

Large Cardinals and Definable Well-Orderings of the Universe

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Abstract

We use a reverse Easton forcing iteration to obtain a universe with a definable well-ordering, while preserving the GCH and proper classes of a variety of very large cardinals. This is achieved by coding using the principle $\diamond_{\kappa^+}^*$ at a proper class of cardinals κ . By choosing the cardinals at which coding occurs sufficiently sparsely, we are able to lift the embeddings witnessing the large cardinal properties without having to meet any non-trivial master conditions.

Key words: Large cardinal preservation, definable well-order, $V = \text{HOD}$, reverse Easton forcing iteration, diamond star, master conditions
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1 Introduction

A major theme of set theory in recent years has been the construction of models of ZFC which contain various large cardinals while at the same time enjoying properties analogous to those of Gödel's constructible universe L . Generally this has been approached via the *inner model programme*, constructing canonical L -like inner models for the large cardinals under consideration. While this approach has had much success, there is a bound on the size of the large cardinals that have thus far been accommodated by such techniques. For this reason, Sy Friedman has proposed the *outer model programme*, in which the goal is to construct L -like outer models containing large cardinals by the method of forcing. In doing so, one may obtain new consistency results for large cardinals beyond the scope of current inner model theory.

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One of the most striking properties of L is the fact that it bears a definable well-order, whose definition moreover involves no parameters. This may be expressed in terms of another well known inner model: V has a well-order definable without parameters if and only if $V = \text{HOD}$, the universe of all hereditarily ordinal definable sets (in V). It is well known that one may force to obtain such a model. Indeed, McAloon [12] shows how to force a model of $\text{ZFC} + \text{GCH} + V = \text{HOD} + V \neq L$ starting from L , or using the continuum function for coding, a model of $\text{ZFC} + V = \text{HOD}$ while preserving all measurables. Sy Friedman [5] shows how to obtain a model of $\text{ZFC} + \text{GCH} + V = \text{HOD}$ while preserving a (single, although the same argument works for boundedly many) hyperstrong or n -superstrong cardinal, starting with techniques of Jensen to make $V = L[A]$ for A a subset of some sufficiently large cardinal.

In this article we improve upon these results, exhibiting a forcing construction that yields a universe satisfying both $V = \text{HOD}$ and the GCH, while preserving proper classes of a variety of large cardinals. The techniques will not in general suffice to preserve all cardinals satisfying a given large cardinal property, but rather those satisfying a combined large cardinal and anti-large cardinal axiom of a certain kind, such as “superstrong but not a limit of huge cardinals”. Of course, this is sufficient to prove the relative consistency of

$$\text{ZFC} + \text{GCH} + V = \text{HOD} + \exists \text{ a proper class of cardinals } \kappa \text{ such that } \varphi(\kappa)$$

for a variety of very strong large cardinal properties φ .

The idea of our forcing is essentially to add unboundedly large Cohen sets and use the fact that every element of $V[G]$ is ordinal definable from A , a class predicate for the added Cohen sets. Of course, we then want to make A itself definable in $V[G]$. To achieve this, we code up the choices made by the generic in terms of whether or not some combinatorial principle holds at various cardinals. Doing this while preserving the GCH puts a heavy constraint on which combinatorial principles can be used for such an encoding; indeed, the GCH itself would otherwise be an ideal principle to use as a coding oracle, as in the work of McAloon [12]. However, the existence of \diamond_{κ}^* -sequences also fits the bill nicely, without disturbing the GCH.

Coming at this from the other direction, we have a property suitable to be used as an oracle (existence of \diamond_{κ}^* -sequences) and we want to use it to encode a definable well-order of the (extension) universe. Instead of using some complicated iteration with lots of bookkeeping, we may simply “let the generic decide” which way to force at each stage. This technique — having an iteration at each stage of which the generic makes an initial decision that determines the rest of the forcing poset at that stage — is not new in other contexts; see for example Theorems 5.27 and 5.33 of [6] and Section 3 of [7].

Finally, we wish to do all this while preserving large cardinals. As usual, we

preserve the large cardinals in question by lifting the witnessing elementary embeddings. Often this entails selecting a generic that lies below a specific *master condition*, and so in general it is problematic to simultaneously preserve many large cardinals, as one will not in general be able to choose a single generic meeting all of the necessary master conditions. The solution we present is to simply avoid the problem, making all of the master conditions in question trivial by only coding at cardinals which will not lead to non-trivial master condition requirements. However, we may not be able to achieve this for all large cardinals of the given kind, and this is what leads to the exceptions in our preservation theorems.

Throughout this article we shall assume that our ground model satisfies the GCH. It is known that that this can be forced while preserving a variety of large cardinals; see for example [5] which specifically deals with the case of n -superstrong cardinals.

2 The coding oracle $\diamond_{\kappa^+}^*$

Recall the following definition.

Definition 1 *Let λ be a regular cardinal, and let $D = \langle D_\alpha \mid \alpha < \lambda \rangle$ be a sequence such that for every $\alpha < \lambda$, $D_\alpha \subset \mathcal{P}(\alpha)$ and $|D_\alpha| \leq |\alpha|$. Then D is said to be a \diamond_λ^* -sequence if for every $X \subset \lambda$, $\{\alpha \in \lambda \mid X \cap \alpha \in D_\alpha\}$ contains a closed unbounded subset of λ . The statement \diamond_λ^* is the statement that a \diamond_λ^* -sequence exists.*

There are known κ^+ -closed, κ^{++} -cc partial orders for forcing $\diamond_{\kappa^+}^*$ to hold or fail while preserving the GCH, for each infinite cardinal κ . Indeed, in the $\kappa = \omega$ case, the two directions are given as exercises in [11] (VII H.18–20 for forcing $\diamond_{\omega_1}^*$ to hold, VIII J.3 for forcing it to fail). For completeness we outline the details (for the general case) here.

To force \diamond_κ^* to hold there are at least two options. Cummings, Foreman and Magidor [3] present one possibility, with an iteration in which the first iterand yields the eventual $\diamond_{\kappa^+}^*$ -sequence and the later iterands shoot clubs through κ^+ to witness that the sequence is indeed a $\diamond_{\kappa^+}^*$ -sequence.

For the more L -inclined, one can follow the lead of [11], using a simpler forcing but a much more involved verification that it yields a $\diamond_{\kappa^+}^*$ -sequence. We use the following lemma.

