

# THE MODEL COMPLETION OF THE THEORY OF MODULES OVER FINITELY GENERATED COMMUTATIVE ALGEBRAS

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ABSTRACT. We find the model completion of the theory modules over  $\mathbb{A}$ , where  $\mathbb{A}$  is a finitely generated commutative algebra over a field  $K$ . This is done in a context where the field  $K$  and the module are represented by sorts in the theory, so that constructible sets associated with a module can be interpreted in this language. The language is expanded by additional sorts for the Grassmanians of all powers of  $K^n$ , which are necessary to achieve quantifier elimination.

The result turns out to be that the model completion is the theory of a certain class of “big” injective modules. In particular, it is shown that the class of injective modules is itself elementary. We also obtain an explicit description of the types in this theory.

## 1. INTRODUCTION

An algebra  $\mathbb{A}$  finitely generated over an algebraically closed field  $K$  corresponds to an affine variety  $V$  over  $K$ , and a module over  $\mathbb{A}$  corresponds to a (quasi-coherent) sheaf over  $V$ . Whereas varieties can be reasonably considered within the framework of model theory (for example, as definable sets in the theory  $ACF$  of algebraically closed fields), modules (or sheaves) do not appear so naturally. For example, basic results about definability of the fibre dimension are proved, using algebraic methods, for algebras and modules alike. On the model theoretic side, the fibre dimension for a map of varieties (or more generally, for definable sets) is well understood, in a much more general framework. However, the analogous statements for modules can not even be phrased. This work represents, we hope, a first step in approaching these questions.

The purpose of this paper is to find the model completion for the theory of modules over a finitely generated commutative  $K$ -algebra ( $K$  a field), and describe the types in that theory. Our initial approach in formulating this theory is to use a two-sorted language, with a sort  $K$  for the field, and another sort  $M$  for the module. In addition to the field structure on  $K$  and the  $K$ -vector space structure on  $M$ , we introduce symbols for  $n$  commuting linear operators on  $M$ , that represent the generators of the algebra.

Our goal is to find the model completion. To estimate the feasibility of our task, we consider the case  $n = 0$ . In this case we simply have a vector space  $M$  over  $K$ . We immediately observe that the most basic relation on  $M$ , that of linear dependence, can not be expressed in this theory without quantifiers. This example leads us to introduce additional sorts for all the Grassmanians of the vector spaces  $K^n$ . The dependence relation on  $M$  then takes values in these Grassmanians. Thus, with this addition to the language, the above problem is resolved, and it turns out

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that this is the only obstacle for the existence of a model completion, even for the case when  $n > 0$ .

Below we give a precise definition of the language and the theory we work with. The rest is divided into the cases  $n = 0$  and  $n > 0$ . Although the first case is not really different, the kind of problems the cases deal with are different and independent: in the first case, we deal with the vector space structure, as well as the new sorts introduced. In the second case, the main interest comes from the action of the operators  $T_i$ .

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**1.1. The theory of modules over a commutative  $K$ -algebra.** Given  $n \geq 0$ , we use the following language  $\mathcal{L} = \mathcal{L}_n$ :

- (1)  $\mathcal{L}_n = (K, +, \cdot, 0, 1,$
- (2)  $M, +, 0, \cdot,$
- (3)  $G_{(i>0)}^i, \pi_{i(i>0)}, P_{\tilde{\varphi}}, D_{i(i>0)}$
- (4)  $T_1, \dots, T_n)$

Where

- (1) is the language of fields
- (2) is the language of abelian groups together with a function symbol  $\cdot : K \times M \rightarrow M$
- each  $G^i$  is a sort, and  $\pi_i : K^{i^2} \rightarrow G^i$  is a function symbol
- $\tilde{\varphi}$  is a quantifier free formula  $\varphi(x_1, \dots, x_N)$  in the language of fields, together with a partition of its variables to sets of sizes  $k_1^2, \dots, k_m^2, l$ . Given such a  $\tilde{\varphi}$ ,  $P_{\tilde{\varphi}}$  is a predicate symbol on  $G^{k_1} \times \dots \times G^{k_m} \times K^l$  (For the meaning of all this, see below)
- Each  $D_i : M^i \rightarrow G^i$  is a function symbol
- Each  $T_i$  is a function symbol  $T_i : M \rightarrow M$

