ_{ij} is continuous at α, and therefore that α is a fixed point of σ_{ij}: This is vacuous if $j = i$. If α is a fixed point of σ_{ij} then the K_j-cofinality of α is not measurable in K_j by elementarity, and therefore $\sigma_{j,j+1}$ is continuous at α; it follows that

$\sigma_{i,j+1}$ is also continuous at α. For limit j, the continuity of σ_{ij} at α follows from the continuity of the σ_{ik} at α for $k < j$. $\square$

Lemma 4 *$(*)$ holds with K replaced by K'.*

Proof. It suffices to show by induction on i that $(*)_i$ holds, where $(*)_i$ is $(*)$ with K replaced by K_i.

Base case: $(*)_0$ is just $(*)$.

Successor case: Suppose that $(*)_i$ holds and that α is K_{i+1}-regular with cofinality at least δ. We may assume that α is greater than κ_i, else α is K_i-regular and we are done by induction. If α is at most $\sigma_{i,i+1}(\kappa_i)$ then α has K_i-cardinality κ_i^+ of K_i, and, as K_{i+1} and K_i contain the same κ_i-sequences of ordinals, α has K_i-cofinality κ_i^+ of K_i. So α and κ_i^+ of K_i have the same cardinality and cofinality, so we are done by induction.

Now suppose that α is greater than $\sigma_{i,i+1}(\kappa_i)$. Represent α in K_{i+1}, the ultrapower of K_i, by $[f]$ where $f : \kappa_i \to \mathrm{Ord}$. We may assume that f is either constant or increasing, and also that $f(\gamma)$ is K_i-regular and greater than κ_i for all $\gamma < \kappa_i$. If f is constant then $\alpha = \sigma_{i,i+1}(\bar{\alpha})$ for some $\bar{\alpha}$ which is regular in K_i and greater than κ_i; then α and $\bar{\alpha}$ have the same cofinality and cardinality, so we are done by induction. Thus we may assume that f is increasing.

Now the K_i-cofinality of α is at least the supremum μ of the $f(\gamma)$'s, as using the regularity of the $f(\gamma)$'s, we can everywhere-dominate any set in K_i of $f(\gamma)$-many functions from κ_i into $\prod_{\gamma' > \gamma} f(\gamma')$ by a single such function in K_i. As μ is K_i-singular, the K_i-cofinality of α is in fact greater than μ. And the K_i-cardinality of α is $\mu^{\kappa_i} = \mu^+$ of K_i. It follows that α and μ^+ of K_i have the same cofinality and cardinality, so we are done by induction.

Limit case: Suppose that i is a limit and α is K_i-regular with cofinality at least δ. By Lemma 3 (a), we can choose $i_0 < i$ such that α equals as $\sigma_{i_0 i}(\bar{\alpha})$, where $\bar{\alpha}$ is regular in K_{i_0} and $\sigma_{i_0 i}$ is continuous at $\bar{\alpha}$. It follows by Lemma 3 (a) that α and $\bar{\alpha}$ have the same cardinality and cofinality, and therefore we are done by induction. $\square$

Lemma 5 *If λ is a limit cardinal then $\mathit{cof}^{K'}(\lambda)$ is not measurable in K'.*

Proof. Let κ denote the K-cofinality of λ. If κ is not measurable in K then by Lemma 3 (c), λ is a fixed point of the iteration $\sigma : K \to K'$ and therefore the result follows by elementarity. Otherwise we claim that κ must equal κ_i for some i: If not, then as κ is a closure point of σ, κ would also be a fixed point of σ and therefore κ is measurable in each K_i. By the definition of the κ_i's, it must be that for each i, κ is either of the form $\sigma_{i_0 i}(\kappa_{i_0})$ for some $i_0 < i$, or κ_i is less than κ. But for sufficiently large i, κ_i cannot be less than κ and so κ is of the form $\sigma_{i_0 i}(\kappa_{i_0})$ for some $i_0 < i$, which implies that κ equals κ_{i_0}, as κ is a fixed point of $\sigma_{i_0 i}$.