Lemma 2 *Let κ^+ be a successor cardinal. If there is a κ^+ -tree T which is a*

subtree of ${}^{<\kappa^+}2$, and an $h \in {}^{\kappa^+}2$ such that

$$\forall f \in {}^{\kappa^+}2 \exists g \in {}^{\kappa^+}2 (f \in L(\{g, h\}) \wedge g \text{ is the union of a cofinal branch in } T)$$

then $\diamond_{\kappa^+}^+$ holds.

Note that the conclusion here is the principle $\diamond_{\kappa^+}^+$, which is stronger than $\diamond_{\kappa^+}^*$. Indeed, we could code using $\diamond_{\kappa^+}^+$ rather than $\diamond_{\kappa^+}^*$, but have elected to use $\diamond_{\kappa^+}^*$ for simplicity. The proof of Lemma 2 is similar to the proof of $\diamond_{\kappa^+}^+$ in L ; see [11] Exercise VI.9 for an outline or [1] Section 4.1.1 for the full details.

With Lemma 2 at our disposal we may now simply force a Kurepa tree T to exist and observe that it will satisfy the conditions of the lemma. We can do this with the following forcing, which is perhaps slightly simpler than that presented (in the ω_1 case) in [11].

Definition 3 For any cardinal κ , let P_{κ^+} denote the partial order whose elements are pairs $\langle X, \alpha \rangle$, where $X \subset {}^{\kappa^+}2$, $\alpha < \kappa^+$, and $|X| \leq \kappa$. For $\langle X, \alpha \rangle, \langle Y, \beta \rangle \in P_{\kappa^+}$, say that $\langle Y, \beta \rangle \leq \langle X, \alpha \rangle$ if and only if $Y \supseteq X$, $\beta \geq \alpha$, and for all $f \in Y$ there is a $g \in X$ such that $f \upharpoonright \alpha = g \upharpoonright \alpha$.

The condition $\langle X, \alpha \rangle$ can be thought of as determining that the initial segment up to level α of the ultimate κ^+ -Kurepa tree T will consist of the α -initial segments of the elements of X , and further, forcing that every element of X will be the union of a cofinal branch of T . Clearly P_{κ} is κ^+ -closed, has the κ^{++} -cc, and has cardinality κ^{++} , so cardinals and the GCH are preserved. To verify that the conditions of Lemma 2 hold for T in the generic extension $V[G]$ by P_{κ} , we take $h \in ({}^{\kappa^+}2)^{V[G]}$ encoding T itself and every element of $\mathcal{P}(\kappa)^{V[G]}$. Every ground model function from κ to 2 appears as a cofinal branch of T up to bounded differences, every extension model function from κ to 2 has a nice name that can be encoded by a ground model function from κ to 2, and T determines the entire generic G , so indeed every $f \in ({}^{\kappa^+}2)^{V[G]}$ lies in $L(\{g, h\})$ for some branch g of T , as required.

Making $\diamond_{\kappa^+}^*$ fail will simply be a matter of adding κ^{++} -many Cohen subsets of κ^+ and observing that $\diamond_{\kappa^+}^*$ will not hold in the extension. We use the common notation $\text{Add}(\lambda, \mu)$ for the partial order with partial functions from $\lambda \times \mu$ to 2 of cardinality less than λ as conditions; thus, our forcing to make $\diamond_{\kappa^+}^*$ fail will be $\text{Add}(\kappa^+, \kappa^{++})$. The arguments in this subsection actually work for any uncountable regular cardinal λ , so we present them at this level of generality, although for later sections we will only need the case when $\lambda = \kappa^+$ for some κ .

The partial order $\text{Add}(\lambda, \lambda^+)$ is of course λ -closed, λ^+ -cc, and has cardinality λ^+ , so it preserves cardinals and the GCH. The verification that it destroys \diamond_{λ}^* is presented in [4] for the case when $\lambda = \omega_1$, but with appropriate modifications

the argument can be lifted to work for any regular uncountable λ . We present this modified argument here.

To avoid the temptation to abuse notation, we make a definition to recast \diamond_λ^* -sequences in terms of characteristic functions, giving notationally more convenient objects.

Definition 4 *Let λ be a regular cardinal and let $D = \langle D_\alpha \mid \alpha < \lambda \rangle$ be a sequence such that for every $\alpha < \lambda$, $D_\alpha \subset \mathcal{P}(\alpha)$ and $|D_\alpha| \leq |\alpha|$. We say that d is a listing of D if d is a function on λ with the property that for each $\alpha < \lambda$, $d(\alpha)$ enumerates the characteristic functions of the elements of D_α in order type $|D_\alpha|$. That is,*

- (i) *for each $\alpha < \lambda$, $d(\alpha)$ is a function from $|D_\alpha|$ to ${}^\alpha 2$; and*
- (ii) *for all $\alpha < \lambda$ and $\beta < |D_\alpha|$, there is some $S \in D_\alpha$ such that for all $\zeta < \alpha$, $d(\alpha)(\beta)(\zeta) = 1$ if and only if $\zeta \in S$; and*
- (iii) *for all $\alpha < \lambda$ and $S \in D_\alpha$, there is a unique $\beta < |D_\alpha|$ such that for all $\zeta < \alpha$, $d(\alpha)(\beta)(\zeta) = 1$ if and only if $\zeta \in S$.*

Proposition 5 *Suppose that $M \models \text{ZFC} + \text{GCH}$ and λ is a regular cardinal of M . Then if G is $\text{Add}(\lambda, \lambda^+)$ -generic over M , $M[G] \models \text{ZFC} + \text{GCH} + \neg \diamond_\lambda^*$.*

PROOF. We first claim that forcing with $\text{Add}(\lambda, \lambda)$ destroys any \diamond_λ^* -sequence of M . Let D be a \diamond_λ^* -sequence of M . For notational convenience, we may assume by expanding the sets D_α if necessary that for every $\alpha < \lambda$, $|D_\alpha| = |\alpha|$. Let d be a listing of D ; the assumption of the last sentence thus becomes the statement that for each $\alpha < \lambda$, $d(\alpha)$ has domain $|\alpha|$.

Let G_λ be $\text{Add}(\lambda, \lambda)$ -generic over M , let \dot{C} be a name for a club subset of λ in $M[G_\lambda]$, and let \dot{F} name $\bigcup G_\lambda$. We claim that in $M[G_\lambda]$, the subset of λ with characteristic function $\bigcup G_\lambda$ is not correctly guessed by D on all elements of \dot{C}_{G_λ} . Suppose to the contrary that there is some $p \in G_\lambda$ such that

$$p \Vdash (\dot{C} \text{ is a club in } \check{\lambda}) \wedge \forall \alpha \in \dot{C} \exists \delta < |\alpha| (\dot{F} \upharpoonright \alpha = \check{d}(\alpha)(\delta)).$$

So that we may smoothly deal with the successor and inaccessible cases simultaneously, let $\gamma = \kappa + 1$ for κ such that $\kappa^+ = \lambda$ if λ is a successor cardinal, and let $\gamma = \lambda$ otherwise. By induction on rank in ${}^{<\gamma} 2$, we may construct conditions $\langle p_s \mid s \in {}^{<\gamma} 2 \rangle$ and ordinals $\langle \alpha_s \mid s \in {}^{<\gamma} 2 \rangle$ such that the following properties hold.

