ON GENERICALLY STABLE TYPES IN DEPENDENT THEORIES

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ABSTRACT. We develop the theory of generically stable types, independence relation based on nonforking and stable weight in the context of dependent (NIP) theories.

1. INTRODUCTION AND PRELIMINARIES

1.1. **Introduction.** The original motivation for this paper was generalizing certain aspects of the theory developed by Haskell, Hrushovski and Macpherson in [4] for stably dominated types to a broader context. We believe that the right framework for most results (at least assuming the theory is dependent) has to do with "stable" types introduced by Shelah in [17]. Since the name "stable" had been used (e.g. by Lascar and Poizat, see [11]) for a different (much stronger) notion before Shelah's paper was written, in order to avoid confusion, we use different terminology suggested by Hrushovski and Pillay and call our main object of study "generically stable" types.

While the paper was being written, other particular cases of generically stable types became important for the study of theories interpretable in o-minimal structures carried out by Hasson, Onshuus, Peterzil and others. For example, notions of "seriously stable", "hereditarily stable" types were investigated in [5]. Numerous conversations with Assaf Hasson and Alf Onshuus slightly changed the character of this work.

We develop a cleaner and a more comprehensible theory of "stable" types than the one found in [17]. In particular we eliminate the need to work with finitely satisfiable types. This has two advantages: first, our approach allows one to avoid considering co-heir sequences (which we call Shelah sequences here) which used to create much confusion. Second, we provide a good picture of types over arbitrary sets, and not only over models or indiscernible sets.

It is important to us, however, to show the connection between our and Shelah's approaches; therefore, sections 3 and 5 of the paper are devoted mostly to a systematic development of "Shelah-stable" types, giving a more complete picture than what is done in [17].

Together with deeper understanding came the realization that nonforking plays a central role in the general theory, and for generically stable types is equivalent to definability and gives rise to a nice independence relation, so we have a very smooth generalization of classical stability. In a sense this provides a complementary picture to the work of Dolich

Date: October 31, 2018.

Research partially supported by FCT grant SFRH / BPD / 34893 / 2007.

[3] which characterizes forking in o-minimal theories (that is, "as unstable as possible" dependent theories).

Some of our results can be found in a different form in a recent preprint by Hrushovski and Pillay [9] which was written simultaneously and independently of our work and is mostly focused on other issues such as invariant types and measures.

Let us make a note on our choice of terminology. Following Lascar and Poizat, some people call a (partial) type $\pi(\bar{x})$ stable if every extension of it is definable. If $\pi(\bar{x})$ is a formula which defines in $\mathfrak{C}$ the set D, a more common terminology is "D is a stable stably embedded (definable) set". If T is dependent, then stability of a definable set has many equivalent definitions, as investigated e.g. by Onshuus and Peterzil in [12]. In particular, a set D is stable if and only if it fails the order property if and only if it fails the strict order property, that is, there is no definable partial order with infinite chains on D. It follows that stable embeddedness comes "for free", that is, if D is stable then every externally definable subset of D is definable with parameters in D. So D is stable if and only if the induced structure (with or without taking into account external parameters) on it is stable. Hence this terminology seems very reasonable to us and we will use it.

It is also quite easy to see that Lascar-Poizat stability of a type p is equivalent (assuming dependence) to the set of realizations of p failing the order property (equivalently, the strict order property). This provides a justification for simply calling such types stable; still, we will restrain from doing so in order to avoid confusion between this and Shelah's terminology. So we'll call stable types in this strong sense "Lascar-Poizat stable" or "hereditarily stable" since in our context p is Lascar-Poizat stable if and only if every extension of it is Lascar-Poizat stable if and only if every extension of it is generically stable.

As for the term "generically stable", we think it captures the concept being studied here pretty well, since generically stable (that is, "stable" according to Shelah) types behave in a stable way "generically", i.e. when one takes nonforking extensions and Morley sequences.

The paper is organized as follows:

Section 2 contains basic definitions and facts on indiscernible sequences, sets, splitting, forking, definability, etc in dependent theories. The main result of the section is Lemma 2.27 which states (among other things) that the global average of a nonforking indiscernible set does not fork over the base set. This is easier if nonforking is replaced with nonsplitting, Lemma 2.24; for the nonforking case one needs to apply a more subtle analysis and understand the connections between different notions of splitting and forking in dependent theories.

Section 3 is devoted to developing the basic theory of finitely satisfiable types, ultrafilters, Shelah sequences, etc. While it is essential for understanding Shelah's approach to generically stable types, it is not at all used in section 4 (where the main theory of generically stable types is developed), hence can be omitted in the first reading.

Section 4 is the central part of the article: we define generically stable types and prove most of their properties (such as definability and stationarity). We also show that generic stability is closed under parallelism.

Section 5 is based on [17], but we give a more complete and wide picture. In particular, we prove that when working over a slightly saturated model, being a generically stable type is equivalent to being both finitely satisfiable in and definable over a small subset. This result does not appear in [17], and in fact was not known to Shelah at the time. We also give an example showing that for this criterion it is essential to consider type over saturated models.

Section 6 presents a summary, connections to previous works on particular cases (stably dominated types [4], seriously stable types and hereditarily stable types introduced by Hasson and Onshuus in [5]) and several examples of generically stable types that do not fall in any of the categories discussed above. These are also a good source of certain curious phenomena, showing the subtleties of working over sets as opposed to saturated models, differences between splitting and forking, which explain some of our earlier choices. In particular we see that nonsplitting extensions of generically stable types do not have to be generically stable, which can not happen with nonforking extensions.

Section 7 is devoted to the original goal, developing the theory of independence for generically stable types, which happens to be quite easy once the general framework is well-understood. Independence relation for generically stable types turns out to be based on both nonforking and definability, generalizing classical stability. We also characterize generically stable types in terms of behavior of forking on the set of their realizations and show that a type stably dominated by a generically stable type is generically stable.

Section 8 is the beginning of the theory of weight for generically stable types. We show that in a strongly dependent theory every generically stable type has finite weight. We also define stable weight of an arbitrary type, hoping that this will help us in understanding the "stable part" of a type in a dependent theory, and show that a strongly dependent type has finite stable weight. The key lemma for proving these results is Lemma 8.10 which says that under certain circumstances indiscernible sequences can be assumed to be mutually indiscernible. We find this interesting on its own. This section is related to more general works on different notions of weight in dependent theories: Onshuus and the author [13] on dp-minimality, strong dependence and weight, [14] on weight based on thorn-forking in rosy theories, and Adler [2] on "burden".

The goal of the Appendix (which had originally been a part of section 3 and was removed for the sake of clarity) is to motivate viewing types as ultrafilters by passing to a more general framework of Keisler measures, in which Shelah's approach to finitely satisfiable types seems very natural.

Acknowledgements. The author thanks Saharon Shelah, Anand Pillay, Itaï Ben Yaacov and Hans Adler for fruitful discussions and correspondence. We are especially grateful to Assaf Hasson and Alf Onshuus for numerous conversations, questions, suggestions and comments on preliminary versions that motivated and improved this work. Many thanks to the anonymous referee for very helpful comments and suggestions.

1.2. **Notations.** In this paper, T will denote a complete theory, τ will denote the vocabulary of T, L will denote the language of T. We will assume that everything is happening in the monster model of T which will be denoted by $\mathfrak{C}$. Elements of $\mathfrak{C}$ will be denoted a, b, c, finite tuples will be denoted $\bar{a}, \bar{b}, \bar{c}$, sets (which are all subsets of $\mathfrak{C}$) will be denoted A, B, C, and models of T (which are all elementary submodels of $\mathfrak{C}$) will be denoted by M, N, etc.

Given an order type O and a sequence $\langle \bar{a}_i : i \in O \rangle$, we often denote $\bar{a}_{<i} = \langle \bar{a}_j : j < i \rangle$, similarly for $\bar{a}_{\leq i}$, $\bar{a}_{>i}$, etc. We will often identify a tuple $\bar{a}$ or a sequence $\langle \bar{a}_i : i \in O \rangle$ with the set which is its union, but it should always be clear from the context what we mean (although sometimes when confusions might arise, we make the distinction, e.g. $\mathrm{Av}(I, \cup I)$ will denote the average type of a sequence I over itself).

By $\bar{a} \equiv_A \bar{b}$ we mean $\mathrm{tp}(\bar{a}/A) = \mathrm{tp}(\bar{b}/A)$.

1.3. **Preliminaries.** Recall that a theory T is called *dependent* if there does not exist a formula which exemplifies the independence property. We are mostly going to use the following equivalent definition:

Fact 1.1. *T is dependent if and only if there do not exist an indiscernible sequence $I = \langle \bar{a}_i : i < \lambda \rangle$, a formula $\varphi(\bar{x}, \bar{y})$ and $\bar{b}$ such that both*

$$\{i \colon \models \varphi(\bar{a}_i, \bar{b})\}$$

and

$$\{i \colon \models \neg\varphi(\bar{a}_i, \bar{b})\}$$

are unbounded in λ.

A type $p \in \mathrm{S}_m(B)$ is said to be *definable* over A if for every formula $\varphi(\bar{x}, \bar{y})$ with $\mathrm{len}(\bar{x}) = m, \mathrm{len}(y) = k$ there exists a formula $d_p\bar{x}\varphi(\bar{x}, \bar{y})$ with free variables $\bar{y}$ such that for every $\bar{b} \in B^k$

$$\varphi(\bar{x}, \bar{b}) \in p \Longleftrightarrow \models d_p\bar{x}\varphi(\bar{x}, \bar{b})$$

A definition schema d_p is said to be *good* is for every set C the set

$$\{\varphi(\bar{x}, \bar{c}) \colon \varphi(\bar{x}, \bar{y}) \text{ is a formula}, \mathrm{len}(\bar{x}) = m, \bar{c} \in C, \models d_p\bar{x}\varphi(\bar{x}, \bar{c})\}$$

is a complete type over C. We call this type *the free extension* of p to C with respect to d and denote it by $p|^d C$. We call a type *properly definable* over a set A if it is definable over A by a good definition.

A type $p \in \mathrm{S}(B)$ said to be *finitely satisfiable* in a set A if for every formula $\varphi(\bar{x}, \bar{b}) \in p$ there exists $\bar{a} \in A$ such that $\models \varphi(\bar{a}, \bar{b})$. If $A \subseteq B$ then we also say that p is a *coheir* of $p{\restriction}A$. Clearly, if $p \in \mathrm{S}(M)$ and M is a model, then p is finitely satisfiable in M.

Recall that a sequence $I = \langle \bar{a}_i : i \in O \rangle$ (where O is a linear ordering) is called *indiscernible* over a set A if the type of $\bar{a}_{i_1}, \dots, \bar{a}_{i_k}$ over A depends only on the order between the indices $i_1, \dots, i_k$ for every k. I is called an indiscernible *set* if the type above depends on k only.

A hyperimaginary element (tuple) $\bar{a}$ is said to be *bounded* over a set A if the orbit of $\bar{a}$ under the action of $\mathrm{Aut}(\mathfrak{C}^{heq}/A)$ is small, i.e. of cardinality less than $|\mathfrak{C}|$. The *bounded closure of* A, denoted by $\mathrm{bdd}^{heq}(A)$, is the collection of all hyperimaginary elements bounded over A. Clearly, this is a generalization of the algebraic closure, and usually is a bigger set. If T is stable, then $\mathrm{bdd}^{heq}(A) = \mathrm{acl}^{eq}(A)$ for every set A. We will not make real use of hyperimaginaries in the paper, hence will not concentrate on these issues.

Let us say that two tuples are of *Lascar distance* 1 over A if there exists an indiscernible sequence over A containing both tuples. Two tuples $\bar{a}$ and $\bar{b}$ are of *Lascar distance k* over A if there exist $\bar{a} = \bar{a}_1, \bar{a}_2, \dots, \bar{a}_{k+1} = \bar{b}$ such that $\bar{a}_i, \bar{a}_{i+1}$ are of Lascar distance 1 over A. Recall that two tuples $\bar{a}, \bar{b}$ are said to have the same *Lascar strong type* over a set A if they are of finite Lascar distance over A. In this case we will often write $\mathrm{Lstp}(\bar{a}/A) = \mathrm{Lstp}(\bar{b}/A)$.

1.4. **Global Assumptions.** All theories mentioned in this paper are *assumed to be dependent* unless stated otherwise. For the sake of clarity of presentation we also assume $T = T^{eq}$.

2. Indiscernible sequences, nonsplitting and stationarity

This section contains a collection of basic definitions and facts some of which are well known, which will be used widely throughout the paper.

Fact 1.1 motivates the following definitions:

Definition 2.1. Let $I = \langle \bar{a}_i : i < \lambda \rangle$ be an indiscernible sequence, B a set. We define the *average type* of I over B, $\mathrm{Av}(I, B)$ as the set of all formulae $\varphi(\bar{x}, \bar{b})$ such that $\{i \colon \neg\varphi(\bar{a}_i, \bar{b})\}$ is bounded in λ.

Remark 2.2. If I, B are as above, then $\mathrm{Av}(I, B) \in \mathrm{S}(B)$.

Note that

Remark 2.3. Let $I = \langle \bar{a}_i : i < \lambda \rangle$ an indiscernible *set*, B a set. Then $\varphi(\bar{x}, \bar{b}) \in \mathrm{Av}(I, B)$ if and only if $\{i \colon \neg\varphi(\bar{a}_i, \bar{b})\}$ is *finite*.

In fact, we can say a bit more. The following definition is motivated by [17], Definition 1.7:

Definition 2.4. Let $I = \langle \bar{b}_i \rangle = \langle \bar{b}_i : i \in O \rangle$ be an infinite indiscernible sequence. We say that a formula $\varphi(\bar{x}, \bar{y})$ is *stable* for I if for every $\bar{c} \in \mathfrak{C}$ the set $\{i \in I \colon \varphi(\bar{b}_i, \bar{c})\}$ is either finite or co-finite.

Observation 2.5. If $I = \langle \bar{b}_i : i \in O \rangle$ is an infinite indiscernible set, then every $\varphi(\bar{x}, \bar{y})$ is stable for I. Moreover, for every $\varphi(\bar{x}, \bar{y})$ there exists $k = k_\varphi < \omega$ such that for every $\bar{c} \in \mathfrak{C}$, either $|\{i \in O \colon \varphi(\bar{b}_i, \bar{c})\}| < k$ or $|\{i \in O \colon \neg\varphi(\bar{b}_i, \bar{c})\}| < k$.

Proof. If not, by indiscernibility we have for every $U, W \subseteq I$ finite disjoint,

$$\{\varphi(\bar{b}_i, \bar{y}) \colon i \in W\} \cup \{\neg\varphi(\bar{b}_i, \bar{y}) \colon i \in U\}$$

is consistent, clearly contradicting dependence of T. $\text{QED}_{2.5}$

It is often useful to consider $\text{Av}(I, \cup I)$, i.e. the average type of an endless indiscernible sequence over itself. Note that $\varphi(\bar{x}, \bar{a}_{<j}) \in \text{Av}(I, \cup I)$ iff $\varphi(\bar{a}_i, \bar{a}_{<j})$ holds for all $i \geq j$. So:

Remark 2.6. Let I be an indiscernible sequence. $\bar{a} \models \text{Av}(I, \cup I)$ if and only if $I^\frown\{\bar{a}\}$ is indiscernible.

Let us recall the definition of nonsplitting:

Definition 2.7.
(i) A type $p \in \text{S}(B)$ *does not split* over a set A if whenever $\bar{b}, \bar{c} \in B$ have the same type over A, we have $\varphi(\bar{x}, \bar{b}) \in p \iff \varphi(\bar{x}, \bar{c}) \in p$ for every formula $\varphi(\bar{x}, \bar{y})$.
(ii) A type $p \in \text{S}(B)$ *does not split strongly* over a set A if whenever $\bar{b}, \bar{c} \in B$ are of Lascar distance 1 over A, we have $\varphi(\bar{x}, \bar{b}) \in p \iff \varphi(\bar{x}, \bar{c}) \in p$ for every formula $\varphi(\bar{x}, \bar{y})$.
(iii) A type $p \in \text{S}(B)$ *does not Lascar-split* over a set A if whenever $\bar{b}, \bar{c} \in B$ have the same Lascar strong type over A, we have $\varphi(\bar{x}, \bar{b}) \in p \iff \varphi(\bar{x}, \bar{c}) \in p$ for every formula $\varphi(\bar{x}, \bar{y})$.