So choose i so that $\kappa = \kappa_i$. Then κ and λ are fixed points of σ_{0i} and therefore λ has K_i-cofinality κ_i. As K_{i+1} has the same κ-sequences as K_i, it follows that λ has cofinality κ in K_{i+1} and therefore also in K'. But since we applied the order 0 measure on κ to form K_{i+1}, κ is not measurable in K_{i+1} and therefore not measurable in K', as desired. $\square$

Now we apply the technique of "dropping along a square sequence" from [2]. Define a function $d : \mathrm{Ord} \to \omega$ as follows. Fix a lightface K'-definable global $\square$-sequence $\langle C_\alpha \mid \alpha \text{ singular in } K' \rangle$: C_α is closed unbounded in α with ordertype less than α for each K'-singular α and $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$ whenever $\bar{\alpha}$ is a limit point of C_α. If α is not K'-singular then $d(\alpha) = 0$. Otherwise define:

$\alpha_0 = \alpha$
$\alpha_1 = \text{ot } (C_{\alpha_0})$
$\alpha_2 = \text{ot } (C_{\alpha_1})$
$\dots$
$\alpha_{n+1} = \text{ot } (C_{\alpha_n})$,

as long as α_n is K'-singular, and let $d(\alpha)$ be the least n such that α_n is not K'-singular. $\alpha_{d(\alpha)}$ is the K'-cofinality of α.

Lemma 6 *(Main Lemma, after [2]) For each n there is a ZFC-preserving class forcing P_n that adds a CUB class C_n of singular cardinals such that for all α in C_n of cofinality at least δ, $d(\alpha)$ is at least n.*

Proof. We use the following.

Lemma 7 *Suppose $k < m$, $\alpha \geq \delta$, α is regular and C is a closed set of ordertype $\alpha^{+m}+1$, consisting of ordinals $\geq \alpha^{+m}$ (where $\alpha^{+0} = \alpha$, $\alpha^{+(p+1)} = (\alpha^{+p})^+$). Then $(C \cap \{\beta \mid d(\beta) \geq k+1\}) \cup \mathit{Cof}\ (< \delta)$ has a closed subset of ordertype $\alpha^{+(m-k-1)} + 1$.*

Proof. The proof is by induction on k, using Lemma 4.

Suppose $k = 0$. Let β be the $\alpha^{+(m-1)}$-st element of C. Then β is K'-singular since its cofinality $(= \alpha^{+(m-1)})$ is at least δ and less than its cardinality $(\geq \alpha^{+m})$. Similarly, each element of Lim $(C \cap \beta)$ of cofinality $\geq \delta$ is K'-singular and therefore Lim $(C \cap \beta)$ is a closed subset of $(C \cap \{\beta \mid d(\beta) \geq 1\}) \cup \text{Cof } (< \delta)$ of ordertype $\alpha^{+(m-1)} + 1$, as desired.

Suppose that the Lemma holds for k and let $m+1 > k+1$, C a closed set of ordertype $\alpha^{+(m+1)}+1$ consisting of ordinals $\geq \alpha^{+(m+1)}$. Then $\mu = \max C$ is K'-singular, as its cofinality is at least δ and less than its cardinality. Let β be the $(\alpha^{+m} + \alpha^{+m})$-th element of $C \cap C_\mu$. β is K'-singular as its cofinality is at least δ and less than its cardinality. Let $\bar{\beta}$ be the α^{+m}-th element of C. Then $\bar{C} = \{\text{ot } C_\gamma \mid \gamma \in C \cap \text{Lim } C_\beta \cap [\bar{\beta}, \beta]\}$ is a closed set of ordertype $\alpha^{+m} + 1$ consisting of ordinals $\geq \alpha^{+m}$. By induction there is a closed $\bar{D}$ contained in $(\bar{C} \cap \{\gamma \mid d(\gamma) \geq k+1\}) \cup \text{Cof } (< \delta)$ of ordertype $\alpha^{+(m-k-1)} + 1$. But then $D = \{\gamma \in C \cap \text{Lim } C_\beta \mid \text{ot } C_\gamma \in \bar{D}\}$ is a closed subset of $(C \cap \{\gamma \mid d(\gamma) \geq k+2\}) \cup \text{Cof } (< \delta)$ of ordertype $\alpha^{+(m-k-1)} + 1$. As $m - k - 1 = (m+1) - (k+1) - 1$, we are done. $\square$ (Lemma 7)