- (i) $p_\emptyset \leq p$ and $p_\emptyset \Vdash \check{\alpha}_\emptyset \in \dot{C}$.
- (ii) For every $s \in {}^{<\gamma} 2$, $\text{dom}(p_s) \in \lambda$.
- (iii) $s \subseteq t$ implies $p_t \leq p_s$ and $\alpha_t \geq \alpha_s$.

(iv) For $i \in 2$ we have $\text{dom}(p_{s \smallfrown \langle i \rangle}) \ni \text{dom}(p_s)$, and

$$p_{s \smallfrown \langle i \rangle}(\text{dom}(p_s)) = i.$$

(v) For $i \in 2$ we have $\alpha_{s \smallfrown \langle i \rangle} > \text{dom}(p_s)$ and $p_{s \smallfrown \langle i \rangle} \Vdash \check{\alpha}_{s \smallfrown \langle i \rangle} \in \dot{C}$.

(vi) If $t \in {}^\beta 2$ then $\text{dom}(p_t) \geq \sup(\{\alpha_s \mid s \in {}^\beta 2\})$.

(vii) $\text{dom}(s) = \text{dom}(t)$ implies $\text{dom}(p_s) = \text{dom}(p_t)$.

(viii) If $s \in {}^\eta 2$ for η is a limit ordinal, then $p_s = \bigcup_{\beta < \eta} p_{s \upharpoonright \beta}$ and

$$\alpha_s = \sup(\{\alpha_{s \upharpoonright \beta} \mid \beta < \eta\}).$$

Indeed, we may construct such $\langle p_s \mid s \in {}^{<\lambda} 2 \rangle$ and $\langle \alpha_s \mid s \in {}^{<\lambda} 2 \rangle$ by first extending p as appropriate for (i) and (ii), at successor stages extending to satisfy (iv), (v), (vi) and (vii) in that order while respecting (ii) and (iii), and at limit stages satisfying (viii). Note that to satisfy (vi) at successor stages, we rely on our assumption of the GCH to give that $\{\alpha_s \mid s \in {}^\beta 2\}$ is bounded below λ .

We claim that for all $s \in {}^{<\gamma} 2$, $p_s \Vdash \check{\alpha}_s \in \dot{C}$. Of course from the definitions we need only check this for s with domain a limit ordinal. But for such s , $p_s \Vdash \check{\alpha}_{s \upharpoonright \beta} \in \dot{C}$ for all $\beta < \text{dom}(s)$, and so since $p \geq p_s$ forces that \dot{C} is a club and $\alpha_s = \sup(\{\alpha_{s \upharpoonright \beta} \mid \beta < \gamma\})$, $p_s \Vdash \check{\alpha}_s \in \dot{C}$.

Note that for any limit ordinal $\zeta < \gamma$, α_s for $s \in {}^\zeta 2$ is independent of the choice of s : for t with domain less than ζ ,

$$\alpha_{t \smallfrown \langle i \rangle} > \text{dom}(p_t) \geq \sup(\{\alpha_s \mid s \in {}^{\text{dom}(t)} 2\}),$$

so

$$\alpha_s = \sup(\{\alpha_{s \upharpoonright \beta} \mid \beta < \zeta\}) = \sup(\{\alpha_t \mid \text{dom}(t) < \zeta\}).$$

Hence, let us denote α_s for $s \in {}^\zeta 2$ by α_ζ . Observe further that because we have terms $\text{dom}(p_t)$ interleaving with terms α_t in the above inequalities, and $p_s = \bigcup_{\beta < \zeta} p_{s \upharpoonright \beta}$ for $s \in {}^\zeta 2$, we have $\text{dom}(p_s) = \alpha_\zeta$ for $s \in {}^\zeta 2$.

But now let μ be the least cardinal such that $|\alpha_\mu| = \mu$; such a $\mu < \lambda$ can easily be found by a typical closure argument. For each $s \in {}^\mu 2$ we have a condition p_s such that $p_s \upharpoonright \lambda \in {}^{\alpha_\mu} 2$. Moreover, (iv) dictates that for $s \neq t \in {}^\alpha 2$, $p_s \upharpoonright \lambda \neq p_t \upharpoonright \lambda$. Thus, we have 2^μ distinct elements of ${}^{\alpha_\mu} 2$, so not all of them can be of the form $d(\alpha_\mu)(\delta)$ for $\delta < |\alpha_\mu| = \mu$. So let $s \in {}^\mu 2$ be such that for all $\delta < \mu$, $p_s \upharpoonright \lambda = p_s \upharpoonright \alpha_\mu \neq d(\alpha_\mu)(\delta)$. But then

$$p_s \Vdash (\check{\alpha}_\mu \in \dot{C}) \wedge \forall \delta < |\check{\alpha}_\mu| (\check{F} \upharpoonright \alpha_\mu = (p_s \check{\upharpoonright} \alpha_\mu) \neq \check{d}(\check{\alpha}_\mu)(\delta))$$

contradicting the fact that $p_s \leq p$. We have therefore shown that $\text{Add}(\lambda, \lambda)$ destroys any ground model \diamond_λ^* -sequences.

Next we claim that all ground model \diamond_λ^* -sequences will be destroyed by our forcing $\text{Add}(\lambda, \lambda^+)$. Since $\text{Add}(\lambda, \lambda^+) \cong \text{Add}(\lambda, \lambda) * \text{Add}(\lambda, \lambda^+)$, it suffices

to show that such a D cannot be resurrected after being killed by the initial $\text{Add}(\lambda, \lambda)$ piece of the forcing. Now the statement that D is not a \diamond_λ^* sequence is equivalent to there being a subset X of λ such that the set of α such that $X \cap \alpha \notin D_\alpha$ is stationary in λ . The partial order $\text{Add}(\lambda, \lambda^+)$ preserves stationary subsets of λ , so D will continue not being a \diamond_λ^* -sequence after subsequently forcing with $\text{Add}(\lambda, \lambda^+)$. Hence, D is not resurrected, and we may conclude that all ground model \diamond_λ^* -sequences are destroyed by the forcing $\text{Add}(\lambda, \lambda^+)$.