Note that a global type doesn't split over a set A if it is invariant under the action of the automorphism group of $\mathfrak{C}$ over A. One can also think of nonsplitting as a weak form of definability.

There are several ways to obtain types which do not split over a set A.

Observation 2.8. (No use of dependence)
(i) If a type over B is finitely satisfiable in $A \subseteq B$, then it does not split over A.
(ii) If a type over B is definable over $A \subseteq B$, then it does not split over A.

Observation 2.9. (No use of dependence) Let M be a $(|A| + \aleph_0)^+$-saturated model containing A, $p \in \text{S}(M)$. Then p does not Lascar-split over A if and only if p does not split strongly over A.

Proof. One direction is clear. For the other one, if p Lascar-splits over A, then there are $\bar{b}, \bar{c} \in M$ of the same Lascar strong type with $\varphi(\bar{x}, \bar{b}) \wedge \neg\varphi(\bar{x}, \bar{c}) \in p$. There are finitely many elements $\bar{b} = \bar{b}_1, \ldots, \bar{b}_k = \bar{c}$ of Lascar distance 1 which witness that $\bar{b}, \bar{c}$ are of finite Lascar distance, and by saturation we may assume they all lie in M (we may even assume that all the indiscernible sequences witnessing Lascar distance 1 are in M). Now if p does not strongly split over A, then by induction $\varphi(\bar{x}, \bar{b}_1) \in p \Rightarrow \varphi(\bar{x}, \bar{b}_i) \in p$ for all i, a contradiction $\text{QED}_{2.9}$

Given a set A, there are boundedly many types which do not split over A:

Observation 2.10. (No use of dependence) Let A be a set. Then there are at most $2^{2^{|A|+|T|}}$ types over $\mathfrak{C}$ which do not split over A. Same is true for splitting replaced with Lascar splitting or strong splitting.

Proof. Let p be a global type which does not split over A. For every formula $\varphi(\bar{x}, \bar{c})$ with parameters, the answer to the question whether or not $\varphi(\bar{x}, \bar{c})$ belongs to p depends only on the type $\mathrm{tp}(\bar{c}/A)$. Since there are at most $2^{|A|+|T|}$ such types, we are done. For Lascar splitting use the same argument with types replaced with Lascar strong types (recall that the number of Lascar strong types over A is also bounded by $2^{|A|+|T|}$ - e.g. Proposition 2.7.5 in [18]); for strong splitting apply in addition Observation 2.9. $\mathrm{QED}_{2.10}$

Recall that a formula *forks* over a set A if it implies a finite disjunction of formulae each of which divides over A. A (partial) type forks over A if it contains a forking formula. In dependent theories forking is strongly related to splitting (see Fact 2.14); still, these notions differ, so we need to state the analogue of Observation 2.8 separately:

Observation 2.11. (No use of dependence)

(i) If a (partial) type over B is finitely satisfiable in $A \subseteq B$, then it does not fork over A. Moreover, if $\varphi(\bar{x}, \bar{b})$ is satisfiable in A, then $\varphi(\bar{x}, \bar{b})$ does not fork over A.

(ii) If a type p over B is definable over $A \subseteq B$, by a good definition d_p then it does not fork over A.

Proof. The first part is very easy. For the second part note that if p forks over A, then every extension q of p to a $(|A| + \aleph_0)^+$-saturated model M containing A divides over A; moreover, the indiscernible sequence exemplifying dividing lies in M. Clearly, taking $q = p|^d M$ we get a contradiction.

$\mathrm{QED}_{2.11}$

The following observation due to Shelah ([15], Observation 5.4) is easy but extremely useful:

Fact 2.12. *In a dependent theory strong splitting implies dividing.*

Proof. Assume $p \in \mathrm{S}(B)$ splits strongly over A, that is, there exists a sequence $I = \langle \bar{b}_i : i < \omega \rangle$ indiscernible over A with $\varphi(\bar{x}, \bar{b}_0), \neg\varphi(\bar{x}, \bar{b}_1) \in p$; then $\psi(\bar{x}, \bar{b}_0\bar{b}_1) = \varphi(\bar{x}, \bar{b}_0) \wedge \neg\varphi(\bar{x}, \bar{b}_1) \in p$ divides over A, since the set

$$\{\varphi(\bar{x}, \bar{b}_{2i}), \neg\varphi(\bar{x}, \bar{b}_{2i+1}) : i < \omega\}$$

is inconsistent by the dependence of T. $\mathrm{QED}_{2.12}$

The other implication is generally not true, as we will see later, unless one works with types over slightly saturated models, in which case the following general fact holds (it will not be of much importance to us):

Fact 2.13. *(No use of dependence) Let A be a set, M be a $(|A|+\aleph_0)^+$-saturated model containing A, $p \in \mathrm{S}(M)$ which forks over A, then it splits strongly over A.*

Proof. Easy. $\mathrm{QED}_{2.13}$

So (recalling also Observation 2.9) we can conclude:

Fact 2.14.
(i) *Let M be a $(|A|+\aleph_0)^+$-saturated model containing A, $p \in \mathrm{S}(M)$. Then p does not split strongly over A if and only if p does not Lascar split over A if and only if p does not fork over A if and only if p does not divide over A.*
(ii) *Let A be such that whenever $\bar{b}_1 \equiv_A \bar{b}_2$, then $\mathrm{Lstp}(\bar{b}_1/A) = \mathrm{Lstp}(\bar{b}_2/A)$ (e.g. A is a model; one can show that in a dependent theory, assuming $A = \mathrm{bdd}^{heq}(A)$ and $\mathrm{tp}(\bar{b}_1/A)$ does not fork over A is enough).*
Let M be a $(|A|+\aleph_0)^+$-saturated model containing A, $p \in \mathrm{S}(M)$. Then p does not split over A if and only if p does not fork over A if and only if p does not divide over A.

One can find much information about the connections between different "pre-independence relations" in dependent theories in Adler [1].

Corollary 2.15. *There are boundedly many global types which do not fork over a given set A.*

Proof. By Observation 2.10 and Fact 2.12. $\mathrm{QED}_{2.15}$

Remark 2.16. Note that while Observation 2.10 is true in any theory, Corollary 2.15 does not have to be true in a theory with the independence property, even if forking behaves nicely in it (e.g. the theory of the random graph). In fact, in a simple theory, a type over a model with a bounded number of global nonforking extensions is stationary, see e.g. [18], Lemma 2.5.15.

Definition 2.17.
(i) Let O a linear order, A a set. We call a sequence $I = \langle \bar{a}_i : i \in O \rangle$ a *nonsplitting/nonforking sequence over A* if it is an indiscernible sequence over A of realizations of p and $\mathrm{tp}(\bar{a}_i/A\bar{a}_{<i})$ does not split (respectively, fork) over A for all $i \in O$.
(ii) If a sequence I is indiscernible over B and nonsplitting/nonforking over $A \subseteq B$, we sometimes say that I is *based on A*.
(iii) Let $p \in \mathrm{S}(B)$ be a type. We call a sequence I a *nonsplitting/nonforking sequence in p* if it is a nonsplitting (respectively, nonforking) sequence over B of realizations of p. We say that it is a sequence in p *nonsplitting/nonforking over A* (or based on A) if it is a sequence of realizations of p indiscernible over B and nonsplitting (respectively, nonforking) over A.

The following fact is a well-known:

Fact 2.18. *(No use of dependence) Let $I = \langle \bar{a}_i : i < \lambda \rangle$ be such that*

- $\mathrm{tp}(\bar{a}_i/A\bar{a}_{<i})$ *does not split over* A
- $\mathrm{tp}(\bar{a}_i/A\bar{a}_{<i}) = \mathrm{tp}(\bar{a}_j/A\bar{a}_{<i})$ *for every* $j \geq i$.

Then I *is a nonsplitting sequence over* A *(that is, it is indiscernible over* A*).*

We will need the following slight modification of the Fact above. We present the short proof for completeness.

Observation 2.19. (No use of dependence) Let $I = \langle \bar{a}_i : i < \lambda \rangle$ be such that

- $\mathrm{tp}(\bar{a}_i/A\bar{a}_{<i})$ does not Lascar-split over A
- $\mathrm{Lstp}(\bar{a}_i/A\bar{a}_{<i}) = \mathrm{Lstp}(\bar{a}_j/A\bar{a}_{<i})$ for every $j \geq i$.

Then I is a indiscernible over A.

Proof. The classical proof works, namely: we prove by induction on k that $\mathrm{Lstp}(\bar{a}_{i_1} \ldots \bar{a}_{i_k}/A) = \mathrm{Lstp}(\bar{a}_{j_1} \ldots \bar{a}_{j_k}/A)$ for every $i_1 < \ldots < i_k, j_1 < \ldots < j_k$. For $k = 1$ this is given.

For $k > 1$, assume wlog $j_k \geq i_k$. By the assumption $\mathrm{Lstp}(\bar{a}_{j_k}/A\bar{a}_{i_1} \ldots \bar{a}_{i_{k-1}}) = \mathrm{Lstp}(\bar{a}_{i_k}/A\bar{a}_{i_1} \ldots \bar{a}_{i_{k-1}})$. By the induction hypothesis $\mathrm{Lstp}(\bar{a}_{i_1} \ldots \bar{a}_{i_{k-1}}/A) = \mathrm{Lstp}(\bar{a}_{j_1} \ldots \bar{a}_{j_{k-1}}/A)$ and by the lack of Lascar splitting $\mathrm{Lstp}(\bar{a}_{j_k}/A\bar{a}_{i_1} \ldots \bar{a}_{i_{k-1}}) = \mathrm{Lstp}(\bar{a}_{j_k}/A\bar{a}_{j_1} \ldots \bar{a}_{j_{k-1}})$, which completes the proof. $\mathrm{QED}_{2.19}$

Observation 2.20. Let $I = \langle \bar{b}_i : i < \omega \rangle$ be a nonforking sequence in $p \in \mathrm{S}(A)$. Then $\mathrm{Av}(I, I \cup A)$ is a nonforking extension of p.

Proof. Let $\varphi(\bar{x}, \bar{b}) \in \mathrm{Av}(I, I \cup A)$ (with $\bar{b} \in I$). By the definition of the average type, $\varphi(\bar{x}, \bar{b}) \in \mathrm{tp}(\bar{b}_k/A\bar{b}_{<k})$ for almost all $k < \omega$. Since I is nonforking, $\varphi(\bar{x}, \bar{b})$ does not fork over A. $\mathrm{QED}_{2.20}$

Fact 2.14 shows that working over slightly saturated models provides us with a very nice picture; unfortunately, this is not the case if one is interested in types over sets (even models), as we shall see for instance in section 6 of the article. This is why we need both nonforking and nonsplitting for slightly different purposes. As the reader will see later, we believe that nonforking plays a deeper role. A major advantage of nonforking over nonsplitting is existence of nonforking extensions, which is well-known and very useful:

Fact 2.21. *(No use of dependence) Let* p *be a partial type over a set* B *which does not fork over* $A \subseteq B$*. Then there exists* $p \in \mathrm{S}(B)$ *which does not fork over* A*.*

Remark 2.22. We will normally use Fact 2.21 when $p \in \mathrm{S}(A')$, $A \subseteq A' \subseteq B$.

It is natural to ask which types have existence and/or uniqueness of nonsplitting extensions. The following general fact will become useful later:

Lemma 2.23. *(No use of dependence)*

Assume $A \subseteq M$*,* M *is* $(|A| + \aleph_0)^+$*-saturated, and* $p \in \mathrm{S}(M)$ *does not split over* A*. Then for every* $M \subseteq B$ *there is a unique extension of* p *to* B *which does not split over* A*.*

Proof. For existence, for every finite tuple $\bar{b} \in B$ introduce a tuple of variables $\bar{y}_{\bar{b}}$ of the same length and denote $r_{\bar{b}}(\bar{y}_{\bar{b}}) = \mathrm{tp}(\bar{b}/M)$.

Let

$$\Sigma = \bigcup_{\bar{b} \in B} r_{\bar{b}}(\bar{y}_{\bar{b}})$$

and

$$\Gamma = p(\bar{x}) \cup \Sigma \cup \{\varphi(\bar{x}, \bar{y}_{\bar{b}}) \leftrightarrow \varphi(\bar{x}, \bar{y}'_{\bar{b}'}) \colon \varphi(\bar{x}, \bar{y}) \in L, \mathrm{tp}(\bar{b}/A) = \mathrm{tp}(\bar{b}'/A)\}$$

For every finite subset B_0 of B find $B'_0 \subseteq M$ satisfying the same type over A, and choose $\bar{a}' \models p{\restriction}A \cup B'_0$. As $p{\restriction}A \cup B'_0$ does not split over A, clearly $\varphi(\bar{a}', \bar{b}') \leftrightarrow \varphi(\bar{a}', \bar{b}'')$ for every $\bar{b}', \bar{b}'' \in B'_0$ satisfying the same type over A. This shows that Γ is finitely satisfiable in M, and therefore consistent. By applying an automorphism over M, we are done.

For uniqueness, let $\varphi(\bar{x}, \bar{y})$ be a formula and $\bar{b} \in B$. Let $\bar{b}' \in M$ realize $\mathrm{tp}(\bar{b}/A)$. Clearly, any nonsplitting extension of p to B chooses $\varphi(\bar{x}, \bar{b})$ if and only if $\varphi(\bar{x}, \bar{b}') \in p$. $\mathrm{QED}_{2.23}$

Note that even the existence in the lemma above can not be taken for granted (if we do not work over an $|A|$-saturated model), even if T is dependent, A itself is a (saturated) model, and $p{\restriction}A$ is generically stable. See more in Discussion 4.9 and Example 6.15.

Another case of uniqueness of nonsplitting extensions occurs for average types:

Lemma 2.24. *Let $I = \langle \bar{b}_i : i < \omega \rangle$ be an indiscernible set over A which is also a nonsplitting sequence. Denote $p = \mathrm{Av}(I, A \cup I)$. Assume that q is a global extension of p which does not split over A. Then $q = \mathrm{Av}(I, \mathfrak{C})$.*

Proof. Denote $B = A \cup I$. Let $\varphi(\bar{x}, \bar{c}) \in q$, and assume towards contradiction $\neg\varphi(\bar{x}, \bar{c}) \in \mathrm{Av}(I, \mathfrak{C})$, so

2.24.1. $\neg\varphi(\bar{b}_i, \bar{c})$ holds for almost all $i < \omega$.

Let $J = \langle \bar{b}'_i : i < \omega \rangle \subseteq M$ be a nonsplitting sequence in q over $BI\bar{c}$ (that is, choose $\bar{b}'_i \models q{\restriction}BI\bar{c}\bar{b}'_{<i}$). Clearly

2.24.2. $\varphi(\bar{b}'_i, \bar{c})$ holds for all $i < \omega$.

Claim 2.24.3. $I^\frown J$ is indiscernible.

Proof. Since I is a nonsplitting sequence and J is nonsplitting over BI, both based on A, it is enough to show that for every $i, j < \omega$ $\bar{b}_i \equiv_{A\bar{b}_{<i}} \bar{b}'_j$, see Fact 2.18. But this is also clear as $\bar{b}'_j \models \mathrm{Av}(I, A \cup I)$ and I is indiscernible over A. $\mathrm{QED}_{2.24}$

Combining all of the above, since I is an indiscernible set, we clearly get a contradiction to dependence.

$\mathrm{QED}_{2.24}$

Following the lemma above, one might want to define stationary types as those having a unique nonsplitting extension over $\mathfrak{C}$. We will see later (e.g. Discussion 4.9) that this definition is wrong, even for generically stable types, one reason being precisely that nonsplitting types do not have to have global nonsplitting (invariant) extensions. Therefore nonforking gives rise to a better notion of stationarity.