Lemma 6 now follows: Let P_n consist of closed sets c of singular cardinals such that

$\alpha \in c$, cof $(\alpha) \geq \delta \to d(\alpha) \geq n$,

ordered by end-extension. Lemma 7 implies that this forcing is κ-distributive for every cardinal κ. $\square$

Now for each n there is an outer model of V containing a real R_n such that in $L[R_n]$:

$(*)_{R_n}$ R_n codes a CUB class C_{R_n} of singular cardinals and an iterable, universal extender model K'_{R_n} such that
a. $d_{R_n}(\alpha) \geq n$ for α in C_{R_n} of sufficiently large cofinality, where $d_{R_n}(\alpha)$ is defined in K'_{R_n} just like $d(\alpha)$ is defined in K'.
b. $\alpha \in C_{R_n} \to$ cof (α) in K'_{R_n} is not measurable in K'_{R_n}.

This is because we can use Lemma 6 to force a CUB class C_n of singular cardinals such that $d(\alpha) \geq n$ for all α in C_n of sufficiently large cofinality, and then L-code the model $\langle V, C_n, K' \rangle$ by a real R_n. The extender model K' is universal in the extension as successors of strong limit cardinals are not collapsed and therefore weak covering holds relative to K' in the extension at all such cardinals of sufficiently large cofinality.

Applying the IMH, there are such reals R_n in V. As each R_n codes a CUB class of singular cardinals, the K of $L[R_n]$ is universal and therefore so is the K_{R_n} arising from $(*)_{R_n}$. Now co-iterate the K_{R_n}'s to a single K^*, resulting in embeddings $\pi_n : K_{R_n} \to K^*$. As singular cardinals in C_{R_n} of sufficiently large cofinality are fixed by π_n (as their K_{R_n}-cofinality is not measurable in K_{R_n}), it follows that there is a single γ belonging to all of the C_{R_n}'s which is fixed by all of the π_n's. But then $d^*(\gamma) \geq n$ for each n, where $d^*(\gamma)$ is defined relative to K^* just like $d(\gamma)$ was defined relative to K'. This is a contradiction. $\Box$

For each real x let M_x, if it exists, be the minimum transitive set model of ZFC containing x. Thus M_x has the form $L_\mu[x]$ for some countable ordinal $\mu = \mu(x)$. If d is a Turing degree we write M_d, $\mu(d)$ for M_x, $\mu(x)$ (x in d).

Theorem 8 *Assume the existence of a Woodin cardinal with an inaccessible above. Then the IMH is consistent. Moreover for all d in a cone of Turing degrees, M_d exists and satisfies the IMH.*

Proof. First we prove the consistency of the IMH by showing that M_d satisfies the IMH for some Turing degree d in a forcing extension of V.

Let κ be Woodin with an inaccessible above in V. Let G be generic over V for the Lévy collapse of κ to ω. Work now in $V[G]$. Σ^1_2 determinacy holds and, as there is still an inaccessible, M_d exists for each Turing degree d. It follows that the theory of $(M_d, \in)$ is constant on a cone of Turing degrees d. Let d be a Turing degree such that the theory of $(M_e, \in)$ is constant for Turing degrees e at least that of d.

We claim that M_d, endowed with its definable classes, witnesses the IMH. Indeed, suppose that φ is a sentence true in some model M of height $\mu(d)$ compatible with M_d. By Jensen coding there is a real y such that d is recursive in y, $\mu(y) = \mu(d)$ and M is a definable inner model of M_y. Let e be the Turing degree of y. Then for some formula ψ, M_e satisfies the sentence

The inner model defined by ψ (with some choice of parameters) satisfies φ.