Now suppose that some D of the right form to be a \diamond_λ^* -sequence (that is, satisfying the assumptions of Definition 4) is added by $\text{Add}(\lambda, \lambda^+)$; we wish to show that D is not in fact a \diamond_λ^* -sequence in $M[G]$. Let $d \in M[G]$ be a listing of D . Since $\text{Add}(\lambda, \lambda^+)$ is λ -closed and hence adds no new $< \lambda$ -tuples of ground model sets, $d(\alpha)$ is an element of M for each $\alpha < \lambda$. Therefore, d can be named by a name \dot{d} which involves for each $\alpha < \lambda$ a single antichain of $\text{Add}(\lambda, \lambda^+)$ to determine $d(\alpha)$. To be precise, if A_α is a maximal antichain of conditions that determine $d(\alpha)$, and for $p \in A_\alpha$ we write f_p for that function such that $p \Vdash \dot{d}(\check{\alpha}) = \check{f}_p$, then we may take \dot{d} to be

$$\dot{d} = \bigcup_{\alpha < \lambda} \left\{ \left\langle \langle \langle \check{\alpha}, 1 \rangle \rangle, 1 \right\rangle, \left\langle \left(\langle \langle \check{\alpha}, 1 \rangle \rangle \cup \{ \langle \check{f}_p, p \rangle \mid p \in A_\alpha \} \right), 1 \right\rangle \right\}.$$

Since $\text{Add}(\lambda, \lambda^+)$ has the λ^+ -chain condition, $|\bigcup_{\alpha < \lambda} A_\alpha| \leq \lambda$, and so

$$\left| \bigcup_{\alpha < \lambda} \bigcup_{p \in A_\alpha} \text{dom}(p) \right| \leq \lambda.$$

Thus, there is some common upper bound $\gamma < \lambda$ on the domains of those conditions p appearing in \dot{d} . Now

$$\text{Add}(\lambda, \lambda^+) \cong \text{Add}(\lambda, \gamma) \times \text{Add}(\lambda, \lambda^+),$$

and if $G_\gamma = G \cap \text{Add}(\lambda, \gamma)$, we have $d \in M[G_\gamma]$. Since $\text{Add}(\lambda, \gamma)$ is λ -closed,

$$\text{Add}(\lambda, \lambda^+)^{M[G_\gamma]} = \text{Add}(\lambda, \lambda^+)^M.$$

So by what we have already shown, if $G^\gamma = G \cap \text{Add}(\lambda, \lambda^+)$, then d does not represent a \diamond_λ^* -sequence in $M[G_\gamma][G^\gamma] = M[G]$. Therefore, there are no \diamond_λ^* -sequences in $M[G]$. \square

3 Forcing a definable well-order

In this section we exhibit our forcing which yields a universe with a definable well order. There is much flexibility in the definition we shall present, a fact

which we will later exploit when trying to preserve various different kinds of large cardinals.

As discussed in the introduction, the general idea of our forcing is to use \diamond_{κ}^* at various κ to act as an oracle, coding up a proper class of ordinals from which our well-order will be defined. In fact, we further obtain that the extension $V[G]$ is of the form $L[A]$ for A a definable class in $V[G]$. In some sense this is the closest to L we can hope to get while trying to preserve very large cardinals — it follows from Kunen’s theorem that $V \neq L(x)$ for any set x if V contains strong cardinals, and of course A cannot be taken to be definable over L as that would give $L[A] = L$. On the other hand, it is possible to have properties very different from those of L coded into A (for example, the failure of $\diamond_{\kappa^+}^*$ for many cardinals $\kappa!$), so in itself this should not be thought of as a resolution of the outer model programme.

We wish to force at various successor cardinals to “switch \diamond^* on or off”, and then use this as an oracle to make the universe well-orderable. Perhaps the most natural sequence of cardinals at which to do this would be simply the class of *all* infinite successor cardinals. However, for consideration of large cardinal preservation, it will be convenient to use more restricted classes of successor cardinals, and we present our results in this generality.

Definition 6 *A definable class C of cardinals is a coding class if there is a definable class B of cardinals such that*

- (1) *C is a proper class, and*
- (2) *every element of C is a successor cardinal, and*
- (3) *if B is a set, then every successor cardinal greater than the supremum of B is in C , and*
- (4) *for every element β of B , $\beta^+ \in C$, and the least successor cardinal greater than β that is not in C (if such exists) is the successor of an inaccessible cardinal, and*
- (5) *B is countably closed.*

Given a coding class C , we will denote by c the increasing enumeration of C . For our present purposes, one may think of C being the class of *all* successor cardinals, with B empty and c being the function $\aleph_{\cdot+1} : \alpha \mapsto \aleph_{\alpha+1}$. Of course there are concerns regarding the absoluteness of C which we shall address in due course; unless otherwise stated, $c(\alpha)$ should be taken to be computed in the ground model V .

So let C be a fixed coding class, with c its increasing enumeration. We retain the notation of Section 2 of P_{κ^+} being our forcing to produce a $\diamond_{\kappa^+}^*$ -sequence. For ease of notation let us set $Q_{\kappa^+} = \text{Add}(\kappa^+, \kappa^{++})$, the forcing that quashes all $\diamond_{\kappa^+}^*$ -sequences. Note again that we assume $V \models \text{GCH}$. For each ordinal α , let $R_{c(\alpha)}$ be the sum of $P_{c(\alpha)}$ and $Q_{c(\alpha)}$, that is, the partial order given by

combining disjoint copies of $P_{c(\alpha)}$ and $Q_{c(\alpha)}$ below a new maximum element in the obvious way. For concreteness, let us set $1_{R_{c(\alpha)}} = \emptyset$, and let

$$R_{c(\alpha)} = \{1_{R_{c(\alpha)}}\} \cup (\{0\} \times P_{c(\alpha)}) \cup (\{1\} \times Q_{c(\alpha)}).$$

For $r_0, r_1 \in R_{c(\alpha)}$, $r_1 \leq r_0$ if and only if either $r_0 = 1_{R_{c(\alpha)}}$, or $r_0 = \langle i, r'_0 \rangle$ and $r_1 = \langle i, r'_1 \rangle$ for some $i \in 2$ and $r'_1, r'_0 \in P_{c(\alpha)} \cup Q_{c(\alpha)}$ such that $r'_1 \leq_{P_{c(\alpha)}} r'_0$ or $r'_1 \leq_{Q_{c(\alpha)}} r'_0$. Clearly $R_{c(\alpha)}$ will have cardinality $c(\alpha)^+$, be $c(\alpha)$ -closed, and have the $c(\alpha)^+$ -cc, since these statements are true of both $P_{c(\alpha)}$ and $Q_{c(\alpha)}$. For $\gamma \in \text{Ord}$ not of the form $c(\alpha)$ for some α , let R_γ be the trivial forcing.