Definition 2.25. We call a type $p \in \mathrm{S}(A)$ *stationary* if it has a unique *nonforking* extension to any superset of A.

Let us prove an analogue of Lemma 2.24 for nonforking. It is probably the central result of this section.

First we need to "improve" Fact 2.12 slightly adjusting it to our purposes. Note that we weaken both the assumption and the conclusion (but forking in the conclusion is really all we need).

Observation 2.26. Lascar splitting implies forking.

Proof. Let $p \in \mathrm{S}(B)$ Lascar split over A, and assume it does not fork over A. By Fact 2.21 there exists a global type q extending p which does not fork over A. Being an extension of p, it clearly Lascar splits over A, hence strongly splits by Observation 2.9; a contradiction to Fact 2.12. $\mathrm{QED}_{2.26}$

Lemma 2.27. (i) *Let $I = \langle \bar{b}_i : i < \omega \rangle$ be a nonforking sequence over A which is also an indiscernible set over A. Denote $p = \mathrm{Av}(I, A \cup I)$. Let p^* be a global extension of p which does not fork over A. Then $p^* = \mathrm{Av}(I, \mathfrak{C})$.*

(ii) *Let I be a nonforking sequence over A which is an indiscernible set over A. Then $\mathrm{Av}(I, \mathfrak{C})$ does not fork over A.*

Proof. The second part follows from the first since $\mathrm{Av}(I, A \cup I)$ does not fork over A by Observation 2.20, hence can be extended to a global type which does not fork over A.

For the first part, we are going to repeat the proof of Lemma 2.24 replacing splitting with Lascar-splitting. Denote $B = A \cup I$. Let $\varphi(\bar{x}, \bar{c}) \in q$, and assume towards contradiction $\neg\varphi(\bar{x}, \bar{c}) \in \mathrm{Av}(I, \mathfrak{C})$, so

2.27.1. $\neg\varphi(\bar{b}_i, \bar{c})$ holds for almost all $i < \omega$.

Let $J = \langle \bar{b}'_i : i < \omega \rangle \subseteq M$ be a nonsplitting sequence in q over $B\bar{c}$ (that is, choose $\bar{b}'_i \models q{\restriction}B\bar{c}\bar{b}'_{<i}$). Clearly

2.27.2. $\varphi(\bar{b}'_i, \bar{c})$ holds for all $i < \omega$.

Claim 2.27.3. $I^\frown J$ is indiscernible.

The claim clearly suffices.

In order to prove the claim, we will have to be a bit more careful than in Lemma 2.24 and apply Observation 2.19. So we have to argue that the sequence is Lascar-nonsplitting and Lascar strong type of an element over the previous ones is "increasing".

Lascar-nonsplitting follows from nonforking by Observation 2.26. I is an A-indiscernible sequence, so clearly Lascar strong type of an element is increasing, same for J. So it is again enough to show that for every $i, j < \omega$ $\mathrm{Lstp}(\bar{b}_i/A\bar{b}_{<i}) = \mathrm{Lstp}(\bar{b}'_j/A\bar{b}_{<i})$. But $\bar{b}'_j \models \mathrm{Av}(I, A \cup I)$, so it continues I; hence $\bar{b}'_j$ and $\bar{b}_i$ are of Lascar-distance 1 over $A\bar{b}_{<i}$. $\mathrm{QED}_{2.27}$

The following definition is standard:

Definition 2.28. We call two types p and q *parallel* if they have a common nonforking extension; that is, if there exists a type r which is a nonforking extension of both p and q.

Since we'll be working a lot with definable types, the notion of a Morley sequence with respect to a given definition will come handy:

Definition 2.29. Let $p \in \mathrm{S}(B)$ be a type definable over $A \subseteq B$ by a definition schema d_p, O an order type. Then $I = \langle \bar{a}_i : i \in O \rangle$ is called a *Morley sequence* in p over B *based on* A (with respect to the definition schema d_p) if for every $i \in O$ we have $\bar{a}_i \models p|^d B_i$, where $B_i = B \cup \{\bar{a}_j : j < i\}$ as usual.

The following is pretty clear:

Observation 2.30. Let I be a Morley sequence in p over B with respect to the definition schema d_p, and assume furthermore that $I = \langle \bar{a}_i : i \in O \rangle$ is an indiscernible set. Then for every $i \in O$ we have $\bar{a}_i \models p|^d B \cup \{\bar{a}_j : j \neq i\}$.

3. Finitely satisfiable types

In this section we develop some basic theory of finitely satisfiable types. Notions introduced here are essential for understanding Section 5, but a reader who is not interested in Shelah's approach to "stable" types can easily skip this section in the first reading and proceed to the next one, where the general theory of generically stable types is developed. Those readers would like to see the connection between the two approaches and intend to read this section, are encouraged to also have a look at the Appendix, where we try to motivate viewing types as ultrafilters by passing to the space of measures on the Boolean algebra of definable sets (Keisler measures).

Definition 3.1. Let A, B sets. We denote the boolean algebra of B-definable subsets of A^m by $\mathrm{Def}_m(A, B)$. Let $\mathrm{Def}(A, B) = \bigcup_{m<\omega} \mathrm{Def}_m(A, B)$. We omit B if $B = A$.

Remark 3.2. Note that an ultrafilter $\mathfrak{U}$ on $\mathrm{Def}(\mathfrak{C}, B)$ precisely corresponds to a complete type over B. In order to be consistent with Shelah's notions and terminology, we call this type the *average type* of $\mathfrak{U}$ and denote it by $\mathrm{Av}(\mathfrak{U}, B)$.

Definition 3.3. Let $A \neq \varnothing$ and B be sets, $\mathfrak{U}$ an ultrafilter on $\mathrm{Def}_m(A, B)$. We define the *average type* of $\mathfrak{U}$ over B by

$$\mathrm{Av}(\mathfrak{U}, B) = \left\{\varphi(\bar{x}, \bar{b}) \colon \bar{b} \in B \text{ and } \{\bar{a} \in A^m \colon \varphi(\bar{a}, \bar{b})\} \in \mathfrak{U}\right\}$$

Observation 3.4. For $A, B, \mathfrak{U}$ as above, $\mathrm{Av}(\mathfrak{U}, B) \in \mathrm{S}_m(B)$ finitely satisfiable in A (so if $A \subseteq B$, then $\mathrm{Av}(\mathfrak{U}, B)$ is a coheir of its restriction to A).

Proof. For $\varphi(\bar{x}, \bar{b})$, a formula with m free variables over B, either $\{\bar{a} \in A^M : \neg\varphi(\bar{a}, \bar{b})\}$ or $\{\bar{a} \in A^M : \varphi(\bar{a}, \bar{b})\}$ is in $\mathfrak{U}$, so the average type is complete. Finite satisfiability in A (hence consistency) is also clear. $\mathrm{QED}_{3.4}$

Remark 3.5. Note that in the definition of $p = \mathrm{Av}(\mathfrak{U}, B)$ above, the set A in which p is finitely satisfiable is given by $\mathfrak{U}$. We can also forget A sometimes, since as a complete type over B, p does not depend on A in the following sense: if $A \subseteq A'$, $\mathfrak{U}, \mathfrak{U}'$ ultrafilters on $\mathrm{Def}(A, B)$ and $\mathrm{Def}(A', B)$ respectively, such that $\mathfrak{U} = \mathfrak{U}' \restriction \mathrm{Def}(A, B)$ in the obvious sense, then $\mathrm{Av}(\mathfrak{U}, B) = \mathrm{Av}(\mathfrak{U}', B)$.

Observation 3.6. Let A, B be sets, $p \in \mathrm{S}_m(B)$. Then p is finitely satisfiable in A if and only if for some ultrafilter $\mathfrak{U}$ on $\mathrm{Def}_m(A, B)$ $p = \mathrm{Av}(\mathfrak{U}, B)$.

Proof. If $p = \mathrm{Av}(\mathfrak{U}, B)$ for some ultrafilter $\mathfrak{U}$ on A^m, then clearly p is finitely satisfiable in A. On the other hand, if p is finitely satisfiable in A, it is easy to see that the collection $\{\varphi^{\mathfrak{C}}(\bar{x}, \bar{b}) \cap A^m : \varphi(\bar{x}, \bar{b}) \in p\}$ is an ultrafilter on $\mathrm{Def}_m(A, B)$. $\mathrm{QED}_{3.6}$

Definition 3.7. (i) Let $A \subseteq B$, O an order type. We say that a sequence $I = \langle \bar{a}_i : i \in O \rangle$ is a *Shelah sequence* over B *supported on* A if (denoting $B_i = B \cup \{\bar{a}_j : j < i\}$)
- $\mathrm{tp}(\bar{a}_i / B_i)$ is finitely satisfiable in A
- I is an indiscernible sequence over B

(ii) Let $A \subseteq B$, $p \in \mathrm{S}(B)$ finitely satisfiable in A. We call a sequence I a *Shelah sequence* in p supported on A if it is a Shelah sequence over B supported on A of realizations of p.

Lemma 3.8. (i) *If $p \in \mathrm{S}^m(B)$ finitely satisfiable in A, then there is an infinite Shelah sequence in p over B supported on A. Moreover, for every ultrafilter $\mathfrak{U}$ on $\mathrm{Def}_m(A, \mathfrak{C})$ satisfying $\mathrm{Av}(\mathfrak{U}, B) = p$ (see Observation 3.6), the sequence defined by $\bar{a}_i \models \mathrm{Av}(\mathfrak{U}, B \cup \langle \bar{a}_j : j < i \rangle)$ is such a sequence.*

(ii) *If $p \in \mathrm{S}^m(B)$ finitely satisfiable in A, then there is a Shelah sequence in p over B supported in A of any order type.*

(iii) *$\langle \bar{a}_i : i < \lambda \rangle$ is a Shelah sequence over B based on A if and only if for some ultrafilter $\mathfrak{U}$ on $\mathrm{Def}(A, \mathfrak{C})$, $\bar{a}_i \models \mathrm{Av}(\mathfrak{U}, B\langle \bar{a}_j : j < i \rangle)$.*

Proof. (i) Denote $B_i = B \cup \langle \bar{a}_j : j < i \rangle$.
It is clear that any sequence obtained in this way (say, of length λ) satisfies:
- $\mathrm{tp}(\bar{a}_i / B_i)$ is finitely satisfiable in A
- for $k \leq j < i$, $\mathrm{tp}(\bar{a}_j / B_k) = \mathrm{tp}(\bar{a}_i / B_k)$

Clearly by Observation 2.8 $\mathrm{tp}(\bar{a}_i / B_i)$ does not split over A. Now by Fact 2.18 $\langle \bar{a}_i : i < \lambda \rangle$ is an indiscernible sequence.

(ii) By the previous clause and compactness.

(iii) Let $\langle \bar{a}_i : i < \lambda \rangle$ be a Shelah sequence. Denote $B_i = B \cup \langle \bar{a}_j : j < i \rangle$. By Observation 3.6, for every i there exists $\mathfrak{U}_i$ an ultrafilter on $\mathrm{Def}(A, B_i)$ such that $\mathrm{tp}(\bar{a}_i/B_i) = \mathrm{Av}(\mathfrak{U}_i, B_i)$. Note that for $i < j$, $\mathfrak{U}_i \restriction \mathrm{Def}(A, B_j) = \mathfrak{U}_j$. Let

$$\mathfrak{U}_\lambda = \bigcup_{i<\lambda} \mathfrak{U}_i$$

Then $\mathfrak{U}_\lambda$ is a pre-filter on $\mathrm{Def}(A, B_\lambda)$, in particular on $\mathrm{Def}(A, \mathfrak{C})$. Extending it to an ultrafilter, we are done.

In other words: the union of types $\mathrm{tp}(\bar{a}_i/B_i)$ is a partial type over B_λ finitely satisfiable in A, so it can be extended to a type q over B_λ finitely satisfiable in A. Now let $\mathfrak{U}$ be such that $q = \mathrm{Av}(\mathfrak{U}, B_\lambda)$.

$\mathrm{QED}_{3.8}$

We would like the reader to compare the definition of a Shelah sequence to Definition 2.29. We will see later that generally Shelah sequences and Morley sequences are not the same object, even if both exist.

Remark 3.9. (i) A Shelah sequence in $p \in \mathrm{S}(B)$ supported in A is a nonsplitting sequence in p based on A.

(ii) A Morley sequence in $p \in \mathrm{S}(B)$ based on A is a nonsplitting sequence in p based on A.

Definition 3.10. Let $p \in \mathrm{S}(B)$ finitely satisfiable in $A \subseteq B$. We say that p has *uniqueness* over A if there is a unique type of a Shelah sequence in p based on A.

Remark 3.11. Stationary types have uniqueness.

Observation 3.12. Let $p \in \mathrm{S}(M)$ finitely satisfiable in $A \subseteq M$, M is $|A|^+$-saturated. Then p has uniqueness over A.

Proof. By Lemma 2.23. $\mathrm{QED}_{3.12}$

In [17] Shelah shows the following:

Fact 3.13. *Let $p \in \mathrm{S}(A)$ finitely satisfiable in A and definable over A. Then p has uniqueness.*

Note that A is just a set, so there is no reason why there would be only one nonforking (or nonsplitting) extension of p to an arbitrary superset; in fact, this is generally false, see Example 5.5 below. Moreover, it is generally false that a Shelah sequence and a Morley sequence in p obtained from extending p by its definition over A have the same type. Still, there is a unique Shelah sequence.

4. Generically stable types

In this section we propose an approach to generically stable types different from [17] which does not require working with finitely satisfiable types. The definition below is more general and might seem weaker than the on given by Shelah, but as it turns out, they give rise to the same notion. See section 5 for more details.

Definition 4.1. We call a type $p \in \mathrm{S}(A)$ *generically stable* if there exists a nonforking sequence $\langle \bar{b}_i : i < \omega \rangle$ in p (over A) which is an indiscernible set.

Lemma 4.2. *Let $I = \langle \bar{b}_i : i < \omega \rangle$ be an indiscernible set over a set A, $C \supseteq A$. Then $p = \mathrm{Av}(I, C)$ is definable over $\cup I$.*

Proof. Let $\varphi(\bar{x}, \bar{y})$ be a formula and let $k = k_\varphi$ be as in Observation 2.5. Now clearly for every $\bar{c} \in C$

$$\varphi(\bar{x}, \bar{c}) \in \mathrm{Av}(I, C)$$

if and only if

$$|\{i < 2k \colon \models \varphi(\bar{b}_i, \bar{c})\}| \geq k$$

if and only if

$$\bigvee_{u \subset 2k, |u| = k} \bigwedge_{i \in u} \varphi(\bar{b}_i, \bar{c})$$

So p is definable over I by the schema

$$d_p \bar{x} \varphi(\bar{x}, \bar{y}) = \bigvee_{u \subset 2k_\varphi, |u| = k_\varphi} \bigwedge_{i \in u} \varphi(\bar{b}_i, \bar{y})$$

as required. $\mathrm{QED}_{4.2}$

Lemma 4.3. *Let $p \in \mathrm{S}(A)$ be generically stable. Then p is properly definable (definable by a good definition) almost over A.*

Proof. Let $I = \langle \bar{b}_i : i < \omega \rangle$ be a nonforking indiscernible (over A) set in p.

Let $\varphi(\bar{x}, \bar{y})$ be a formula, then p is definable over I as in Lemma 4.2 by

$$\vartheta(\bar{y}, \bar{b}_{<2k}) = d_p \bar{x} \varphi(\bar{x}, \bar{y}) = \bigvee_{u \subset 2k_\varphi, |u| = k_\varphi} \bigwedge_{i \in u} \varphi(\bar{b}_i, \bar{y})$$

.

Claim 4.3.1. $\vartheta(\bar{x}, \bar{b}_{<2k})$ as above is almost over A.

Note that once we have proven the Claim we are done: p is definable almost over A by a definition which is clearly good (it defines $\mathrm{Av}(I, \mathfrak{C})$).