It follows that there is an inner model of M_d which satisfies φ, as desired. This proves the consistency of the IMH.

To say that a countable M, together with its countable collection of definable classes, satisfies IMH is simply a Π^1_1-statement with a real coding M as parameter, since one only needs to quantify over outer models of M of height $M \cap \mathrm{Ord}$. Thus the assertion that there exists a Turing degree d such that M_d (with its definable classes) satisfies IMH is a Σ^1_2-statement and hence absolute. So the existence of a Woodin cardinal with an inaccessible above implies that such an M_d exists in V (and indeed in L).

To prove the stronger statement that in V, M_d satisfies the IMH for a cone of d's, one argues as follows. Say that a set of reals X is *absolutely* Δ^1_2 iff there is a pair of Σ^1_2 formulas $\varphi(x)$, $\psi(x)$ such that X consists of all solutions to $\varphi(x)$ in V and φ is equivalent to the negation of ψ both in V and all of its forcing extensions.

Claim. Assume that there is a Woodin cardinal. Then determinacy holds for absolutely Δ^1_2 sets.

Proof of Claim. As before let G be generic for the Lévy collapse of the Woodin cardinal to ω. Then Σ^1_2 determinacy holds in $V[G]$. By the Moschovakis Third Periodicity theorem, ([7] Theorem 6E.1), if X is Σ^1_2 in $V[G]$ there is a definable winning strategy in $V[G]$ for one of the players in the game G_X. By the homogeneity of the Lévy collapse, it follows that absolutely Δ^1_2 sets are determined in V. This proves the Claim.

As there is an inaccessible in V, M_d exists for each Turing degree d in V. Now it follows from the Claim that in V, for any sentence φ, either for a cone of Turing degrees d,

$$M_d \models \varphi$$

or for a cone of Turing degrees d,

$$M_d \models \neg\varphi,$$

since the relevant games are absolutely Δ^1_2. Therefore in V the theory of $(M_d, \in)$ is constant for a cone of Turing degrees d. We can then apply the argument used earlier in $V[G]$ to conclude that also in V, M_d satisfies IMH for d in a cone of Turing degrees. □

Parameters and the Strong Inner Model Hypothesis

How can we introduce parameters into the Inner Model Hypothesis? The following result shows that inconsistencies arise without strong restrictions on the type of parameters allowed.

Proposition 9 *([5]) The Inner Model Hypothesis with arbitrary ordinal parameters or with arbitrary real parameters is inconsistent.*

So instead we consider *absolute parameters*, as in [4]. For any set x, the *hereditary cardinality* of x, denoted hcard (x), is the cardinality of the transitive closure of x. If V^* is an outer model of V, then a parameter p is *absolute between* V *and* V^* iff V and V^* have the same cardinals $\leq$ hcard (p) and some parameter-free formula has p as its unique solution in both V and V^*.

Inner Model Hypothesis with locally absolute parameters Suppose that p is absolute between V and V^* and φ is a first-order sentence with parameter p which holds in an inner model of V^*. Then φ holds in an inner model of V.

For a singular cardinal κ, a $\square_\kappa$ *sequence* is a sequence of the form $\langle C_\alpha \mid \alpha < \kappa^+, \alpha \text{ limit}\rangle$ such that each C_α has ordertype less than κ and for $\bar{\alpha}$ in Lim C_α, $C_{\bar{\alpha}} = C_\alpha \cap \bar{\alpha}$. *Definable* $\square_\kappa$ is the assertion that there exists a $\square_\kappa$ sequence which is definable over $H(\kappa^+)$ with parameter κ. We will be interested in the special case $\kappa = \beth_\omega$, in which case the parameter κ is superfluous.