Definition 7 *The \diamond^* Oracle Partial Order S is the reverse Easton iteration of \dot{R}_α as above for $\alpha \in \text{Ord}$.*

Note that with only trivial forcings used between cardinals, Easton support is the same as taking direct limits at inaccessibles and inverse limits elsewhere. In particular, this implies that S can be factored as $S_\alpha * \dot{S}^\alpha$ for any stage α — see for example Lemma 21.8 of [9].

Lemma 8 *If $V \models \text{ZFC} + \text{GCH}$ and G is generic for the \diamond^* Oracle Partial Order S over V , then $V[G]$ satisfies $\text{ZFC} + \text{GCH}$ and has the same cardinals as V .*

PROOF. As is generally the case for reverse Easton iterations used in practice, S is tame because the iterands are increasingly closed (see Lemmata 2.22 and 2.31 of [6]) so ZFC is preserved. To prove that cardinals and the GCH are preserved, we argue by induction on the length of the iteration. Successor stages are immediate from the fact that R_{κ^+} is κ^+ -closed and κ^{++} -cc. For limit stages λ , cardinals and the GCH are preserved below λ by the closure of the tail parts of the iteration S_λ . There is a dense suborder of S_λ of size at most λ^+ if λ is singular or λ is regular, so the GCH and cardinals are preserved above λ^+ , or λ in the regular case. It therefore only remains to show that in the λ singular case, the GCH holds at λ and λ^+ is preserved, and this follows by considering nice names for subsets of λ built up as the union of nice names for subsets of smaller cardinals. \square

In particular, note that the class of successor cardinals is unchanged at each stage of the iteration, so if $c = \aleph_{+1}$, then c is absolute.

Considering the factorisation of S as $S_\kappa * S^\kappa$, with S^κ being κ -closed, also gives the following.

Lemma 9 *Forcing with S preserves inaccessible cardinals.* \square

Another basic property of this forcing we shall need is that after applying it, \diamond^* holds at exactly those points in the sequence c where we expect it to: those $c(\alpha)$ such that $\langle 0, 1_{P_{c(\alpha)}} \rangle \in G(c(\alpha))$.

Lemma 10 *Let $V \models ZFC + GCH$ and let G be S -generic over V . Then for every $\alpha \in \text{Ord}$, $V[G] \models \diamond_{c(\alpha)}^*$ if and only if $\langle 0, 1_{P_{c(\alpha)}} \rangle \in G(c(\alpha))$.*

PROOF. Let κ^+ be of the form $c(\alpha)$, and consider the factorisation of S as $S_{\kappa^+} * R_{\kappa^+} * S^{\kappa^++1}$. Clearly $V[G_{\kappa^+} * G(\kappa^+)] \models \diamond_{\kappa^+}^*$ if and only if $\langle 0, 1_{P_{c(\alpha)}} \rangle \in G(c(\alpha))$, and S^{κ^++1} is κ^{++} -closed, so any $\diamond_{\kappa^+}^*$ -sequence of $V[G_{\kappa^+} * G(\kappa^+)]$ remains a $\diamond_{\kappa^+}^*$ -sequence of $V[G]$, and no new $\diamond_{\kappa^+}^*$ -sequences are added by G^{κ^++1} . \square

Theorem 11 *Let $V \models ZFC + GCH$, and let S be the \diamond^* Oracle Partial Order as defined above. If G is S -generic over V , then there is a definable class of ordinals A of $V[G]$ such that $V[G] = L[A]$. In particular, $V[G] = HOD^{V[G]}$, and there is a definable well-order on $V[G]$.*

PROOF. The class A will of course be $\{\alpha \in \text{Ord} \mid \diamond_{c(\alpha)}^* \text{ holds}\}$. Clearly A is definable in $V[G]$; the question will be what the relationships between A , $C^{V[G]}$ and C^V are, because of course C need not be absolute.

Actually in many specific cases of interest, C will be absolute. As mentioned above, the class of all successor cardinals will be absolute, and in context of the next section one could get absoluteness from the fact that we are preserving the large cardinals from which B is defined, along with Hamkins' Gap Forcing Theorem [8]. But in any case, the requirements we have placed on B will give C^V and $C^{V[G]}$ sufficient agreement to show that every set in $V[G]$ is encoded into A . Namely, since $V[G]$ satisfies the Axiom of Replacement with respect to formulas involving a predicate for V (see [6] Lemma 2.19), the usual argument to show that countably closed unbounded sets have countably closed unbounded intersection goes through for $B^V \cap B^{V[G]}$, and we see that there are unboundedly many cardinals in $B^V \cap B^{V[G]}$. We claim that this agreement is sufficient for our purposes.

So suppose $x \in V[G]$, and let X be a subset of $\mu = |\text{trcl}(\{x\})|$ coding up x . We claim that for every β such that $c(\beta) \geq \mu$, X appears in the choices made by the generic between $c(\beta)$ and the least inaccessible greater than $c(\beta)$ (if one exists), where in each case $c(\beta)$ is to be computed in V . To see this, let $\iota_{c(\beta)}$ denote the least inaccessible greater than $c(\beta)$ if such exists or Ord otherwise, and consider the class

$$D_{X,\beta} = \left\{ s \in S^{\mu+1} \mid \exists \gamma \in \iota_{c(\beta)} \left(\gamma \geq \beta \wedge \forall \alpha < \mu \right. \right. \\ \left. \left. \begin{aligned} & \left(\alpha \in X \rightarrow \Vdash_{S^{\mu+1}} s(c(\gamma + \alpha)) \leq_{\dot{R}_{c(\gamma+\alpha)}} \langle 0, \dot{i}_{P_{c(\gamma+\alpha)}} \rangle \wedge \right. \right. \\ & \left. \left. \alpha \notin X \rightarrow \Vdash_{S^{\mu+1}} s(c(\gamma + \alpha)) \leq_{\dot{R}_{c(\gamma+\alpha)}} \langle 1, \dot{i}_{Q_{c(\gamma+\alpha)}} \rangle \right) \right) \right\}.$$

Because a direct limit is taken at $\iota_{c(\beta)}$, we have for any $s \in S^{\mu+1}$ that $\text{supp}(s) \cap \iota_{c(\beta)}$ is bounded in $\iota_{c(\beta)}$, so we may extend $s \upharpoonright \iota_{c(\beta)}$ to an element of $D_{X,\beta} \cap S^{\mu+1, \iota_{c(\beta)}}$, and then “re-attach the tail of s ” to get an extension of s in $D_{X,\beta}$. Hence, for each β with $c(\beta) \geq \mu$, the class $D_{X,\beta}$ is dense in $S^{\mu+1}$, and so has non-empty intersection with $G^{\mu+1}$.