For the proof of the Claim note that otherwise we would have unboundedly many pairwise nonequivalent automorphic copies of ϑ over A. In other words, we would have an unbounded sequence of automorphisms $\langle \sigma_\alpha \rangle$ over A such that $\{\vartheta_\alpha = \sigma_\alpha(\vartheta)\}$ are pairwise nonequivalent. Let $I_\alpha = \sigma_\alpha(I)$, $p_\alpha = \mathrm{Av}(I_\alpha, I_\alpha \cup A)$. By Lemma 2.27 $q_\alpha = \mathrm{Av}(I_\alpha, \mathfrak{C})$ all do not fork over A. Note that q_α is definable by ϑ_α and therefore are all distinct. So $\langle q_\alpha \rangle$ is an unbounded sequence of global types all of which do not fork over A, which is a contradiction to Corollary 2.15.

$\text{QED}_{4.3}$

Note that all we used in the proof of Lemma 4.3 is that there exists an indiscernible set I in p such that $\text{Av}(I, \mathfrak{C})$ does not fork (or split) over A. So the following is a corollary of the proof of Lemma 4.3:

Corollary 4.4. *Let $p \in \text{S}(A)$, I an indiscernible set in p such that $\text{Av}(I, \mathfrak{C})$ does not fork/split over A. Then p is properly definable almost over A.*

Let is summarize the Lemmas above:

Corollary 4.5. *Let $p \in \text{S}(A)$ be generically stable, I a nonforking indiscernible set in p. Then there exists a definition schema d_p over I, almost over A, such that for every set C, $\text{Av}(I, C \cup I) = p|^d(C \cup I)$.*

This allows us to speak about definitions and free extensions instead of averages, which makes our lives quite a bit simpler. One important consequence is stationarity of generically stable types. Recall that we defined stationarity using *nonforking*.

First, let us recall and slightly rephrase Lemma 2.27:

Corollary 4.6. *Let $p \in \text{S}(A)$ be generically stable, I a nonforking indiscernible set in p. Then $\text{Av}(I, A \cup I)$ has a unique extension to $\mathfrak{C}$ which does not fork over A. This extension equals $\text{Av}(I, \mathfrak{C})$.*

Now we proceed to the main stationarity result.

Proposition 4.7. *Let $p \in \text{S}(A)$ be a generically stable type witnessed by a nonforking indiscernible set I such that the definition schema d_p as in Corollary 4.5 is* over A *(e.g. $A = \text{acl}(A)$). Then p is stationary.*

Proof. We aim to show that p has a unique nonforking extension to any superset of A. By existence of nonforking extensions (Fact 2.21) and Corollary 4.6, it is enough to show that the only nonforking extension of p to $A \cup I$ is $\text{Av}(I, A \cup I)$. By Fact 2.12 it is enough to show that $\text{Av}(I, A \cup I)$ is the only extension of p to $A \cup I$ which does not *split strongly* over A. Denote $B = A \cup I$, $B_k = A \cup \langle \bar{b}_i : i < k \rangle$ for $k \leq \omega$.

Let $\bar{b}' \models p$, $\text{tp}(b'/B)$ does not split strongly over A. We show by induction on k that $\text{tp}(\bar{b}'/B_k) = \text{Av}(I, B_k)$. There is nothing to show for $k = 0$.

Assume the claim for k, and suppose $\varphi(\bar{b}', \bar{b}_0, \ldots, \bar{b}_k, \bar{a})$ holds. Let $\psi(\bar{x}, \bar{b}_{<k}, \bar{b}', \bar{a}) = \varphi(\bar{b}', \bar{b}_0, \ldots, \bar{b}_{k-1}, \bar{x}, \bar{a})$, so

4.7.1. $\psi(\bar{b}_k, \bar{b}_{<k}, \bar{b}', \bar{a})$ holds.

Note that since $\text{tp}(\bar{b}'/B)$ doesn't split strongly over A, the set $\langle \bar{b}_i \colon i \geq k \rangle$ is indiscernible over $B_k \bar{b}'$: for every $i_1, \ldots, i_\ell$ and $j_1, \ldots, j_\ell$ all greater or equal to k, we have

$$\text{Lstp}(\bar{b}_{i_1} \ldots \bar{b}_{i_\ell}/B_k) = \text{Lstp}(\bar{b}_{j_1} \ldots \bar{b}_{j_\ell}/B_k)$$

(moreover, their Lascar distance is 1) and therefore by the lack of strong splitting

$$\bar{b}'\bar{b}_{i_1}\dots\bar{b}_{i_\ell} \equiv_{B_k} \bar{b}'\bar{b}_{j_1}\dots\bar{b}_{j_\ell}$$

which precisely means

$$\bar{b}_{i_1}\dots\bar{b}_{i_\ell} \equiv_{B_k\bar{b}'} \bar{b}_{j_1}\dots\bar{b}_{j_\ell}$$

So by 4.7.1 we see that $\psi(\bar{b}_\ell, \bar{b}_{<k}, \bar{b}', \bar{a})$ holds for all ℓ big enough, and therefore

4.7.2. $\psi(\bar{x}, \bar{b}_{<k}, \bar{b}', \bar{a}) \in \mathrm{Av}(I, B\bar{b}')$.

Therefore (denoting $q = \mathrm{Av}(I, \mathfrak{C})$), $d_q\bar{x}\psi(\bar{x}, \bar{b}_{<k}, \bar{b}', \bar{a})$ holds, where the definition is over A (by the assumptions of the Proposition). So we get $\theta(\bar{y}) = d_q\bar{x}\psi(\bar{x}, \bar{b}_{<k}, \bar{y}, \bar{a})$ is in $\mathrm{tp}(\bar{b}'/B_k)$ and therefore (by the induction hypothesis) is in $\mathrm{Av}(I, B_k)$, which we think now as of a type in $\bar{y}$. This means that $d_q\bar{x}\psi(\bar{x}, \bar{b}_{<k}, \bar{b}_\ell, \bar{a})$ holds for almost all ℓ, and therefore (since d_q defines $\mathrm{Av}(I, \mathfrak{C})$) we have $\psi(\bar{x}, \bar{b}_{<k}, \bar{b}_\ell, \bar{a}) \in \mathrm{Av}(I, B)$ for almost all ℓ. Let ℓ be such, so by the definition of average type, there exists an m such that $\psi(\bar{b}_m, \bar{b}_{<k}, \bar{b}_\ell, \bar{a})$, that is, $\varphi(\bar{b}_\ell, \bar{b}_{<k}, \bar{b}_m, \bar{a})$ holds. Since I is an indiscernible set, we get $\varphi(\bar{b}_m, \bar{b}_{<k}, \bar{b}_k, \bar{a})$ for all m big enough, and therefore

4.7.3. $\varphi(\bar{x}, \bar{b}_{\leq k}, \bar{a}) \in \mathrm{Av}(I, B)$

as required.

$\mathrm{QED}_{4.7}$

Corollary 4.8. *A generically stable type over an algebraically closed set is stationary.*

Discussion 4.9. From examining the proofs it might seem like we have shown that a generically stable type p over an algebraically closed set A has a unique nonsplitting (or not strongly splitting) extension over any set, and therefore in particular every nonsplitting sequence in p is an indiscernible set, etc; but this is not the case. The reason is that if $I = \langle \bar{b}_i : i < \omega \rangle$ is a nonsplitting indiscernible set in p, a nonsplitting extension of p to $A\bar{b}_0$ does not need have an extension over I which does not split over A. We will come back to this phenomenon in section 6 while discussing examples of generically stable types. Let us formulate precise statements that do follow from the analysis above:

Corollary 4.10. *Let $p \in \mathrm{S}(A)$ be a generically stable type which is definable over A, $C \supseteq A$ is a set containing an infinite Morley sequence I in p (or just a nonforking sequence in p which is an indiscernible set). Then p has a unique extension to C which does not split strongly over A. This extension equals $q = \mathrm{Av}(I, C)$.*

In particular, p has a unique extension to any $(|A|+\aleph_0)^+$-saturated model M containing A which does not split strongly (equivalently, Lascar split) over A. This unique extension is definable over A and equals $p|^d M$.

If A is e.g. a model (can be weakened to $A = \mathrm{bdd}^{heq}(A)$) then strong splitting above can be replaced with splitting.

The following is an easy consequence of stationarity:

Corollary 4.11. *A nonforking extension of a generically stable type is generically stable.*

Proof. Clearly, every extension of a generically stable type $p \in \mathrm{S}(A)$ to the algebraic closure of A is generically stable; now use stationarity. $\mathrm{QED}_{4.11}$

Recall that we call two types *parallel* if they have a common nonforking extension.

Lemma 4.12. *Let $p \in \mathrm{S}(A)$ be a type, then p is generically stable if and only if every q parallel to p is generically stable.*

Proof. Let $q \in \mathrm{S}(B)$ be parallel to p and let $r \in \mathrm{S}(C)$ witness this, that it, $A, B \subseteq C$, r extends p and q and does not fork over both A and B. Without loss of generality $C = \mathrm{acl}(C)$.

By the previous Corollary, r is generically stable. Fix M a $(|C| + |T|)^+$-saturated model containing C. Let I be a nonforking sequence (set) in r contained in M, then r has a unique nonforking extension to M which equals $\mathrm{Av}(I, M)$.

Since r does not fork over B, by Fact 2.21 there exists $q^* \in \mathrm{S}(M)$ extending r which does not fork over B. Clearly q^* does not fork over C and therefore $q^* = \mathrm{Av}(I, M)$.

So we've shown that $\mathrm{Av}(I, M)$ does not fork over B; applying Corollary 4.4, $\mathrm{Av}(I, M)$ is definable almost over B, so I is a Morley (and therefore nonforking) sequence in q which is an indiscernible set, hence q is generically stable. $\mathrm{QED}_{4.12}$

Corollary 4.13. *(Transitivity of forking for generically stable types) Let $p \in \mathrm{S}(C)$, $A \subseteq B \subseteq C$, one of $p, p\restriction B, p\restriction A$ is generically stable, p does not fork over B and $p\restriction B$ does not fork over A. Then all of the above three types are generically stable and p does not fork over A.*

Proof. Easy at this point. $\mathrm{QED}_{4.13}$

Recall that by a well-known result of Kim, transitivity of forking implies simplicity of T, therefore one can't expect forking to be transitive in general in a dependent unstable theory.

Note that combining all the results of this section one can quite easily deduce properties of stable independence relation based on forking for realizations of generically stable types; we will come back to this issue in section 7.

5. Generically stable types - Shelah's approach

The following definition is given in [17]:

Definition 5.1. A type $p \in \mathrm{S}(B)$ is called *Shelah-stable* if there exists an infinite Shelah sequence in p which is an indiscernible set.

Remark 5.2. Note that in particular p is finitely satisfiable in B.

Shelah shows in [17] that:

Fact 5.3. *If p is a Shelah-stable type, then p is definable, hence has uniqueness, so every Shelah sequence in it is an indiscernible set.*

A natural particular case of a finitely satisfiable type is a type over a model. The following lemma will help us understand Shelah-stable types over slightly saturated models:

Lemma 5.4. *Let M be $|A|^+$-saturated, and $p \in \mathrm{S}(M)$ finitely satisfiable in A. Assume furthermore that p is definable over A. Then p is Shelah-stable.*

Proof. Let $\langle \bar{a}_i : i < \omega \rangle$ a Morley sequence in p based on A (recall that p is definable over A). Clearly it is a nonplitting sequence. Since by Lemma 2.23 p has a unique extension to any superset of M which does not split over A, $\langle \bar{a}_i : i < \omega \rangle$ is also a Shelah sequence.

Let us show for instance that $\bar{a}_0\bar{a}_1 \equiv \bar{a}_1\bar{a}_0$, and even $\bar{a}_0\bar{a}_1 \equiv_M \bar{a}_1\bar{a}_0$. Since the type $\mathrm{tp}(\bar{a}_1/Ma_0)$ is an heir of p (as it is definable by the same definition scheme), and since M is a model, it follows that the type $\mathrm{tp}(\bar{a}_0/M\bar{a}_1)$ is a co-heir of p; moreover it is *the* coheir of p since p has uniqueness by Observation 3.12. In other words, $\bar{a}_1\bar{a}_0$ start a Shelah sequence in p, and by uniqueness $\mathrm{tp}(\bar{a}_1\bar{a}_0/M) = \mathrm{tp}(\bar{a}_0\bar{a}_1/M)$, as required. QED$_{5.4}$

In Lemma 5.4 it is necessary to assume that M is saturated over A. In general it is not true that if p is both definable over A and finitely satisfiable in it then p is Shelah-stable:

Example 5.5. Let T be the theory of dense linear orderings with no endpoints, $A = \mathbb{Q}$, $p \in \mathrm{S}(A)$ "the type at infinity", i.e. $[x > a] \in p$ for all $a \in A$. Then p is clearly finitely satisfiable in A and definable over A (in fact, over the empty set), but is not Shelah-stable. It still has uniqueness, and any Shelah sequence in p based on A is simply a descending sequence. A Morley sequence in p is, on the other hand, an increasing sequence.

So in order to obtain an "if and only if" criterion, we will have to strengthen Fact 5.3 slightly (the proof can be extracted from Shelah's proof of Fact 5.3):

Theorem 5.6. *Let $p \in \mathrm{S}(A)$ finitely satisfiable in A. Then the following are equivalent:*

(i) *p is Shelah-stable*
(ii) *Any coheir of p over $\mathfrak{C}$ is definable over A*
(iii) *Some coheir of p over some $(|A|+|T|)^+$-saturated model containing A is definable over A*

Proof. $(ii) \Rightarrow (iii)$ is clear and $(iii) \Rightarrow (i)$ follows from Lemma 5.4. So we only need to prove $(i) \Rightarrow (ii)$. So assume p is Shelah-stable, and let q be a coheir of p over $\mathfrak{C}$; then $q = \mathrm{Av}(\mathfrak{U}, \mathfrak{C})$ for some ultrafilter $\mathfrak{U}$ on $\mathrm{Def}(\mathfrak{C}, A)$.

Let $\langle \bar{b}_i : i < \omega \rangle$ be a $\mathfrak{U}$-Shelah sequence in p over B. Since p is Shelah-stable, $\langle \bar{b}_i \rangle$ is an indiscernible set. Let $\varphi(\bar{x}, \bar{y})$ be a formula, and let Δ be a finite set of formulae containing $\exists \bar{y} \varphi(\bar{x}, \bar{y})$. Let $k = k_\varphi < \omega$ be as in Observation 2.5.

Since p is finitely satisfiable in A (using e.g. [17], 1.16(1)) we can find $\langle \bar{a}_i : i < 2k \rangle$ in A such that the sequence $\bar{a}_0 \ldots \bar{a}_{2k-1}$ has the same Δ-type as $\bar{b}_0 \ldots \bar{b}_{2k-1}$, and so $\langle \bar{a}_i : i < 2k \rangle^\frown \langle \bar{b}_i : i \in I \rangle$ is a Δ-indiscernible set, and $k = k_\varphi$ is still as in Observation 2.5 for this prolonged sequence.

Claim 5.6.1. Let $\bar{c} \in \mathfrak{C}$, then the following are equivalent

(*a*) $|\{i < 2k\colon \models \varphi(\bar{a}_i, \bar{c})\}| \geq k$
(*b*) $|\{i < 2k\colon \models \varphi(\bar{b}_i, \bar{c})\}| \geq k$
(*c*) $\varphi(\bar{x}, \bar{c}) \in \operatorname{Av}(\mathfrak{U}, \mathfrak{C})$

Proof. $(a) \iff (b)$ is true by the choice of k and the sequence $\langle \bar{a}_i \rangle$.

Let $\langle \bar{b}'_i : i < \omega \rangle$ be a $\mathfrak{U}$-Shelah sequence in p over $B\bar{c}$. Then

$$\varphi(\bar{x}, \bar{c}) \in \operatorname{Av}(\mathfrak{U}, \mathfrak{C})$$

if and only if

$$\{\bar{a} \in A\colon \models \varphi(\bar{a}, \bar{c})\} \in \mathfrak{U}$$

if and only if

$$\varphi(\bar{b}'_i, \bar{c}) \text{ for all } i$$

Note that the sequences $\langle \bar{b}_i \rangle$ and $\langle b'_i \rangle$ have the same type over B (by uniqueness, and even without it: both are Shelah sequences with respect to the same ultrafilter).