Theorem 10 *The Inner Model Hypothesis with locally absolute parameters is inconsistent.*

Proof. We first show that definable $\square_\kappa$ fails, where κ is $\beth_\omega$. Let $\langle C_\alpha \mid \alpha < \kappa^+, \alpha \text{ limit}\rangle$ be a $\square_\kappa$ sequence definable over $H(\kappa^+)$ without parameters. For each n let S_n be the stationary set of all limit $\alpha < \kappa^+$ such that the ordertype of C_α is greater than $\beth_n$.

Claim. Let P_n be the forcing that adds a CUB subset of S_n using closed bounded subsets of S_n as conditions, ordered by end extension. Then P_n is κ^+-distributive, i.e., does not add κ-sequences.

Proof of Claim. It is enough to show that P_n is $\beth_m^+$ distributive for each $m < \omega$. Assume that m is at least n. Suppose that p is a condition and $\langle D_i \mid i < \beth_m\rangle$ are dense. Let $\langle M_i \mid i < \kappa^+\rangle$ be a continuous chain of size κ elementary submodels of some large $H(\theta)$ such that M_0 contains $\kappa \cup \{\langle C_\alpha \mid \alpha < \kappa^+, \alpha \text{ limit}\rangle, p\}$ and for each $i < \kappa^+$, $\langle M_j \mid j \leq i\rangle$ is an element of M_{i+1}. Let κ_i be $M_i \cap \kappa^+$ and C the set of such κ_i's. Then a final segment D of $C \cap \text{Lim } C_\gamma$ is contained in S_n, where $\gamma = \kappa_{\beth_m^+}$. Write

D as $\langle \kappa_{\alpha_i} \mid i < \text{ordertype } D\rangle$. We can then choose a descending sequence $\langle p_i \mid i < \beth_m\rangle$ of conditions below p such that p_{i+1} meets D_i and belongs to $M_{\kappa_{\alpha_i}+1}$ for each i. Then the greatest lower bound of this sequence meets each D_i. This proves the Claim.

It follows that for each n the forcing P_n does not alter $H(\kappa^+)$. By the Inner Model Hypothesis with locally absolute parameters S_n has a CUB subset C_n in V for each n. But this is a contradiction, as the intersection of the C_n's is empty.

Now we refine the above argument. As not every real has a #, there exist reals R such that κ^+ equals κ^+ of $L[R]$, where κ is $\beth_\omega$. Let X be the set of such reals and for each R in X let $\langle C^R_\alpha \mid \alpha < \kappa^+,\ \alpha \text{ limit}\rangle$ be the $L[R]$-least $\square_\kappa$ sequence. Now for limit $\alpha < \kappa^+$, define C^*_α to be the intersection of the C^R_α, $R \in X$. Then $\langle C^*_\alpha \mid \alpha < \kappa^+,\ \alpha \text{ limit}\rangle$ is definable in $H(\kappa^+)$ without parameters and has the properties of a $\square_\kappa$ sequence with the sole exception that C^*_α is only guaranteed to be unbounded in α if α has cofinality greater than $2^{\aleph_0}$. Now repeat the above argument using $\langle C^*_\alpha \mid \alpha < \kappa^+,\ \alpha \text{ limit}\rangle$ in place of $\langle C_\alpha \mid \alpha < \kappa^+,\ \alpha \text{ limit}\rangle$, to obtain a contradiction. $\square$

To obtain the Strong Inner Model Hypothesis, we require more absoluteness. We say that the parameter p is *(globally) absolute* iff there is a parameter-free formula which has p as its unique solution in all outer models of V which have the same cardinals $\leq$ hcard (p) as V.

Strong Inner Model Hypothesis (SIMH) Suppose that p is absolute, V^* is an outer model of V with the same cardinals $\leq$ hcard (p) as V and φ is a first-order sentence with parameter p which holds in an inner model of V^*. Then φ holds in an inner model of V.

Remark. If above we assume that the sentence φ holds not just in an inner model of V^* but in V^* itself, then in the conclusion we may demand that in an inner model of V witnessing φ, p is definable via the same formula ψ witnessing the absoluteness of p. (This inner model may, however, fail to have the same cardinals $\leq$ hcard (p) as V.) This is because we can replace the sentence φ by: "φ holds and p is defined by ψ".