Given an ideal  $I \subset \mathbb{Z}[T_1, \dots, T_n]$ , the theory  $\mathfrak{T} = \mathfrak{T}_I$  says the following:

- (1) is a field, and (2) is a vector space over it
- For each  $i$ ,  $(G^i, \pi_i)$  is the set of all linear subspaces (the *Grassmanian*) of  $K^i$ . Thus, we view  $K^{i^2}$  as an  $i$ -tuple of row vectors in  $K^i$ , and the theory says that  $\pi_i : K^{i^2} \rightarrow G^i$  is surjective, and two elements belong to the same fibre if and only if the corresponding row vectors span the same linear subspace.
- For any  $\tilde{\varphi}$ ,  $P_{\tilde{\varphi}}$  is (using the notation above) the set induced on  $G^{k_1} \times \dots \times G^{k_m} \times K^l$  by  $\varphi$ , i.e., we have the formula

$$\begin{aligned} \forall p_1 \in G^{k_1}, \dots, p_m \in G^{k_m}, \bar{x} \in K^l \\ (P_{\tilde{\varphi}}(p_1, \dots, p_m, \bar{x}) \iff \\ \exists \bar{x}_1 \in K^{k_1^2}, \dots, \bar{x}_m \in K^{k_m^2} (\varphi(\bar{x}_1, \dots, \bar{x}_m, \bar{x}) \wedge \\ \bigwedge_i \pi(\bar{x}_i) = p_i)) \end{aligned}$$

Note that this is not an additional structure, but part of  $K^{eq}$ . However, as explained above, this addition (together with the operators  $D_i$ ) is essential to achieve quantifier elimination.

- For any  $\bar{v} = v_1, \dots, v_m \in M$ ,  $D_m(\bar{v})$  is the subspace  $p \in G^m$  of all  $\bar{x} \in K^m$  such that  $\sum x_i v_i = 0$ .
- The  $T_i$  represent generators of the algebra. Thus they are commuting linear operators on  $M$ , and any  $p(T_1, \dots, T_n) \in I$  is the 0 operator (See remark 17 about the non-commutative case.)

Models of this theory are (determined by) pairs  $(K, M)$ , where  $K$  is a field, and  $M$  is a module over  $K[T_1, \dots, T_n]/\hat{I}$ . Here,  $\hat{I}$  is the ideal generated by  $I$  in  $K[T_1, \dots, T_n]$ .

*Notation 1.* For the sake of readability, we use the following conventions: the letter  $G$  is used to denote a  $G^i$  with an unspecified  $i$ . Unless otherwise mentioned,  $x, y, z$  are field variables,  $p, q, r$  are  $G$  variables, and  $u, v, w$  are module variables.  $X, Y, Z$  are used for tuples of field variables when considered as matrices. Also, since the number of operators  $T_i$  is fixed in every situation,  $n$  is released for other uses.

## 2. THE CASE $n = 0$

We are looking for a model completion of the theory above, so in particular the field part  $K$  should eliminate quantifiers. Since in  $\mathcal{L}$ , the quantifier free subsets of the field are only those defined in the field language, this leads us to the requirement that  $K$  is algebraically closed. This requirement makes the theory complete, up to the characteristic of  $K$  and the dimension of  $M$  as a vector space over  $K$  (in fact, fixing the characteristic of  $K$  and the dimension of  $M$ , the theory we get is  $\aleph_1$  categorical). Let  $\tilde{\mathfrak{T}}$  be  $\mathfrak{T}_0$ , together with the axioms saying that  $K$  is algebraically closed, and  $M$  is of a given dimension over  $K$  (which might be infinity). Our first goal is:

**Proposition 2.** *The theory  $\tilde{\mathfrak{T}}$  has elimination of quantifiers*

We begin with a few remarks concerning only the relation between  $K$  and the  $G^i$ . For  $\varphi(\bar{x}, p_1, \dots, p_k)$  a formula, let

$$\varphi^*(\bar{x}, Y_1, \dots, Y_k) \stackrel{\text{def}}{=} \varphi(\bar{x}, \pi(Y_1), \dots, \pi(Y_k))$$

Such formulas will be called homogeneous (in  $Y_1, \dots, Y_k$ ).