So since $\langle \bar{a}_i \rangle \subseteq A$, $(a) \iff (b)$ is still true for all $\bar{c}$ if $\langle \bar{b}_i \rangle$ is replaced with $\langle \bar{b}'_i \rangle$ (as the sequence $\langle \bar{a}_i : i < 2k \rangle^\frown \langle \bar{b}'_i : i < \omega \rangle$ is Δ-indiscernible and has the same type over A as $\langle \bar{a}_i : i < 2k \rangle^\frown \langle \bar{b}_i : i < \omega \rangle$). Therefore,

$$\varphi(\bar{x}, \bar{c}) \in \operatorname{Av}(\mathfrak{U}, \mathfrak{C})$$

if and only if

$$\varphi(\bar{b}'_i, \bar{c}) \text{ for all i}$$

if and only if

$$\varphi(\bar{a}_i, \bar{c}) \text{ for the majority of } \bar{a}_i\text{'s}$$

which shows $(a) \iff (c)$. $\text{QED}_{5.6}$

So clearly the φ-type $\operatorname{Av}_\varphi(\mathfrak{U}, \mathfrak{C})$ ($\operatorname{Av}(\mathfrak{U}, \mathfrak{C})$ restricted to φ) is definable over both the sequence $\langle \bar{b}_0, \ldots, \bar{b}_{2k-1} \rangle$ and the sequence $\langle \bar{a}_0, \ldots, \bar{a}_{2k-1} \rangle$ by a φ-formula: more specifically, $\varphi(\bar{x}, \bar{c}) \in \operatorname{Av}(\mathfrak{U}, \mathfrak{C})$ if and only if

$$\bigvee_{u \subseteq 2k, |u| = k} \bigwedge_{i \in u} \varphi(\bar{b}_i, \bar{c})$$

if and only if

$$\bigvee_{u \subseteq 2k, |u| = k} \bigwedge_{i \in u} \varphi(\bar{a}_i, \bar{c})$$

This shows that p is definable both over the sequence $\langle \bar{b}_i : i < \omega \rangle$ and over A. $\text{QED}_{5.6}$

Corollary 5.7. *A Shelah-stable type is stationary.*

Proof. Note that in the proof of the theorem we showed precisely that the average type of a Shelah sequence in p (which is an indiscernible set) is definable over A. So the conclusion follows by Proposition 4.7. $\text{QED}_{5.7}$

Remark 5.8. Note that this doesn't follow immediately from Corollary 4.8: a Shelah-stable type does not have to defined be over an algebraically closed set.

Lemma 5.9. *Let p be a generically stable type definable over A and finitely satisfiable in A, then a Morley sequence in p is a Shelah sequence in p and vice versa.*

Proof. The desirable conclusion is an easy consequence of stationarity, Proposition 4.7. $\text{QED}_{5.9}$

We conclude with a simple and natural characterization of Shelah-stable types:

Corollary 5.10. *A type $p \in \mathrm{S}(A)$ is Shelah-stable if and only if it is generically stable and finitely satisfiable in A.*

Proof. The "only if" direction is clear. So assume p is finitely satisfiable in A and generically stable; let p' be an extension of p to $\mathrm{acl}(A)$ finitely satisfiable in A. Since $\mathrm{Aut}(\mathfrak{C}/A)$ acts transitively on the set of extensions of p to $\mathrm{acl}(A)$ and p is properly definable over $\mathrm{acl}(A)$, we have that p' is definable over $\mathrm{acl}(A)$, and there exists a Morley sequence I with respect to this definition which is an indiscernible set; by by Lemma 5.9 it is also a Shelah sequence, so we're done. $\text{QED}_{5.10}$

Discussion 5.11. So we have just shown that the two approaches to "stable" types coincide, and from now on will basically stop using the term "Shelah-stable", although sometimes it is convenient in order to indicate that the type is finitely satisfiable in its domain.

6. Generically stable types - summary and examples

We have chosen to define generically stable types using nonforking indiscernible sets. There are two reasons for this. First, we find this definition compact and elegant. The second reason is that it is general: we do not require the sequence to be of any specific kind. There was a price to the generality: we had to work in order to show important properties (such as definability), which required understanding to some extent general behavior of nonforking in dependent theories. But now the picture is much more complete, and we would like to begin this section by stating several of possible alternative definitions, some of which provide us with powerful machinery, while others are easier to check:

Theorem 6.1. *Let $p \in \mathrm{S}(A)$. The Following Are Equivalent:*

(i) *p is generically stable, that is, there exists a nonforking sequence in p which is an indiscernible set.*
(ii) *Every nonforking sequence in p is an indiscernible set.*
(iii) *p is definable over $\mathrm{acl}(A)$ and some Morley sequence in p is an indiscernible set.*
(iv) *p is definable over $\mathrm{acl}(A)$ and every Morley sequence in p is an indiscernible set.*
(v) *There exists an indiscernible set I in p such that $\mathrm{Av}(I, \mathfrak{C})$ does not fork over A.*
(vi) *There exists an indiscernible set I in p such that $\mathrm{Av}(I, \mathfrak{C})$ does not split over A.*

(vii) *There is a nonforking extension of* p *to a* $(|A|+|T|)^+$*-saturated model* M *which is definable over* $\mathrm{acl}(A)$ *and finitely satisfiable in some* $M_0 \prec M$ *satisfying* $A \subseteq M_0$ *and* $|M_0| = |A| + |T|$.

(viii) *There is a nonforking extension of* p *to a* $(|A|+|T|)^+$*-saturated model* M *which is both definable over and finitely satisfiable in some countable indiscernible set contained in* M.

(ix) *Every nonforking extension of* p *to a model containing* A *is Shelah-stable.*

(x) *Some nonforking extension of* p *to a model containing* A *is Shelah-stable.*

Proof. All the equivalences are easy at this point, we will sketch the proofs:

(i) and (ii) are equivalent by stationarity and the fact that $\mathrm{Aut}(\mathfrak{C}/A)$ acts transitively on the set of extensions of p to $\mathrm{acl}(A)$.

(i) $\Rightarrow$(iii) is basically Corollary 4.5.

(iii) and (iv) are again equivalent by stationarity.

(iii) $\Rightarrow$(i) is clear.

(i) $\Rightarrow$(v), (i) $\Rightarrow$(vi) are easy by Corollary 4.5.

(v) $\Rightarrow$(iii), (vi) $\Rightarrow$(iii): Corollary 4.4.

(i) $\Rightarrow$(vii),(i) $\Rightarrow$(viii) are again easy.

(vii) $\Rightarrow$(i), (viii)$\Rightarrow$(i) follow from Lemma 5.4 and Lemma 4.12.

The equivalences with (ix) and (x) use stationarity, Lemma 4.12 and Corollary 5.10.

$\mathrm{QED}_{6.1}$

Let us now summarize our knowledge on generically stable types:

Summary 6.2. *Let* $p \in \mathrm{S}(A)$ *generically stable.*

- p *is definable by a good definition almost over* A.
- *There are boundedly many (at most* $2^{|T|}$*) global extensions of* p *which do not split strongly/fork over* A.
- *Any global extension* p' *of* p *which does not split strongly/fork over* A *is definable over* $\mathrm{acl}(A)$. *If* p *is finitely satisfiable in* A, *then* p' *is definable over* A.
- *If* $A = \mathrm{acl}(A)$ *then* p *is stationary and its unique global extension* p' *which does not split strongly/fork over* A *is a free extension with respect to some/any good definition over* A.
- *If* p *is finitely satisfiable in* A *then* p *is stationary, and its unique global extension which does not split strongly/fork over* A, *is both its coheir and a free extension with respect to some/any good definition over* A.
- *Any nonforking extension of* p *is generically stable. Moreover, any* q *which is parallel to* p *is generically stable.*
- *Any nonsplitting extension of* p *to a set containing an indiscernible set in* p *(in particular a slightly saturated model) is generically stable.*
- *Any nonforking sequence in* p *is an indiscernible set and a Morley sequence. Any two nonforking sequences in* p *have the same type.*

- *If p is finitely satisfiable in A, any Morley sequence in p is a Shelah sequence and vice versa.*
- *The unique global nonsplitting extension of a stationary generically stable type p is finitely satisfiable in and definable over any Morley (equivalently, Shelah) sequence in p.*
- *If $A = M$ a $|T|^+$-saturated model, then p is generically stable if and only if p is both definable over and finitely satisfiable in some $A_0 \subseteq A$ of cardinality $|T|$. (Note that in general being finitely satisfiable in and definable over M does not imply generic stability).*

We would like to point out particular cases which have been studied in more detail and have become central in the recent study of theories interpretable in o-minimal structures and in the theory of algebraically closed valued fields. Although some of the notions below have been extensively studied by many people over the years (and we try to mention this), we will adopt the more recent terminology, partially due Hasson and Onshuus from [5].

We begin with the strongest version of stability which is based on the notion of a "stable set". "Stable partial types" are originally due to Lascar and Poizat [11]. Let us recall the definition (since we restrain from using the term "stable type", we'll call this notion "Lascar-Poizat stable"):

Definition 6.3. A partial type $\pi(\bar{x})$ is called *Lascar-Poizat stable* (*LP-stable*) if every extension of it to a global type is definable.

We will come back to this general concept later. The most common terminology in case $\pi(\bar{x})$ is finite (that is, a single formula) is "a stable and stably embedded set". We give a definition which in our opinion justifies the name "stable" very well:

Definition 6.4. A definable set D defined by a formula $\theta(\bar{x})$ (maybe with parameters) is said to be *stable* if the induced structure on D (including all the relations definable on D with external parameters) is stable.

We state the following fact without a proof. Most of the equivalences are well-known. Some (which are true in any theory) were already explored by Lascar and Poizat. Others (which require dependence) have been discovered more recently. All references and some proofs can be found in Onshuus and Peterzil [12]. Note that Proposition 6.9 below provides a generalization of some of the following equivalences.

Fact 6.5. *Let D be a definable set defined by a formula $\theta(\bar{x})$. The Following Are Equivalent:*

(i) *D is stable.*
(ii) *D is stable and stably embedded (that is, every externally definable subset of D is definable with parameters from D).*
(iii) *D is Lascar-Poizat stable, that is, every global type extending $\theta(\bar{x})$ is definable.*
(iv) *For every formula $\varphi(\bar{x}, \bar{y})$, the formula $\theta(\bar{x}) \wedge \varphi(\bar{x}, \bar{y})$ is a stable formula.*

(v) *D equipped with all the relations on it which are D-definable in $\mathfrak{C}$ is a stable structure*

(vi) *D equipped with all the externally definable relations does not have the strict order property. That is, there is no definable (maybe, with external parameters) partial order with infinite chains on D.*

(vii) *D equipped with all the D-definable relations does not have the strict order property. That is, there is no definable (with no external parameters) partial order with infinite chains on D.*

The following notion is due to Hasson and Onshuus, see [5].

Definition 6.6. A type $p \in \mathrm{S}(A)$ is called *seriously stable* if there exists a stable set D defined by a formula $\theta(\bar{x})$ (maybe with parameters) such that $\theta(\bar{x}) \in p$.

Obviously, this is a very strong version of stability for a type. We'll see later that this is stronger than (and not equivalent to) the type being Lascar-Poizat stable.

In [5] Onshuus and Hasson work with the generalization of the notion of a stable set in a dependent theory based on Fact 6.5(vi). We'll see in Proposition 6.9 that just like in the case of stable sets, this definition is equivalent to LP-stability. The choice of the name might seem peculiar at first, a more natural term would probably be "p does not admit the strict order property"; it will be justified by clauses (iii) and (iv) of Proposition 6.9.

Definition 6.7. A type $p \in \mathrm{S}(A)$ is called *hereditarily stable* if there is no definable (maybe with external parameters) partial order with infinite chains on the set of realizations of p.

In order to show that this definition is equivalent to what one normally thinks of as stability, we first have to recall that in a dependent theory the order property implies the strict order property:

Fact 6.8. *Let $\varphi(\bar{x}, \bar{y})$ be an unstable formula witnessed by indiscernible sequences $I = \langle \bar{a}_i : i \in \mathbb{Q} \rangle$, $J = \langle \bar{b}_i : i \in \mathbb{Q} \rangle$. Then there exists a formula $\vartheta(\bar{y}_1, \bar{y}_2, \bar{c})$ such that*

- *ϑ defines on $\mathfrak{C}$ a quasi-order*
- *There exists an infinite subsequence $J' \subseteq J$ which is linearly ordered by ϑ*
- *$\bar{c} \subseteq \cup J$*

In fact,

$$\vartheta(\bar{y}_1, \bar{y}_2) = \forall \bar{x}[\psi(\bar{x}, \bar{y}_1) \to \psi(\bar{x}, \bar{y}_2)]$$

for some $\psi(\bar{x}, \bar{y}, \bar{c})$ such that

- *$\psi(\bar{x}, \bar{y}, \bar{c})$ implies $\varphi(\bar{x}, \bar{y})$*
- *ψ has the strict order property*
- *$\bar{c} \subseteq \cup J$*

Proof. This is all contained in the proof of Shelah's classical theorem that in a dependent theory an unstable formula gives rise to the strict order property, but we would rather refer the reader to the slightly more general result by Onshuus and Peterzil, Lemma 4.1 in [12]. It states that if $\varphi(\bar{x}, \bar{y})$ is unstable then there exists a strengthening of it (which we call here $\psi(\bar{x}, \bar{y}, \bar{c})$) with the strict order property; reading the proof carefully, one sees both that the additional parameters are taken from J and that the strict order property is exemplified by an indiscernible sequence which is an infinite subsequence of J. Now defining $\vartheta(\bar{y}_1, \bar{y}_2)$ as above, we're clearly done. $\text{QED}_{6.8}$

We can now state the non-surprising analogue (and generalization) of Fact 6.5. Some of the equivalences below appear also in [5].

Proposition 6.9. *Let $p \in \mathrm{S}(A)$. The Following Are Equivalent:*

(i) *p is LP-stable.*
(ii) *For every $B \supseteq A$, p has at most $|B|^{\aleph_0}$ extensions in $\mathrm{S}(B)$.*
(iii) *Every extension of p is LP-stable.*
(iv) *Every extension of p is generically stable.*
(v) *Every indiscernible sequence in p is an indiscernible set.*
(vi) *There is no formula $\varphi(\bar{x}, \bar{y})$ (with parameters from $p^{\mathfrak{C}}$) and an indiscernible sequence $\langle \bar{a}_i : i < \omega \rangle$ in $p^{\mathfrak{C}}$ such that*

$$i < j < \omega \Rightarrow \varphi(\bar{a}_i, \bar{a}_j) \wedge \neg\varphi(\bar{a}_j, \bar{a}_i)$$

(vii) *There is no formula $\varphi(\bar{x}, \bar{y})$ (with parameters from $\mathfrak{C}$) exemplifying the order property with respect to indiscernible sequences $I = \langle \bar{a}_i : i < \omega \rangle$ and $J = \langle \bar{b}_i : i < \omega \rangle$ with $\cup J \subseteq p^{\mathfrak{C}}$. We call this "$p$ does not admit the order property".*
(viii) *p is hereditarily stable as in Definition 6.7; that is, p does not admit the strict order property.*
(ix) *On the set of realizations of p there is no definable (with no external parameters) partial order with infinite chains.*

Proof. The equivalence of (i) and (ii) is well-known (Theorem 4.4 in [11]).

(i) $\Longleftrightarrow$(iii) is trivial.

(ii) $\Rightarrow$(vii): assume p admits the order property. Using the standard argument, for every infinite $\lambda \geq |A|$ one can easily construct a collection of λ^+ extensions of p over a set of cardinality $\leq \lambda$; clearly, this contradicts (ii).

(vii) $\Rightarrow$(vi): Clear.