Theorem 11 *([5]) Assume the SIMH. Then CH is false. In fact, $2^{\aleph_0}$ cannot be absolute and therefore cannot be $\aleph_\alpha$ for any ordinal α which is countable in L.*

Theorem 12 *The SIMH implies the existence of an inner model with a strong cardinal.*

Proof. Assume not, and let K be the core model below a strong cardinal (see [8]). As in the proof of Theorem 2, we let K' denote the iterate of K obtained by applying each order 0 measure exactly once. Then by the argument of Lemma 5, if λ is a cardinal then the K'-cofinality of λ is not measurable in K'. And by weak covering relative to K, if λ is a singular cardinal, then λ^+ is computed correctly in K (i.e., $(\lambda^+)^K = \lambda^+$).

Lemma 13 *For any singular cardinal λ, λ^+ is computed correctly in K'.*

Proof of Lemma 13. This is clear if the K-cofinality of λ is not measurable in K, for then λ is a fixed point of the iteration from K to K' and $(\lambda^+)^{K'} = (\lambda^+)^K = \lambda^+$. Otherwise let $\langle K_i \mid i \in \mathrm{Ord}\rangle$ result from the iteration of K to K' and choose i so that the ultrapower map $\sigma_i : K_i \to K_{i+1}$ applies the order 0 measure at $\kappa = \mathrm{cof}^{K_i}(\lambda)$. If $\langle \lambda_j \mid j < \kappa\rangle$ is a continuous and increasing sequence in K_i with supremum λ, then λ^+ of K_{i+1} is represented in the ultrapower of K_i by $\langle \lambda_j^+ \mid j < \kappa\rangle$. In K_i, the product of the λ_j^+'s contains a subset of size $(\lambda^+)^{K_i}$, consisting of functions well-ordered by dominance on a final segment of κ. It follows that $(\lambda^+)^{K_{i+1}}$ has cardinality λ^+ and therefore K_{i+1} computes λ^+ correctly. As λ^+ is a fixed point of the remaining iteration from K_{i+1} to K', it follows that K' computes λ^+ correctly. This proves Lemma 13.

We say that λ is a *cut point* of K' iff no extender on the K' sequence with critical point less than λ has length at least λ (i.e., λ is *not overlapped* in K'). The class of cut points of K' is clearly closed. If the class of cut points of K' is bounded, then for sufficiently large λ we can choose $f(\lambda)$ less than λ which is the critical point of an extender on the K' sequence whose length is at least λ; but then by Fodor's theorem, there would be a fixed κ and extenders on the K' sequence with critical point κ of arbitrarily large length, which implies that κ is a strong cardinal in K'. Thus as we have assumed that there is no strong cardinal in K', the class of cut points of K' is closed and unbounded.

Let $\langle \lambda_n \mid n \in \omega\rangle$ be the first ω-many limit cardinals of V which are cut points of K', and let λ_ω be their supremum. Then each λ_n, and of course λ_ω, has cofinality ω.

Lemma 14 *Each λ_n, and λ_ω as well, is an absolute parameter.*

Proof of Lemma 14. We first show that λ_0 is absolute. Let V^* be an outer model of V with the same cardinals as V up to λ_0. Note that for some real R in V, no $L_\alpha[R]$ satisfies ZFC and therefore $R^\#$ does not exist in V^*. It follows that for any singular cardinal λ of V^*, λ is singular in V and λ^+ is computed correctly in V. In particular, V^* and V have the same cardinals up to λ_0^+ and the same singular cardinals up to λ_0.

It follows by Lemma 13 that for any singular cardinal λ of V^*, λ^+ is computed correctly in both K' and $(K^*)'$, where $(K^*)'$ denotes the K' of V^*, obtained from K^*, the K of V^*, by applying each order 0 measure exactly once. As both K' and $(K^*)'$ are universal in V^*, it follows that they coiterate simply to a common model W.