Now because inaccessibles are absolute between V and $V[G]$ (Lemma 9), if $\lambda \in B^V \cap B^{V[G]}$ is greater than μ and ι_λ is the least inaccessible greater than λ , then $[\lambda, \iota_\lambda) \cap \text{Succ} \subset C^V \cap C^{V[G]}$, where Succ denotes the class of successor cardinals. Therefore, taking β , s and γ such that β is least with $c(\beta) \geq \lambda$, $s \in D_{X,\beta} \cap G^{\mu+1}$, and γ is as in the definition of $D_{X,\beta}$ witnessing that $s \in D_{X,\beta}$, we have that $c^V \llbracket \gamma, \gamma + \mu \rrbracket = c^{V[G]} \llbracket \zeta, \zeta + \mu \rrbracket$ for some ordinal ζ , and so indeed, $X \in L[A]$. Hence, we have shown that $V[G] = L[A]$, as required. \square

4 Preserving large cardinals

As mentioned in the introduction, if we wish to preserve large cardinals while forcing, we will generally have master conditions to be hit by our generic, which will be problematic if we wish to preserve many large cardinals. In other settings this can be overcome by making the partial order sufficiently homogeneous that generics containing particular master conditions can be constructed in *any* generic extension — see for example [2] for the case of forcing gap-1 morasses to exist at every regular cardinal. We need another approach, however, as our forcing partial order is inherently inhomogeneous — indeed, any forcing that yields a model of $V = \text{HOD}$ must be inhomogeneous, as HOD of the generic extension by a homogeneous forcing must be contained in V (see [11], Exercise VII E1).

The solution to this problem comes from the extra flexibility we have because we are interested in forcing a global principle, rather than a local principle at, say, every regular cardinal. We can “thin out” our forcing partial order, still obtaining a definable well-order of the extension universe, but finessing the issue of master conditions by making the forcing trivial at every point where master conditions might be required.

To facilitate this thinning out, we make the following definitions.

Definition 12 *Suppose that φ is a formula in one variable, and more specif-*

ically, that:

- (1) $\varphi(\kappa) \equiv$ “ κ is measurable”, or
- (2) for some ordinal η , $\varphi(\kappa) \equiv$ “ κ is η -strong”, or
- (3) $\varphi(\kappa) \equiv$ “ κ is Woodin”, or
- (4) for some $n \in \omega + 1$, $\varphi(\kappa) \equiv$ “ κ is n -superstrong”, or
- (5) $\varphi(\kappa) \equiv$ “ κ is hyperstrong”, or
- (6) for some definable function g , $\varphi(\kappa) \equiv$ “ κ is $g(\kappa)$ -supercompact”, or
- (7) for some ordinal η , $\varphi(\kappa) \equiv$ “ κ is η -extendible”, or
- (8) for some $m \in \omega$, $\varphi(\kappa) \equiv$ “ κ is m -huge”.

A cardinal λ is a φ -bound if λ is an infinite cardinal, λ is not Mahlo, and if $\varphi(\kappa)$ holds for any $\kappa < \lambda$, then there is an elementary embedding j with critical point κ witnessing the fact that $\varphi(\kappa)$ holds, such that

- (i) if $\varphi(\kappa) \equiv$ “ κ is measurable” then $\kappa < \lambda$ (that is, no extra requirement),
- (ii) if $\varphi(\kappa) \equiv$ “ κ is η -strong” then $\beth_{\kappa+\eta}^+ < \lambda$,
- (iii) if $\varphi(\kappa) \equiv$ “ κ is Woodin” then for all $f : \kappa \rightarrow \kappa$ there is an $\alpha \in \kappa$ and a $j : V \rightarrow M$ elementary such that $f^{\alpha} \subset \alpha$, $\text{crit}(j) = \alpha$, $V_{j(f)(\alpha)} \subseteq M$, and $\beth_{j(f)(\alpha)}^+ < \lambda$.
- (iv) if $\varphi(\kappa) \equiv$ “ κ is n -superstrong” then $\beth_{j^n(\kappa)} < \lambda$,
- (v) if $\varphi(\kappa) \equiv$ “ κ is hyperstrong” then $\beth_{j(\kappa)+1} < \lambda$,
- (vi) if $\varphi(\kappa) \equiv$ “ κ is $g(\kappa)$ -supercompact” then $g(\kappa)^{<\kappa} < \lambda$,
- (vii) if $\varphi(\kappa) \equiv$ “ κ is η -extendible” then $\zeta < \lambda$ for the ζ such that $j : V_{\kappa+\eta} \rightarrow V_{\zeta}$, and
- (viii) if $\varphi(\kappa) \equiv$ “ κ is m -huge” then $2^{j^m(\kappa)} < \lambda$.

We say that λ is a minimal φ -bound if for every cardinal $\nu < \lambda$ there is a cardinal κ which is not a φ -bound such that $\nu \leq \kappa < \lambda$.

Clearly one may construct minimal φ -bounds by the usual methods for obtaining fixed points. The non-Mahloness assumption is simply a convenient way to ensure that the large cardinals we are interested in are not considered φ -bounds. Also note that we have stuck with \beth notation for clarity in the arguments to come, even though we are assuming the GCH and so \aleph notation would be equivalent.

Unless otherwise specified, φ shall henceforth denote one of the listed large cardinal properties 1–8, and for convenience we shall refer to cardinals κ satisfying $\varphi(\kappa)$ as φ -cardinals. This list of large cardinal properties, which will be the ones that are preserved in Theorem 15, should by no means be thought of as encompassing all large cardinals for which the techniques of this chapter are applicable. Rather, it is a representative list of well-known large cardinals each witnessed by boundedly many elementary embeddings so that bounds may be constructed for them. For reasons that will become apparent, it would also be of interest (and moreover straightforward) to include large cardinals

of the form “ φ a limit of φ ” (for example, a measurable limit of measurables). However, if we do not wish to move to a more general statement, we must draw the line somewhere!

Once we have a φ -bound, the succeeding cardinals will remain φ -bounds for some time. The following lemma in this direction will be sufficient for our purposes.