(vi) $\Rightarrow$(v) is standard: e.g., taking an indiscernible sequence in $\langle \bar{a}_i : i < \omega + \omega \rangle$ in p which is not an indiscernible set, we may assume that for some formula $\varphi(\bar{z}, \bar{x}, \bar{y}, \bar{z}')$ and $n < \omega$ we have $\varphi(\bar{a}, \bar{a}_n, \bar{a}_{n+1}, \bar{a}')$ and $\neg\varphi(\bar{a}, \bar{a}_{n+1}, \bar{a}_n, \bar{a}')$ where $\bar{a} = \bar{a}_{<n}$, $\bar{a}' \subseteq \cup\bar{a}_{>\omega}$. Now adding $\bar{a}\bar{a}'$ to the parameters, we obtain the sequence $\langle \bar{a}_i \colon n \leq i < \omega \rangle$ as required.

(v) $\Rightarrow$(iv): Clear.

(iv) $\Rightarrow$(i): Clear.

So we showed (i) $\Longleftrightarrow$(ii) $\Longleftrightarrow$(iii) $\Rightarrow$(vii) $\Rightarrow$(vi) $\Rightarrow$(v) $\Rightarrow$(iv) $\Rightarrow$(i). This completes all the equivalences except (viii) and (ix).

(vii) $\Rightarrow$(viii), (viii) $\Rightarrow$(ix) are trivial.

(ix) $\Rightarrow$(vii) Let $\varphi(\bar{x}, \bar{y}), I, J$ be as in $\neg$(vii), without loss of generality both I and J are of order type $\mathbb{Q}$. By Fact 6.8 there exists $\vartheta(\bar{x}, \bar{y})$ (maybe with additional parameters from $\cup J \subseteq p^{\mathfrak{C}}$) which defines a partial order on $\mathfrak{C}$ and linearly orders an infinite subsequence of J which lies in $p^{\mathfrak{C}}$; so we're done.

$\text{QED}_{6.9}$

Remark 6.10. A curious point: since we do not use clauses (viii) and (ix) of Proposition 6.9 in the proof of the equivalence of (i) – (vii), we also obtain an alternative proof of Proposition 4.2 in [12] (weak stability implies stability, even for a type) that goes through generically stable types.

We now proceed to the third version of stability which is due to Haskell, Hrushovski and Macpherson and is studied in great detail in [4]. We give an equivalent definition which appears in Hrushovski [7].

Definition 6.11. A type $p \in \mathrm{S}(A)$ is called *stably dominated* if there exists a collection of stable sets $\bar{D} = \langle D_i \colon i < \alpha \rangle$ and definable functions $f_i \colon p^{\mathfrak{C}} \to D_i$ such that for every set $B \supseteq A$ and $\bar{a} \models p$, if $f_i(a) \mathop{\smile\!\!\!\!\!\!\perp}^{st}_A B$ for all i (which in this context just means that $\mathrm{tp}(f_i(a)/B)$ is definable over A), then (denoting $\bar{f} = \langle f_i : i < \alpha \rangle$) $\mathrm{tp}(B/A\bar{f}(\bar{a})) \vdash \mathrm{tp}(B/A\bar{a})$.

In this case we also say that p is stably dominated *by* $\bar{D}$ *via* $\bar{f}$.

Observation 6.12. (i) A seriously stable type is stably dominated.
(ii) A seriously stable type is hereditarily stable.
(iii) A stably dominated type is generically stable.
(iv) A hereditarily stable type is generically stable.

Proof. The only nontrivial statement here is (iii); but it is easy to deduce using properties of independence of stably dominated types, see e.g. Proposition 2.8 in [7], that a Morley sequence in a stably dominated type is an indiscernible set. $\text{QED}_{6.12}$

The following examples show that the notions "hereditarily stable" and "stably dominated" are "orthogonal", that is, none of them implies the other. Both of these examples were used by Hasson and Onshuus in [6] for different purposes.

Example 6.13. Let us consider the theory of $\mathbb{Q}$ with a predicate P_n for every interval $[n, n+1)$ ($n \in \mathbb{Z}$) and the natural order $<_n$ on P_n. It is easy to see that the "generic" type "at infinity" (that is, the type of an element not in any of the P_n's) is hereditarily stable. It is not stably dominated since there are no stable sets. In particular, it is not seriously stable.

Note that this theory is interpretable in the o-minimal theory $(\mathbb{Q}, +, <)$ and therefore dependent.

Example 6.14. Let us consider the theory of a two-sorted structure (X, Y): on X there is an equivalence relation $E(x_1, x_2)$ with infinitely many infinite classes and each class densely linearly ordered, while Y is just an infinite set such that there is a definable function f from X onto Y with $f(a_1) = f(a_2) \iff E(a_1, a_2)$.

In other words, Y is the sort of imaginary elements corresponding to the classes of E. Clearly Y is stable and stably embedded.

Let M a model and p the "generic" type in X over M, that is, a type of an element in a new equivalence class. Pick $a \models p$ and $B \supseteq M$ such that $a \mathop{\smile\hskip -0.9em ^| \ }^{st}_M B$, that is, $\mathrm{tp}(a/B)$ is definable over M, which necessarily means $\mathrm{tp}(a/B)$ is generic in the sense above, that is, B does not contain any elements of the equivalence class of a. So clearly $\mathrm{tp}(B/Ma)$ is completely determined by $\mathrm{tp}(B/Mf(a))$.

This shows that p is stably dominated via f and Y. It is clearly not hereditarily stable (e.g. admits the strict order property).

We will give now several examples of generically stable types which are not hereditarily stable or stably dominated.

The following example is basically due to Kobi Peterzil. A version of it discussed in more detail by Hasson and Onshuus in [5].

Example 6.15. Let FDO ("FDO" stands for "Finite Dense Orders") be the theory of $\mathbb{Q}$ equipped with predicate symbols $<_n$ for $n \in \mathbb{N}$ such that $<_n$ defines an order on rational numbers of distance at most n. That is, $\models q_1 <_n q_2$ if and only if $|q_1 - q_2| \leq n$ and $q_1 < q_2$. Clearly $q_1 <_n q_2$ implies $q_1 <_m q_2$ for all $m > n$.

Let $M \models FDO$ and p be the "infinity" type, that is, the type of an element which is not comparable to any element of M with respect to any of the finite orders. Clearly p is generically stable and a Shelah/Morley sequence in p is just a set of pairwise incomparable elements which are "infinitely far" from each other. On the other hand, there exists an indiscernible sequence in p which is increasing with respect to (for example) $<_{45}$ but not $<_{44}$.

So there are many different extensions of p which are not generically stable, hence p is not hereditarily stable. Just like in Example 6.13, there are no stable sets in FDO and therefore no stably dominated types.

Note that a similar phenomenon can be obtained by starting with the theory from Example 6.13 expanded with the group structure on $\mathbb{Q}$ and taking a reduct. Just like the theory in 6.13, FDO is interpretable in $(\mathbb{Q}, +, <)$.

The second example arises in a more natural context:

Example 6.16. Let RV be a two-sorted theory of a real closed (ordered) field R and an infinite dimensional vector space V over it. There is a definable partial order on V:

$$v_1 \leq v_2 \iff \exists r \in R, r \geq 1_R \text{ such that } v_2 = r \cdot v_1$$

Let M be a model and $p \in \mathrm{S}(M)$ be the type of a generic vector. Then p is generically stable and every Morley/Shelah sequence is an indiscernible linearly independent set. On the other hand, there are (for example) increasing indiscernible sequences in p, so p is not hereditarily stable. Like in the previous examples, there are no stable sets, and therefore no stably dominated types.

Note that one could define a more general notion of stable domination, using a hereditarily stable type instead of a collection of stable sets, as is done in [9]:

Definition 6.17. We call $p \in \mathrm{S}(A)$ is called *stably dominated* if there exists a collection of LP-stable partial types $\bar{\pi} = \langle \pi_i \colon i < \alpha \rangle$ and definable functions $f_i \colon p^{\mathfrak{C}} \to \pi_i$ such that for every set $B \supseteq A$ and $\bar{a} \models p$, if $f_i(a) \mathop{\smile\!\!\!\!\!\!|}\nolimits^{st}_{A} B$ for all i (which just means that $\mathrm{tp}(f_i(a)/B)$ is definable over A), then (denoting $\bar{f} = \langle f_i : i < \alpha \rangle$) $\mathrm{tp}(B/A\bar{f}(\bar{a})) \vdash \mathrm{tp}(B/A\bar{a})$.

Clearly, working with this definition, every hereditarily stable type is stably dominated. Still, generically stable types given in Examples 6.15 and 6.16 are not stably dominated even in this stronger sense (there are no hereditarily stable types).

Discussion 6.18. We would like to point out a phenomenon which can be seen in both examples 6.15 and 6.16. Let us consider e.g. $T = FDO$. Let M be a model, $p \in \mathrm{S}(M)$ the generically stable type "at infinity", $I = \langle b_i : i < \omega \rangle$ a Morley sequence in p.

Let $J = \langle b'_i : i < \omega \rangle$ be a $<_1$-increasing sequence in p with $b'_0 = b_0$. Note that $q = \mathrm{tp}(b'_1/Mb_0)$ is not generically stable and does not split over M. This shows that nonsplitting extensions of generically stable types over arbitrary sets are not necessarily generically stable, even if the domain of the original type is a model (making it saturated wouldn't help).

Clearly q does not have an extension to $B = M \cup I$ which doesn't split over M; otherwise, I would be indiscernible over Mb'_1, that is, b'_1 would be $<_1$-bigger than all elements of I, which is absurd since elements of I are $<_n$ incomparable for all n. Of course we know another reason that suggests that such an extension doesn't exist: any such extension must be generically stable, and as a matter of fact, it is unique and equals to $\mathrm{Av}(I, M \cup I)$.

Moreover, note that J is a nonsplitting sequence in p over M. Obviously it is not a Morley or a Shelah sequence. This shows that it is not true that a generically stable type has a unique nonsplitting sequence, or even that every nonsplitting sequence in a generically stable type must be an indiscernible set.

Generically stable types which are not hereditarily stable or stably dominated are generally difficult to handle because they do not have to be at all related to the "stable" part of the theory; in fact, the theory does not even have to have any "stable" part, like in Examples 6.15 and 6.16. Still, our results apply in the most general case. The next section generalizes the independence relation developed for stably dominated types in [4] to generically stable types.

7. Generically stable types and forking independence

Let $p \in \mathrm{S}(A)$ be a generically stable type. In particular it is properly definable over $\mathrm{acl}(A)$ by a definition schema d_p. We will denote the free extension of p to $\mathfrak{C}$ by $p|^d\mathfrak{C}$, or $p|\mathfrak{C}$ when d is clear from the context. For a set B let $p|B = p|\mathfrak{C}{\restriction}B$.

Definition 7.1. For $\bar{a} \models p$ we say that it is *stably forking independent* (or just *forking independent*) of B over A, $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_A B$, if $\mathrm{tp}(\bar{a}/AB)$ does not fork over A.

Observation 7.2. Let $\mathrm{tp}(\bar{a}/A)$ be generically stable, then $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_A B$ if and only if $a \models p|^d B$ with respect to one of the (boundedly many) definitions of p over $\mathrm{acl}(A)$.

Proof. By Observation 2.11 and stationarity. $\mathrm{QED}_{7.2}$

Remark 7.3. Observation 7.2 implies in particular that forking independence generalizes independence relation developed for stably dominated types in [4].

Lemma 7.4. *(Symmetry Lemma) Let $p, q \in \mathrm{S}(A)$ be generically stable types, $\bar{a} \models p$, $\bar{b} \models q$. Then $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_A \bar{b} \iff \bar{b} \mathop{\smile\!\!\!\!\!\!|}_A \bar{a}$.*

Proof. Suppose not. Assume for example that $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_A \bar{b}$ and $\bar{b} \not\mathop{\smile\!\!\!\!\!\!|}_A \bar{a}$. By Observation 7.2 there is $\varphi(\bar{x}, \bar{y})$ such that $\varphi(\bar{a}, \bar{b})$, $\models d_p\bar{x}\varphi(x, \bar{b})$, but $\models \neg d_q\bar{y}\varphi(\bar{a}, \bar{y})$. Let $\bar{a}_0 = \bar{a}$, $\bar{b}_0 = \bar{b}$. Now construct sequences $\langle \bar{a}_i \rangle$, $\langle \bar{b}_i \rangle$ for $i < \omega + \omega$ as follows:

$$\bar{a}_i \models p|A\langle \bar{a}_j : j < i\rangle\langle \bar{b}_j : j < i\rangle$$
$$\bar{b}_i \models q|A\langle \bar{a}_j : j < i+1\rangle\langle \bar{b}_j : j < i\rangle$$

These sequences exemplify the order property for $\varphi(\bar{x}, \bar{y})$. But they are indiscernible sets (as p and q are generically stable), so $\varphi(\bar{x}, \bar{y})$ is supposed to be stable with respect to them, see Observation 2.5, and this is clearly not the case, take e.g. $\varphi(\bar{a}_\omega, \bar{b}_i)$ which holds for $i < \omega$ and fails for $i > \omega$. $\mathrm{QED}_{7.4}$

Remark 7.5. Note that for the proof of Symmetry Lemma we only need one of the types to be generically stable, and another one to only be definable (since it is enough to get a contradiction to stability of φ with respect to one of the sequences). In fact, a slight modification of the proof shows that even definability is not necessary; see Lemma 8.5 for a strong symmetry result.

Theorem 7.6. *Let $p, q \in S(A)$ be generically stable, $\bar{a}, \bar{b}$ realize p, q respectively, and let $\bar{c}, \bar{d}$ be any tuples (maybe infinite). Then:*

- Irreflexivity $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_A \bar{a}$ *if and only if p is algebraic*
- Monotonicity *If $a \mathop{\smile\!\!\!\!\!\!|}_A \bar{b}\bar{c}\bar{d}$, then $a \mathop{\smile\!\!\!\!\!\!|}_A \bar{c}\bar{b}$.*
- Symmetry $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_A \bar{b}$ *if and only if* $\bar{b} \mathop{\smile\!\!\!\!\!\!|}_A \bar{a}$
- Transitivity $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_A \bar{c}\bar{d}$ *if and only if* $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_{A\bar{c}} \bar{d}$ *and* $\bar{a} \mathop{\smile\!\!\!\!\!\!|}_A \bar{c}$
- Existence *Let $B \supseteq A$, then there exists $\bar{a}' \equiv_A \bar{a}$ such that* $\mathrm{tp}(\bar{a}'/B)$ *is generically stable and $\bar{a}' \mathop{\smile\!\!\!\!\!\!|}_A B$.*

- Uniqueness *If* $\bar{a} \downarrow_A \bar{c}$, $\bar{a}' \downarrow_A \bar{c}$ *and* $\bar{a}' \equiv_{\mathrm{acl}(A)} \bar{a}$*, then* $\bar{a} \equiv_{A\bar{c}} \bar{a}'$
- Local Character *If* $\bar{a} \downarrow_A \bar{c}$*, then for some subset* A_0 *of* A *of cardinality* $|T|$*,* $\bar{a} \downarrow_{A_0} \bar{c}$*. If* $A = M$ *is an* $\aleph_1$*-saturated model, there exists a countable* $A_0 \subseteq M$ *such that* $\bar{a} \downarrow_{A_0} \bar{b}$*.*

Proof. By the definitions and previous results: e.g., Transitivity is clear from Observation 7.2, Existence is just existence of nonforking extensions, for Uniqueness use stationarity of generically stable types, for Local Character take A_0 to be a Morley sequence in p over $\bar{b}\bar{a}$.

$\mathrm{QED}_{7.6}$

As usual, we will call a set $\mathcal{B}$ of tuples realizing generically stable types *forking independent* over a set A if for every $\mathcal{B}_0 \subseteq \mathcal{B}$ we have $\cup\mathcal{B} \downarrow_A \cup(\mathcal{B} \setminus \mathcal{B}_0)$. We call a sequence $I = \langle \bar{b}_i : i \in O \rangle$ of realizations of generically stable types *forking independent* if the set $\{\bar{b}_i : i \in O\}$ is forking independent. Just like in stable theories, using the properties of stable forking independence, I is forking independent if and only if for every $i \in O$ we have $\bar{b}_i \downarrow_A \bar{b}_{<i}$.