Claim. The co-iteration of K' with $(K^*)'$ fixes singular cardinals of V^* which are cut points either of K' or of $(K^*)'$.

Proof of Claim. Let λ be a singular cardinal of V^* (and therefore also a singular cardinal of V). First assume that λ is a cut point of K'. Then as λ has non-measurable cofinality in K', λ is fixed by the iteration on the K'-side. And as λ has non-measurable cofinality in $(K^*)'$, λ can only move on the $(K^*)'$-side if an extender overlapping λ were applied. The assumption that λ is not overlapped in K' implies that λ is not overlapped in W and therefore the least extender overlapping λ was applied on the $(K^*)'$-side. But then λ^+ is not computed correctly in the resulting ultrapower and therefore is computed correctly neither in W nor in K'. This contradicts Lemma 13. As the same argument applies with K' and $(K^*)'$ switched, this proves the Claim.

It follows from the Claim that λ_0 is the least limit cardinal of V^* which is a cut point of $(K^*)'$. As V^* is an arbitrary outer model of V with the same cardinals as V up to λ_0, we have shown that λ_0 is an absolute parameter. The same argument shows that each λ_n is absolute, and therefore so is λ_ω, the supremum of the first ω limit cardinals which are cut points of K'. This proves Lemma 14.

Now let $\langle C_\alpha \mid \alpha < \lambda_\omega^+,\ \alpha \text{ limit}\rangle$ be the least $\Box_{\lambda_\omega}$ sequence of K'; this is also a $\Box_{\lambda_\omega}$ sequence in V, as $(\lambda_\omega^+)^{K'} = \lambda_\omega^+$. As in the proof of Theorem 10, there are generic extensions of V preserving $H(\lambda_\omega^+)$ which add CUB subsets to each $S_n = \{\alpha < \lambda_\omega^+ \mid \text{ordertype } C_\alpha > \lambda_n\}$. It follows from the Strong Inner Model Hypothesis (and the Remark immediately following its statement) that for each n there is an inner model M_n, with the correct

λ_ω^+ and λ_n, in which $S_n^{M_n}$ contains a CUB subset C_n, where $S_n^{M_n}$ is defined using the least $\square_{\lambda_\omega}$ sequence of $(K')^{M_n}$. The latter may of course differ from the least $\square_{\lambda_\omega}$ sequence of K'. However as λ_ω^+ is computed correctly in each $(K')^{M_n}$ and λ_ω is a cut point of non-measurable cofinality in each $(K')^{M_n}$, it follows that the $(K' \mid \lambda_\omega^+)^{M_n}$'s compare to a common K'' of height λ_ω^+ with all ordinals in some CUB subset C of λ_ω^+ as closure points. But if α is such a closure point in the intersection of the C_n's and α_n is the image of α under the comparison embedding of $(K' \mid \lambda_\omega^+)^{M_n}$ into K'', then C_{α_n} as defined in K'' contains elements cofinal in α and therefore C_α as defined in K'', an initial segment of C_{α_n}, has ordertype at least that of C_α as defined in $(K' \mid \lambda_\omega^+)^{M_n}$. It follows that C_α as defined in K'' has ordertype greater than λ_n for each n, which is a contradiction. $\square$

Remarks. (a) It is likely that Theorem 12 can be improved to obtain an inner model with a Woodin cardinal. But it is not possible to obtain an iterable inner model with a Woodin cardinal and an inaccessible above it (unless the SIMH is inconsistent): Otherwise every real would be generic for Woodin's extender algebra defined in an iterate of such an inner model, implying that for every real R there is an inaccessible in $L[R]$; this contradicts Theorem 1.
(b) David Asperó and the first author observed that the consistency of the SIMH for the parameter ω_1 follows as in the proof of Theorem 8 from that of a Woodin cardinal with an inaccessible above. In particular this yields the consistency of the natural extension of Lévy absoluteness asserting Σ_1 absoluteness with parameter ω_1 for arbitrary ω_1-preserving extensions.

Question. Is the Strong Inner Model Hypothesis consistent relative to large cardinals?

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