Lemma 13 *Suppose \aleph_β is a φ -bound. Then for all γ less than the least inaccessible greater than \aleph_β (or all γ if no such inaccessible exists), $\aleph_{\beta+\gamma}$ is a φ -bound.*

PROOF. This is immediate from the fact that the least cardinal κ which is *not* a φ -bound above a given φ -bound will satisfy $\varphi(\kappa)$, and hence be inaccessible. \square

With this fact, we are ready to define the points at which we shall perform our coding.

Definition 14 *A cardinal λ is a φ -coding point if*

- (1) λ is a successor cardinal, and
- (2) λ is a φ -bound, and
- (3) if there is a cardinal $\kappa > \lambda$ such that $\varphi(\kappa)$, there is a minimal φ -bound $\mu \leq \lambda$ such that λ is less than the least inaccessible cardinal greater than μ .

Thus, our coding points come after each minimal φ -bound, going on until the next inaccessible cardinal, or indefinitely if there is no next φ -cardinal. Clearly the class C of all φ -coding points is a coding class, witnessed by the class B of minimal φ -bounds.

Theorem 15 *Let $V \models ZFC + GCH$, and let φ be one of the large cardinal properties 1–8 listed in Definition 12. Let C be the coding class of all φ -coding points of V , and let S be the \diamond^* Oracle Partial Order defined from C . Suppose G is S -generic over V . Then there is $V[G]$ -definable class of ordinals A such that $V[G] = L[A]$. Further, if κ is a φ -cardinal in V that is not a limit of φ -cardinals, then in $V[G]$ κ remains a φ -cardinal.*

So for example, we can preserve all measurable cardinals that are not limits of measurables; see below for a discussion of extensions strengthening this.

PROOF. As before, we denote by c the increasing enumeration of C , and let B denote the class of minimal φ -bounds in V . Theorem 11 gives that

$V[G] = L[A]$, and it only remains to show that any φ -cardinal κ of V that is not a limit of φ -cardinals remains a φ -cardinal in $V[G]$. We prove this by lifting embeddings witnessing $\varphi(\kappa)$, and moreover taking these embeddings to be given by ultrapowers or extenders where appropriate. This will allow us to use representation results about the codomain of such embeddings; a good reference for these is [10]. We deal with each large cardinal property separately.

Measurable Cardinals. Let $j : V \rightarrow M$ be an ultrapower embedding witnessing the measurability of κ with $j(\kappa)$ least. We shall construct an S^M -generic G^* over M in $V[G]$, such that we can lift j to $j^* : V[G] \rightarrow M[G^*]$. Note that the φ -coding points of V less than κ are in fact bounded below κ since the class of measurable cardinals is. Hence, S^M is S^V up to stage κ , and is trivial from κ to $j(\kappa)$. We may therefore take $G_\kappa^* = G_\kappa$, trivially extend to $G_{j(\kappa)}^*$, and have a lift of j to $j' : V[G_\kappa] \rightarrow M[G_{j(\kappa)}^*]$. To define $G^{*j(\kappa)}$, note that every element of M has the form $j(f)(\kappa)$, where $f : \kappa \rightarrow V$ is a function in V , and so every element of $M[G_{j(\kappa)}^*]$ has the form $\sigma_{G_{j(\kappa)}^*}$, where σ has the form $j(f)(\kappa)$. We claim that the filter on $S^{j(\kappa)}$ generated by $j'^{**}G^{j(\kappa)}$ is $(S^{j(\kappa)})^M$ -generic over M .

So suppose that D is a dense class in $M[G_{j(\kappa)}^*]$, defined (in $M[G_{j(\kappa)}^*]$) relative to the parameter $d \in M[G_{j(\kappa)}^*]$ by $D = \{x \mid \psi(x, d)\}$. Let σ be an $S_{j(\kappa)}^M$ -name in M such that $d = \sigma_{G_{j(\kappa)}^*}$, and let $f : \kappa \rightarrow V$ in V be such that $\sigma = j(f)(\kappa)$. Since κ is not a measurable-coding point, $(S^\kappa)^V$ is κ^+ -closed, and we see that it is dense for $s \in (S^\kappa)^V$ to extend an element of the class $D_\alpha = \{x \mid \psi(x, f(\alpha)_{G_\kappa})\}$ of V whenever $\alpha \in \kappa$ with $f(\alpha)$ an S_κ -name and D_α dense in $(S^\kappa)^V$. Therefore, we may take such an s lying in G^κ . By elementarity, it follows that $j(s)$ extends an element of D . Hence, the filter generated by $j'^{**}G^\kappa$ is indeed $(S^{j(\kappa)})^{M[G_{j(\kappa)}^*]}$ -generic over M . By the Lifting Lemma, it follows that there is an elementary embedding $j^* : V[G] \rightarrow M[G^*]$ lifting j , and so κ is measurable in $V[G]$.

η -Strong Cardinals. We may assume that our η -strong embedding $j : V \rightarrow M$ is an extender ultrapower embedding, with every element of M having the form $j(f)(a)$, with a a finite tuple from $|V_{\kappa+\eta}|^+$ and f a function in V from $[\kappa]^{|a|}$ to V . As in the measurable cardinal case, we show that the filter generated by $j'^{**}G^\kappa$ is $(S^{j(\kappa)})^{M[G_{j(\kappa)}^*]}$ -generic, observing that by the definition of a φ -bound, S^κ is trivial up to at least stage $|V_{\kappa+\eta}|^{++}$, and so we have the requisite closure to make the argument go through. We can therefore lift j to $j^* : V[G] \rightarrow M[G^*]$. We may also conclude that $V_{\kappa+\eta}^{V[G]} \subseteq M[G^*]$ from a nice names argument, since $V_{\kappa+\eta}^V \subseteq M$, $G_{\kappa+\eta} = G_{\kappa+\eta}^*$, and $S_{\kappa+\eta}$ is trivial beyond some bound below κ . Hence, κ is η -strong in $V[G]$.