Let $A = \mathrm{acl}(A)$, $p, q \in \mathrm{S}(A)$ generically stable types. We denote by $p \otimes q$ the unique (by stationarity) type of an independent (over A) pair $(\bar{a}, \bar{b})$ of realizations of p and q respectively. If $p = q$ we also write $p^{\otimes 2}$ for $p \otimes p$, and generally denote $p^{\otimes n} = p \otimes \ldots \otimes p$ n times, which is well-defined by the properties of forking independence.

It is easy to see that

Observation 7.7. Given $p, q \in \mathrm{S}(A)$ generically stable, $p \otimes q \in \mathrm{S}(A)$ is generically stable. Similarly for a product of any number (finite or infinite) of generically stable types.

Like in stable theories, we have

Remark 7.8. Let $I = \langle \bar{b}_i : i \in O \rangle$ be a sequence forking independent over a set $A = \mathrm{acl}(A)$ of realizations of the same generically stable type p over A. Then I is an indiscernible set over A, the type of an n-tuple $\bar{b}_{i_1} \ldots \bar{b}_{i_n}$ being $p^{\otimes n}$.

One can characterize generically stable types in terms of the properties of forking on the set of their realizations. Of course there are many different such characterizations, we start with the simplest ones, which also come handy in Lemma 7.10:

Observation 7.9. Let $p \in \mathrm{S}(A)$ be a type. Then the following are equivalent:

(i) p is generically stable
(ii) For every $\bar{a}, \bar{a}' \models p$ and $B, C, D \subseteq p^{\mathfrak{C}}$ we have ($B \downarrow_A C$ stands for "tp(B/AC) does not fork over A"):
  - Symmetry: $B \downarrow_A C \iff C \downarrow_A B$
  - Transitivity: $B \downarrow_A C$ and $B \downarrow_{AC} D \Rightarrow B \downarrow_A CD$
  - Uniqueness: $\bar{a} \downarrow_A B, \bar{a}' \downarrow_A B \Rightarrow \bar{a} \equiv_{AB} \bar{a}'$

(iii) p is properly definable over $\mathrm{acl}(A)$ by a definition scheme d and for every $\bar{a}_1, \ldots, \bar{a}_n \models p$ we have:
- Symmetry: if $\bar{a}_i \models p|^d A\bar{a}_{<i}$ for all i then for every permutation $\sigma \in S_n$ we have $\bar{a}_{\sigma(i)} \models p|^d A\bar{a}_{\sigma(<i)}$ for all i.

Proof. (ii)⇒(i): one chooses a nonforking sequence in p and shows using symmetry, transitivity and uniqueness that it is an indiscernible set.

(iii)⇒(i) is even easier: for any definable type, a Morley sequence is indiscernible. Symmetry does the rest.

QED$_{7.9}$

Following Definition 6.17, one could try to generalize stable domination to generically stable types. The following lemma shows that this does not lead to anything new, which confirms our perception of generic stability as the most general notion of stability for a type.

Lemma 7.10. *Let $q \in \mathrm{S}(A)$ be a generically stable type. Assume that $p \in \mathrm{S}(A)$ is* stably dominated by q via f*, that is, assume that f is a definable function from $p^{\mathfrak{C}}$ to $q^{\mathfrak{C}}$ such that for every set $B \supseteq A$ and $\bar{a} \models p$, if $f(a) \mathop{\smile\!\!\!\!\!\!\mid}^{st}_A B$ then $\mathrm{tp}(B/Af(\bar{a})) \vdash \mathrm{tp}(B/A\bar{a})$.*

Then p is generically stable.

Proof. Using Lemma 3.12 in [4], it is easy to see that p is properly definable over $\mathrm{acl}(A)$. Moreover, "pulling back" to p via f properties of stable forking independence on q, one shows that definable extensions satisfy Symmetry as in Observation 7.9(iii). Hence p is generically stable.

QED$_{7.10}$

Remark 7.11. In [14] Onshuus and the author provide a generalization of this Lemma, replacing stable domination with forking domination.

It would be interesting to investigate properties mentioned in Observation 7.9 on their own: which ones imply each other, which imply generic stability, etc. We do not pursue this direction much further here and only make a few remarks.

Observation 7.12. Let $A = \mathrm{acl}(A)$, $p \in \mathrm{S}(A)$ be definable. Then p is generically stable if and only if p is stationary.

Proof. Taking a free extension we may assume A is a model. Now every nonforking sequence in p is by stationarity both a Morley and a Shelah sequence. It is easy to see that this implies generic stability. QED$_{7.12}$

Lemma 7.13. (i) *Let A be a model (or just $A = \mathrm{bdd}^{heq}(A)$), $p \in \mathrm{S}(A)$ be a type satisfying Uniqueness of nonforking as in Observation 7.9(ii). Then p is stationary.*

(ii) *Let p be a Lascar strong type satisfying Uniqueness of nonforking as in Observation 7.9(ii) with "type" replaced by "Lascar strong type". Then p is stationary.*

(iii) *Let $p \in \mathrm{S}(A)$ be a definable type which satisfies Uniqueness of definable extensions in the following sense: any two definitions of it over A agree on the set of realizations of p. Then p has a unique global extension definable over A.*

Proof. (i) Assume that p has two global nonforking extensions q_1 and q_2. Then there is $\bar{c} \in \mathfrak{C}$ and a formula $\varphi(\bar{x}, \bar{y})$ such that $\varphi(\bar{x}, \bar{c}) \in q_1$, $\neg\varphi(\bar{x}, \bar{c}) \in q_2$. Now define a sequence of realizations of p, $I = \langle \bar{a}_i : i < \omega \rangle$ as follows:

$$\bar{a}_{2i} \models q_1 {\restriction} A\bar{c}\bar{a}_{<2i}$$

$$\bar{a}_{2i+1} \models q_2 {\restriction} A\bar{c}\bar{a}_{<2i+1}$$

This is a nonforking sequence. Since A is a model, it is also nonsplitting. By Uniqueness of nonforking extensions on the set of realizations of p (that is, by the assumption) and Fact 2.18, it is indiscernible. Now $\varphi(\bar{x}, \bar{c})$ and I clearly contradict dependence.

(ii) Same proof with splitting replaced by Lascar splitting (using Observation 2.26) and Fact 2.18 replaced with Observation 2.19.

(iii) Similar.

$\mathrm{QED}_{7.13}$

Corollary 7.14. (i) *Let A be a model (or just $A = \mathrm{bdd}^{heq}(A)$), $p \in \mathrm{S}(A)$ be a definable type which satisfies Uniqueness of nonforking as in Observation 7.9(ii). Then p is generically stable.*

(ii) *Let p be a definable Lascar strong type over A which satisfies Uniqueness of nonforking as in Observation 7.9(ii) with "type" replaced with "Lascar strong type". Then p is generically stable.*

Proof. By Lemma 7.13 and Observation 7.12. $\mathrm{QED}_{7.14}$

Note that uniqueness of definable extensions does not imply generic stability: the type "at infinity" in the theory of a dense linear order $(\mathbb{Q}, <)$ is definable and has a unique global definable extension. It is not, of course, stationary or generically stable.

8. Strong stability and stable weight

In this section we develop the basic theory of stable weight of a type, that is, weight with respect to generically stable types. Our hope is that in a dependent theory it is possible to "analyze" an arbitrary type with respect to its "stable-like" part and a "partial order". The goal of stable weight is to provide certain understanding of the "stable" part.

We aim to connect finiteness of stable weight to strong dependence introduced by Shelah in [15] and studied more intensively in [16]. The following definitions are motivated by those notions.

Definition 8.1. (i) A *randomness pattern* of depth κ for a (partial) type p over a set A is an array $\langle \bar{b}^\alpha_i : \alpha < \kappa, i < \omega \rangle$ and formulae $\varphi_\alpha(\bar{x}, \bar{y}_\alpha)$ for $\alpha < \kappa$ such that

(a) The sequences $J_\alpha = \langle \bar{b}^\alpha_i \colon i < \omega \rangle$ are mutually indiscernible over A; more precisely, J_α is indiscernible over $AJ_{\neq\alpha}$
(b) $\mathrm{len}(\bar{b}^\alpha_i) = \mathrm{len}(\bar{y}_\alpha)$
(c) for every $\eta \in {}^\kappa\omega$, the set

$$\Gamma_\eta = \{\varphi_\alpha(\bar{x}, \bar{b}^\alpha_\eta(\alpha) \colon \alpha < \kappa\} \cup \{\neg\varphi_\alpha(\bar{x}, \bar{b}^\alpha_i) \colon \alpha < \kappa, i < \omega, i \neq \eta(\alpha)\}$$

is consistent with p.

(ii) A (partial) type p over a set A is called *strongly dependent* if there do not exist formulae $\varphi_\alpha(\bar{x}, \bar{y}_\alpha)$ for $\alpha < \omega$ and sequences $\langle \bar{b}^\alpha_i \colon i < \omega \rangle$ for $\alpha < \omega$ mutually indiscernible over A such that for every $\eta \in {}^\omega\omega$, the set

$$\Gamma_\eta = \{\varphi_\alpha(\bar{x}, \bar{b}^\alpha_\eta(\alpha) \colon \alpha < \omega\} \cup \{\neg\varphi_\alpha(\bar{x}, \bar{b}^\alpha_i) \colon \alpha < \omega, i \neq \eta(\alpha)\}$$

is consistent with p.

In other words, p is called strongly dependent if there does not exist a randomness pattern for p of depth $\kappa = \omega$.

(iii) *Dependence rank* (dp-rk) of a (partial) type p over a set A is the supremum of all κ such that there exists a randomness pattern for p of depth κ.

(iv) A (partial) type over a set A is called *dp-minimal* if dp-rank of p is 1.

In other words, p is dp-minimal if there does not exist a randomness pattern for p of depth 2.

(v) A theory is called strongly dependent/dp-minimal if the partial type $x = x$ is.

(vi) Let T be dependent. A type p is called *strongly stable* if it is strongly dependent and generically stable.

Remark 8.2. For a partial type p, $\text{dp-rk}(p) \geq 1$ iff p is nonalgebraic.

Proof. The "only if" direction is obvious. For the "if" direction, by non-algebraicity, the formula $x = y$ does the trick. $\mathrm{QED}_{8.2}$

Remark 8.3. A very close relative of dp-rk is called "burden" by Hans Adler in [2]. He also studies "strong" theories which is a class containing strongly dependent theories, but also some independent ones, e.g. supersimple theories, and more.

We can define the *stable weight* of p, $\mathrm{swt}(p)$ as weight of p with respect to generically stable types:

Definition 8.4. (i) Let $p \in \mathrm{S}(A)$ be a type. We define the *stable pre-weight* of p, $\mathrm{spwt}(p)$, to be the supremum of all α such that there exist $\bar{a} \models p$, generically stable types $\langle q_i : i < \alpha \rangle$ over A and $\bar{b}_i \models q_i$ such that:

- $\{\bar{b}_i \colon i < \alpha\}$ is an independent set over A
- $\mathrm{tp}(\bar{a}/A\bar{b}_i)$ divides over A for all i

(ii) The *stable weight* of p, $\mathrm{swt}(p)$ is the supremum of the stable pre-weights of all nonforking extensions of p.

The main goal of this section is to show that a strongly dependent type has finite stable weight. We will need the following slightly surprising strengthening of the Symmetry Lemma. Recall that Remark 7.5 states that for the proof of Lemma 7.4 it is enough to assume that one of the types is generically stable and the other one is definable. We intend to eliminate definability from the assumptions.

For simplicity of notation, we will denote "$\mathrm{tp}(B/AC)$ does not fork over A" by "$B \mathop{\smile\!\!\!\!\!\mid}_A C$" even if $\mathrm{tp}(B/A)$ is not generically stable (and so the relation above does not need to be symmetric).

Lemma 8.5. *(Strong Symmetry Lemma) Let $p \in \mathrm{S}(A)$ be generically stable, $q \in \mathrm{S}(A)$ does not fork over A, $\bar{a} \models p$, $\bar{b} \models q$. Then*

(i) $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_A \bar{b} \Longrightarrow \bar{b} \mathop{\smile\!\!\!\!\!\mid}_A \bar{a}$. *Moreover, if $A = \mathrm{acl}(A)$ and $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_A \bar{b}$, then there exists a unique nonforking extension of q to $\mathrm{S}(A\bar{a})$ which equals $\mathrm{tp}(\bar{b}/A\bar{a})$.*

(ii) $\bar{b} \mathop{\smile\!\!\!\!\!\mid}_A \bar{a} \Longrightarrow \bar{a} \mathop{\smile\!\!\!\!\!\mid}_A \bar{b}$.

Proof. (i) Clearly, it is enough to prove the lemma for $A = \mathrm{acl}(A)$. Let q^* be a global nonforking extension of q. We will show that $q^* \restriction A\bar{a} = \mathrm{tp}(\bar{b}/A\bar{a})$, proving the moreover part as well.

Suppose not. Then there is a formula $\varphi(\bar{x}, \bar{y})$ such that $\varphi(\bar{a}, \bar{b})$ (so $d_p\bar{x}\varphi(\bar{x}, \bar{b})$ holds), but $\neg\varphi(\bar{a}, \bar{y}) \in q^*$.

Let $\bar{a}_0 = \bar{a}$, $\bar{b}_0 = \bar{b}$. Construct sequences $\langle \bar{a}_i \rangle$, $\langle \bar{b}_i \rangle$ for $i < \omega + \omega$ as follows:

$$\bar{a}_i \models p|A\langle \bar{a}_j : j < i\rangle\langle \bar{b}_j : j < i\rangle$$
$$\bar{b}_i \models q^* \restriction A\langle \bar{a}_j : j < i+1\rangle\langle \bar{b}_j : j < i\rangle$$

Now note:

- $j < i \Rightarrow \varphi(\bar{a}_i, \bar{b}_j)$: since $\models \bar{d}_p\bar{x}\varphi(\bar{x}, \bar{b})$, $\bar{b} \equiv_A \bar{b}_j$ and $\bar{a}_i$ is chosen generically over $A\bar{b}_j$
- $j \geq i \Rightarrow \neg\varphi(\bar{a}_i, \bar{b}_j)$: since $\neg\varphi(\bar{a}, \bar{y}) \in q^*$, q^* does not fork hence does not Lascar split over A, $\bar{a} \equiv_{Lstp,A} \bar{a}_i$ (in fact, they are of Lascar distance 1) and $\bar{b}_j$ was chosen to realize q^* over $A\bar{a}_i$.

As in the proof of Lemma 7.4, this is a contradiction to generic stability of p (that is, $\langle \bar{a}_i : i < \omega + \omega\rangle$ being an indiscernible set).

(ii) Now assume $\bar{a} \not\mathop{\smile\!\!\!\!\!\mid}_A \bar{b}$, so there is a formula $\varphi(\bar{x}, \bar{y})$ such that $d_p\bar{x}\varphi(\bar{x}, \bar{b})$ but $\neg\varphi(\bar{a}, \bar{b})$ holds. Again without loss of generality $A = \mathrm{acl}(A)$.

Let $\bar{a}' \models p$ such that $\bar{a}' \mathop{\smile\!\!\!\!\!\mid}_A \bar{b}$, so $\varphi(\bar{a}', \bar{b})$. By clause (i) of the Lemma we have

8.5.1. The only nonforking extension of q to $A\bar{a}'$ is $\mathrm{tp}(\bar{b}/A\bar{a}')$.

On the other hand, $\neg\varphi(\bar{a}', \bar{y})$ is consistent with q (by applying an automorphism of $\mathrm{tp}(\bar{b}/A\bar{a})$ over A taking $\bar{a}$ to $\bar{a}'$). So $q \cup \{\neg\varphi(\bar{a}', \bar{y})\}$ forks over A (by 8.5.1), hence so does $q \cup \{\neg\varphi(\bar{a}, \bar{y})\}$, which is a subset of $\mathrm{tp}(\bar{b}/A\bar{a})$, as required.

$\mathrm{QED}_{8.5}$

We will make use of the following well-known fact (due to Morley):

Fact 8.6. *Let λ be a cardinal. Then there exists $\mu > \lambda$ such that for every set A of cardinality λ and a sequence of tuples $\langle a_i : i < \mu \rangle$ there exists an ω-type $q(x_0, x_1, \cdots)$ of an A-indiscernible sequence such that for every $n < \omega$ there exist $i_1 < i_2 < \ldots < i_n < \mu$ such that the restriction of q to the first n variables equals $\operatorname{tp}(a_{i_1} \ldots a_{i_n}/A)$.*

We will sometimes denote μ as above by $\mu(\lambda)$.