Woodin Cardinals. The situation for Woodin cardinals is somewhat differ-

ent from that for the other large cardinals listed here, since Woodinness is witnessed by multiple embeddings. However, since there are only boundedly many, this point will not present a problem. Let ι denote the supremum of the Woodin-coding points less than κ . Note that for any two functions $f, g : \kappa \rightarrow \kappa$, if $f(\alpha) \leq g(\alpha)$ for all $\alpha \in \kappa$, then

$$\{\alpha \in \kappa \mid f \text{``}\alpha \subseteq \alpha\} \subseteq \{\alpha \in \kappa \mid g \text{``}\alpha \subseteq \alpha\}.$$

Now given a name \dot{f} for a function from κ to (ι, κ) , we can find a function $\bar{f} : \kappa \rightarrow (\iota, \kappa)$ in V such that $1_S \Vdash \dot{f} \leq \bar{f}$, since the forcing iterands are trivial from ι to κ . To prove that Woodinness is preserved, then, it is sufficient to show that for every function f in V from κ to the interval (ι, κ) , there is an α with $f \text{``}\alpha \subseteq \alpha$ and a j^* from $V[G]$ to N an inner model of $V[G]$ such that $\text{crit}(j^*) = \alpha$ and $V_{j^*(f)(\alpha)} \subseteq N$. Since κ is Woodin in V , we have for each such f an α with $f \text{``}\alpha \subseteq \alpha$ and an elementary embedding $j : V \rightarrow M$ such that $\text{crit}(j) = \alpha$ and $V_{j(f)(\alpha)} \subseteq M$; that is, an η -strong embedding for η such that $\alpha + \eta = j(f)(\alpha)$. But this is simply a case of η -strength, so we can lift j to j^* as above. Since $j^* \upharpoonright V = j$, $j^*(f)(\alpha) = j(f)(\alpha)$, and we are done.

n -Superstrong Cardinals and Hyperstrong Cardinals. The argument is analogous to the measurable and η -strong cases, this time with our extender models having elements of the form $j(f)(a)$ with a in $V_{j^n(\kappa)}$ and f with domain $V_{j^{n-1}(\kappa)}$ in the case of n -superstrong cardinals, and $a \in V_{j(\kappa)+1}$ and f with domain $V_{\kappa+1}$ in the case of hyperstrong cardinals. The required level of agreement between $V[G]$ and $M[G^*]$ again follows from a nice names argument, noting in the hyperstrong case that $V_{j(\kappa)+1} \in M \leftrightarrow H_{j(\kappa)+1} \in M$.

$g(\kappa)$ -Supercompact Cardinals. In this case we may take the elements of M to be of the form $j(f)(j \text{``} g(\kappa))$, so in $V[G_\kappa]$ we consider all $x \in \mathcal{P}_\kappa(g(\kappa))$; again, because of our definition of a $g(\kappa)$ -supercompact coding point, the argument goes through without difficulty. To show that $M[G^*]$ is closed under taking $g(\kappa)$ -tuples, note that for any $g(\kappa)$ -tuple from $M[G^*]$ in $V[G]$, we may consider an $g(\kappa)$ -tuple t of names for its elements in $V[G]$, where the names are in M . All $g(\kappa)$ -tuples in $V[G]$ of elements of V are in $V[G_{g(\kappa)}]$ by closedness of the tail of the iteration, and so since $S_{g(\kappa)}$ is trivial beyond some bound below κ , there is a nice name \dot{t} for t with only $g(\kappa)$ elements. Therefore, since M is closed with respect to taking $g(\kappa)$ -tuples in V , $\dot{t} \in M$, and since G_κ^* is the same as G_κ , $\dot{t} \in M[G_{j(\kappa)}^*]$. But then the original $g(\kappa)$ -tuple of elements from $M[G^*]$ is in $M[G^*]$, as desired.

η -Extendible Cardinals. On the domain and range of j , the nontrivial part of S is bounded below κ , so this is trivial.

m -Huge Cardinals. This is much like the $g(\kappa)$ -supercompact case. The elements of M may be taken to be of the form $j(f)(j^{m}(j^m(\kappa)))$ where the domain of f is $\mathcal{P}(j^m(\kappa))$, so the assumed $2^{j^m(\kappa)}$ -closure of S^κ is what we need to construct the M -generic for the lifting. Closure of $M[G^*]$ with respect to taking $j^m(\kappa)$ -tuples is exactly as in the $g(\kappa)$ -supercompact case.

This completes the verification. \square

One may wonder if the restriction on which φ -cardinals are preserved (that is, only those that are not limits of φ -cardinals) can be lifted. However, some kind of restriction like this is necessary for our technique. We are using the fact that the set of coding points is bounded below every cardinal that we lift. Hence, by Fodor's theorem, we cannot hope to lift all φ -cardinals from a universe where they form a stationary set in Ord.

On the other hand, we can always mollify this problem by restricting it to smaller and smaller classes of large cardinals. By thinning out the class of coding point while keeping it unbounded in Ord, the above arguments will still go through at all of the cardinals that were previously preserved, but with new cardinals added to the list of large cardinals that stay large. For example, if there are boundedly many measurable limits of measurables, and we thin out the measurable-coding points to only use those "directly after" a measurable limit of measurables, until there are no more, then we will still preserve all measurable cardinals that are not limits of measurables, but we will also preserve those measurable limits of measurables that are not limits of measurable limits of measurables. If there is a proper class of measurable limits of measurables, the "thinned out" class of coding points is even easier to describe: it is simply the set of φ -coding points for $\varphi \equiv$ " κ is a measurable limit of measurables". Indeed, this can be done not just for φ limits of φ -cardinals, but for any proper class sequence of cardinals at whose limits we don't mind preservation failing. So for example, we may deduce the following.

Theorem 16 *Suppose there is a proper class of φ_0 -cardinals, and let δ be an arbitrary ordinal. Then a definable well-order of the universe may be forced while preserving all measurable, η -strong for $\eta < \delta$, Woodin, n -superstrong for $n \in \omega + 1$, hyperstrong, $\kappa^{+\eta}$ -supercompact for $\eta < \delta$, η -extendible for $\eta < \delta$, and m -huge for $m \in \omega$ cardinals that are not limits of φ_0 -cardinals.*

PROOF. We take φ to be the (size $|\delta|$) disjunction of all of the stated large cardinal properties for the sake of defining φ -bounds, but for defining the coding points we take B to only contain those minimal φ -bounds that are minimal above a φ_0 -cardinal or are a limit of such (and as before, take the

block of coding points starting at such minimal φ -bounds to have length the next inaccessible). The arguments from Theorem 15 for each individual case will all go through unaffected, as long as the cardinal in question is not a limit of φ_0 -cardinals. \square

Of course it makes sense to choose a very strong large cardinal property as φ_0 in this theorem, for the simple reason that generally, stronger large cardinals *are* limits of weaker large cardinals. The choice can be calibrated to the tastes of the reader — if the assumption of, say, a proper class of ω -superstrong cardinals seems unpalatably strong, one can use some other large cardinal property as φ_0 and the theorem will remain true, albeit vacuous in some cases.

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