We will also need some basic facts about nonforking calculus and preservation of independence in dependent theories. Recall that "$B \mathop{\smile\!\!\!\!\!\!|}_A C$" stands for "$\operatorname{tp}(B/AC)$ does not fork over A".

Fact 8.7. *(Shelah) Let A, B be sets and assume that $I = \langle a_i : i < \lambda \rangle$ is a nonforking sequence based on A, that is ,$a_i \mathop{\smile\!\!\!\!\!\!|}_A Ba_{<i}$ for all $i < \lambda$. Then $I \mathop{\smile\!\!\!\!\!\!|}_A B$, that is, $\operatorname{tp}(I/AB)$ does not fork over A.*

Proof. This is Claim 5.16 in [15]. $\text{QED}_{8.7}$

Corollary 8.8. *Let $\{A_i : i < \lambda\}$ be a nonforking (independent) set over A, that is, $A_i \mathop{\smile\!\!\!\!\!\!|}_A A_{\neq i}$ for all i. Then for every $W, U \subseteq \lambda$ disjoint we have $A_{\in W} \mathop{\smile\!\!\!\!\!\!|}_A A_{\in U}$.*

Proof. Monotonicity and transitivity on the left. $\text{QED}_{8.8}$

Observation 8.9. Suppose I is an indiscernible sequence over A and $B \mathop{\smile\!\!\!\!\!\!|}_A I$. Then I is indiscernible over AB.

Proof. By Fact 2.14 $\operatorname{tp}(B/AI)$ does not split strongly over A. Recall that this implies that for every $\bar{a}_1, \bar{a}_2 \in I$ which are on the same A-indiscernible sequence we have $B\bar{a}_1 \equiv_A B\bar{a}_2$, which is precisely what we want. $\text{QED}_{8.9}$

The following lemma is the key to the proof of the main theorem. It shows that indiscernible sequences which start with generically stable independent elements can be assumed to be mutually indiscernible.

Lemma 8.10. *Let A be an extension base (that is, no type over A forks over A; e.g. A is a model). Let $\{\bar{a}_i : i < \alpha\}$ be an A-independent set of elements satisfying generically stable types over A, and let $\langle I_i : i < \alpha \rangle$ be a sequence of A-indiscernible sequences starting with $\bar{a}_i$ respectively. Then there exist sequences $\langle I'_i : i < \alpha \rangle$ such that*

- $I'_i \equiv_A I_i$
- I'_i *starts with* $\bar{a}_i$
- I'_i *is indiscernible over* $AI'_{\neq i}$

Proof. By compactness it is enough to take care of $\alpha = k$ finite. So we will prove the lemma by induction on k, the case $k = 1$ being trivial. In fact, we will prove more, that is, we will prove by induction on k that there there are $\langle I'_i : i < k \rangle$ such that for all $i < k$

8.10.1.

- $I'_i \equiv_A I_i$
- I'_i starts with $\bar{a}_i$
- I'_i is indiscernible over $AI'_{\neq i}$
- $I'_i \downarrow_A I'_{<i}$

Let $\{\bar{a}_i : i < k+1\}$, $\langle I_i : i < k+1 \rangle$ be as in the assumptions of the Lemma. By the induction hypothesis we may assume that $\langle I_i : i < k \rangle$ are as in 8.10.1 above.

Since $\bar{a}_{<k} \downarrow_A \bar{a}_k$, by Fact 2.21 there exist $\bar{a}'_{<k} \equiv_{A\bar{a}_k} \bar{a}_{<k}$ such that $\bar{a}'_{<k} \downarrow_A I_k$. By applying an automorphism over $A\bar{a}_k$, we may assume $\bar{a}'_{<k} = \bar{a}_{<k}$. More specifically, let $\sigma \in \mathrm{Aut}(\mathfrak{C}/A\bar{a}_k)$ take $\bar{a}'_{<k}$ to $\bar{a}_{<k}$. Denote $I''_k = \sigma(I_k)$. Then

- $I''_k \equiv_A I_k$
- I''_k starts with $\bar{a}_k$ (σ does not move $\bar{a}_k$)

So without loss of generality $I''_k = I_k$.

By Observation 7.7, $\mathrm{tp}(\bar{a}_{<k}/A)$ is generically stable. By the Strong Symmetry Lemma 8.5 (note that $I_k \downarrow_A A$ since A is an extension base) we have $I_k \downarrow_A \bar{a}_{<k}$.

By Fact 2.21 again there is $I''_k \equiv_{A\bar{a}_{<k}} I_k$ such that $I''_k \downarrow_A I_{<k}$. Applying an automorphism over $A\bar{a}_{<k}$ like before, we may assume that $I''_k = I_k$ (this time the sequences $I_{<k}$ might change, but they still have all the desired properties). Note that it is still the case that $\bar{a}_{<k} \downarrow_A I_k$, hence by Observation 8.9 the sequence I_k is indiscernible over $A\bar{a}_{<k}$.

Since we could make I_k as long as we wish to begin with, by Fact 8.6 there is I''_k indiscernible over $AI_{<k}$ such that every n-type of I''_k over $AI_{<k}$ "appears" in I_k. In particular, we have $I''_k \downarrow_A I_{<k}$ and (since I_k is $A\bar{a}_{<k}$-indiscernible) $I''_k \equiv_{A\bar{a}_{<k}} I_k$.

Let $\sigma \in \mathrm{Aut}(\mathfrak{C}/A\bar{a}_{<k})$ taking I''_k onto (an initial segment of) I_k. Denote $I'_k = \sigma(I''_k)$, $I'_{<k} = \sigma(I_{<k})$. Clearly we still have

(i) $\langle I'_i : i < k \rangle$ are as in 8.10.1
(ii) I'_k is $AI'_{<k}$-indiscernible
(iii) $I'_k \downarrow_A I'_{<k}$
(iv) I'_k starts with $\bar{a}_k$

For $i < k$ let $B_i = A \cup \bigcup\{I_j : j < k, j \neq i\}$. By (iii) above, for every $i < k$ we have $I'_k \downarrow_{B_i} I'_i$. By (i) above I'_i is B_i-indiscernible, hence by Observation 8.9 I'_i is indiscernible over $AI'_{\neq i}$.

Combining this with (ii) and (iii) above, we see that $\langle I'_i : i < k+1 \rangle$ are as in 8.10.1, which completes the induction step.

$\mathrm{QED}_{8.10}$

Theorem 8.11. *Let A be an extension base (that is, no type over A forks over A; e.g. A is a model). Then for every type $p \in \mathrm{S}(A)$, $\mathrm{dp\text{-}rk}(p) \geq \mathrm{spwt}(p)$*

Proof. Let $\bar{a} \models p$, $\langle q_i : i < \alpha \rangle$ generically stable, $\langle \bar{b}_i : i < \alpha \rangle$ ($\bar{b}_i \models q_i$) exemplify $\mathrm{spwt}(p) \geq \alpha$. Since $\mathrm{tp}(\bar{a}/A\bar{b}_i)$ divides over A, this is exemplified by an A-indiscernible ω-sequence I_i starting with $\bar{b}_i$. By Lemma 8.10, without loss of generality the sequence I_i is indiscernible over $AI_{\neq i}$ for all i.

Suppose $\mathrm{tp}(\bar{a}/A\bar{b}_i)$ k_i-divides over A, and k_i is minimal such for p; that is, there is a formula $\varphi_i(\bar{x}, \bar{y})$ such that $\varphi_i(\bar{a}, \bar{b}_i)$ holds, and the set $\{\varphi(\bar{x}, \bar{b}) : \bar{b} \in I_i\}$ is $k_i - 1$-consistent with p, but k_i-inconsistent. Clearly $k_i \geq 2$. Let

$$\psi_i(\bar{x}, \bar{y}_0, \dots, \bar{y}_{k_i-2}) = \bigwedge_{\ell < k_i - 1} \varphi_i(\bar{x}, \bar{y}_\ell)$$

and let J_i be the sequence of $k_i - 1$-tuples of elements of I_i ("chunks" of size $k_i - 1$ from I_i), formally:

$$J_i = \langle \bar{b}_{i,\ell}\bar{b}_{i,\ell+1} \dots \bar{b}_{i,\ell+k_i-2} : \ell < \omega \rangle$$

Denote the first element of J_i by $\bar{c}_i$. Clearly

- J_i is indiscernible over $AJ_{\neq i}$
- $\psi(\bar{a}, \bar{c}_i)$ holds
- The set $\{\psi(\bar{x}, \bar{c}) : c \in J_i\}$ is 2-inconsistent with p

Now the sequences $\langle J_i : i < \alpha \rangle$ form an array which together with formulas ψ_i give a randomness pattern of depth α for p, as required.

$\mathrm{QED}_{8.11}$

Corollary 8.12. (i) *In a strongly dependent theory every type has finite stable weight.*

(ii) *A strongly dependent type has finite stable weight.*

(iii) *A stable theory is strongly stable if and only if every type has finite weight.*

Corollary 8.12(iii) was observed independently by Hans Adler in [2].

Further properties of stable weight will be investigated elsewhere. On the different notions of weight etc in dependent theories see also works by Adler [2], Alf Onshuus and the author [13], [14]. In section 4 of [14] different attempts are made in order to remove the assumption of generic stability and prove an analogue of Theorem 8.11 for weight and not stable weight, which leads to several general results on mutual indiscernibility (some generalize Lemma 8.10) and the behavior of forking in dependent and strongly dependent theories, but the main goal has not yet been achieved.

Appendix A. Ultrafilters and measures

The following definitions are motivated by [17], [10] and [8]. Our hope is that they might clarify the connections between ultrafilters used by Shelah in [17], coheirs and Shelah sequences, Definition 3.7 (see Discussion A.9).

In order to make the appendix more self-contained, we will repeat here certain definitions and remarks from Section 3. Note that it is convenient and natural to define these notions in this generality.

Definition A.1. Let A, B, C sets.

(i) Recall that we denote the boolean algebra of C-definable subsets of A^m by $\mathrm{Def}_m(A, C)$, $\mathrm{Def}(A, C) = \bigcup_{m<\omega} \mathrm{Def}_m(A, C)$.

(ii) A (C,m)-*Keisler measure* on A, or an m-measure *on A over C* is a finitely additive probability measure on $\mathrm{Def}_m(A,C)$. We omit C if $C=A$. We omit m if it is clear from the context.
(iii) We will call a (C,m)-Keisler measure on $\mathfrak{C}$ a *full* m-Keisler measure over C. A *global* m-measure is a full measure over $\mathfrak{C}$, i.e. a measure on $\mathrm{Def}_m(\mathfrak{C})$.
(iv) A Keisler measure μ on $\mathrm{Def}_m(B,C)$ is *supported on* $A\subseteq B$ if there exists a measure μ_0 on $\mathrm{Def}_m(A,C)$ such that for every formula $\varphi(\bar{x},\bar{c})$ over C with $\mathrm{len}(\bar{x})=m$ we have $\mu(\varphi^{\mathfrak{C}}(\bar{x},\bar{c}))=\mu_0(\varphi^{\mathfrak{C}}(\bar{x},\bar{c})\cap A^m)$. We denote μ_0 (if exists) by $\mu\downarrow A$.

Remark A.2. (i) Note that a full $\{0,1\}$-measure over C (i.e. an ultrafilter $\mathfrak{U}$ on $\mathrm{Def}(\mathfrak{C},C)$) precisely corresponds to a complete type over C.
(ii) Note that a type $p\in \mathrm{S}(C)$ is finitely satisfiable in A iff the appropriate ultrafilter (measure) $\mathfrak{U}$ on $\mathrm{Def}(\mathfrak{C},C)$ is supported on A. In this case the measure $\mathfrak{U}\downarrow A$ (see the definition of "supported on A" above) defines over C the same type as $\mathfrak{U}$, that is,$\mathrm{Av}(\mathfrak{U}\downarrow A,C)=p$.

Definition A.3. (i) Let $A\subseteq B$, and let μ be a Keisler measure on A^m over C, i.e. μ a measure on $\mathrm{Def}_m(A,C)$. We define the *lifted measure* of μ over B (denoted $\mu\uparrow B$) on $\mathrm{Def}_m(B,C)$ by $\mu\uparrow B(\varphi(\bar{x},\bar{b})^{\mathfrak{C}}\cap B^m)=\mu(\varphi^{\mathfrak{C}}(\bar{x},\bar{b})\cap A^m)$.

In particular, for $B=\mathfrak{C}$, call $\mu\uparrow\mathfrak{C}$ the *full lifted measure* of μ.
(ii) Let A,C sets, $\mathfrak{U}$ an ultrafilter on $\mathrm{Def}_m(A,C)$. Recall that we define the *average type* of $\mathfrak{U}$ over C) by

$$\mathrm{Av}(\mathfrak{U},C)=\big\{\varphi(\bar{x},\bar{c})\colon \bar{c}\in C \text{ and } \{\bar{a}\in A^m\colon \varphi(\bar{a},\bar{c})\}\in\mathfrak{U}\big\}$$

Observation A.4. For A,B,C,μ as in (i) above, $\mu\uparrow B$ is a Keisler measure on $\mathrm{Def}_m(B,C)$ supported on A, $(\mu\uparrow B)\downarrow A=\mu$. Moreover, for $A\subseteq A'\subseteq B$, $(\mu\uparrow B)\downarrow A'=\mu\uparrow A'$.

Remark A.5. Note that in clause (ii) of Definition A.3 we obtain a complete type over C, therefore a full measure over C, i.e. an ultrafilter $\mathfrak{V}$ on $\mathrm{Def}(\mathfrak{C},C)$. Clearly, this ultrafilter corresponds to the full lifted measure of $\mathfrak{U}$ over C, that is, $\mathfrak{V}=\mathfrak{U}\uparrow\mathfrak{C}$. So the second definition is a particular case of the first one.

One can also generalize the definition of nonsplitting to measures:

Definition A.6. A Keisler measure μ on $\mathrm{Def}(B,D)$ *does not split* over A if $\mu(\varphi(\bar{x},\bar{b}))$ depends only on $\mathrm{tp}(\bar{b}/A)$ for every formula $\varphi(\bar{x},\bar{y})$ and $\bar{b}\in B$.

So a global measure doesn't split over a set A if it is invariant under the action of the automorphism group of $\mathfrak{C}$ over A.

Many properties of nonsplitting types remain true when passing to measures. We will just make a few small observations.

A notion of a definable measure (which is the analogue of a definable type) was studied in [8]. The following is the analogue of Observation 2.8:

Observation A.7. (i) If a Keisler measure over B is supported on $A \subseteq B$, then it does not split over A.

(ii) If a Keisler measure over B is definable over $A \subseteq B$, then it does not split over A.

Given a set A, there are boundedly many measures which do not split over A:

Observation A.8. Let A be a set. Then there are at most $\beth_2(|A| + |T|) = 2^{2^{|A|+|T|}}$ Keisler measures μ over $\mathfrak{C}$ which do not split over A.

Proof. For each type $r(\bar{y})$ over A and each formula $\varphi(\bar{x}, \bar{y})$, we have (at most) continuum many options for $\mu(\varphi(\bar{x}, \bar{c}))$ where $\bar{c} \models r$. This determines μ completely. So we have $2^{\aleph_0(2^{|A|+|T|})} = 2^{2^{|A|+|T|}}$ nonsplitting measures. QED$_{A.8}$

Discussion A.9. Note that in [17] Shelah works with types of the form $p \in \mathrm{S}(C)$ finitely satisfiable in some A, which corresponds to the following situation: an ultrafilter $\mathfrak{U}$ on $\mathrm{Def}(\mathfrak{C}, C)$ which is supported on A, i.e. comes from a measure on $\mathrm{Def}(A, C)$. (Shelah works with ultrafilters on the algebra of all subsets of A, but of course it is enough to restrict oneself to the definable subsets).

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