For  $p_i \in G^{l_i}$  ( $1 \leq i \leq k$ ), let  $[p_1, \dots, p_k] \in G^{\sum l_i}$  be the subspace  $\oplus p_i$  of  $\oplus K^{l_i}$ .

We are going to make assertions regarding linear transformations on spaces like  $K^n$  and  $M^n$ . Since we usually need the claims for matrices with arbitrary terms (and not just constants), we redefine ‘linear map’ to mean any definable map from  $K^m$  to the set of  $n_1 \times n_2$  matrices (some  $m, n_i$ ), considered as linear transformations acting on the right for  $K$  and on the left for  $M$ .

**Lemma 3.** *The following facts hold in  $\tilde{\mathfrak{T}}$ :*

- Any quantifier free formula is equivalent to a quantifier free formula without  $\pi$ .*
- Let  $\varphi(\bar{x}, p_1, \dots, p_k)$  be a quantifier free formula. Then*

$$(5) \quad \varphi \equiv P_{\varphi^*}$$

- The map  $(p_1, \dots, p_k) \mapsto [p_1, \dots, p_k]$  is definable without quantifiers.*

- d. The theory  $\tilde{\mathfrak{L}}$  restricted to  $K$  and the  $G^i$  eliminates quantifiers. Thus, the equation (5) holds for any formula  $\varphi$  in the restricted theory. Any formula in this theory is equivalent to the formula  $P_\varphi$  for some field formula  $\varphi$ .
- e. For any linear map  $A : K^m \rightarrow K^n$ , and for subspaces  $p \subseteq K^m$  and  $q \subseteq K^n$ , the image  $pA$  and the inverse image  $qA^{-1}$  are again linear subspaces. Thus we have induced maps  $A : G^m \rightarrow G^n$  and  $A^{-1} : G^n \rightarrow G^m$  (these maps will be written on the left).

*Proof.* a. It's enough to prove this for atomic formulas. There are two kinds of these:

- $P_\varphi(\pi(\bar{x}), \dots)$ . This holds, by definition, iff

$$\exists \bar{y}, \dots (\varphi(\bar{y}, \dots) \wedge \pi(\bar{y}) = \pi(\bar{x}) \wedge \dots)$$

where  $\bar{y}$  does not appear in any of the  $\dots$  parts. The expression  $\pi(\bar{y}) = \pi(\bar{x})$  just says that  $\bar{x}$  and  $\bar{y}$  span the same vector space, which can be expressed using only the language of fields. Therefore, the above formula is equivalent to  $\exists \dots (\exists \bar{y}(\varphi'(\bar{x}, \bar{y}, \dots)) \wedge \dots)$  where  $\varphi'$  is in the language of fields. By elimination of quantifiers in *ACF*,  $\exists \bar{y}(\varphi'(\bar{x}, \bar{y}, \dots))$  is equivalent to some quantifier free  $\varphi''$ , so our original formula is equivalent to  $P_{\varphi''}$ , where we got rid of one  $\pi$ .

- $\pi(\bar{x}) = p$ . This one is equivalent to  $P_{\bar{x}=\bar{y}}(\bar{x}, p)$ , with the corresponding partition of the variables.
- b. For  $\varphi = P_\psi$ , this follows directly from the definitions. By **a**, this is the only kind of atomic formulas we should check. On the other hand,  $*$  is a homomorphism of boolean algebras, and so is  $P$  restricted to formulas of the form  $\psi^*$ , so the result follows.
- c. This map is  $P_{\bar{x}=\bar{y}}$  for an appropriate partition of the variables, and with  $\bar{x}$  padded with zeroes in the right places (more precisely,  $\bar{x}$  is a matrix with  $k$  matrices of the right sizes on the diagonal, and 0 elsewhere).
- d. By **b**, we need to show there is a quantifier free formula equivalent to

$$\exists A_1(P_\varphi(\bar{x}, p_1, \dots, p_k))$$

where  $A$  is either  $x$  or  $p$ . Unravelling the definition, this amounts to the fact that existential quantifiers commute, together with quantifier elimination for algebraically closed fields. The rest is just a summary of the previous items, together with the fact (clear from inspecting the proofs) that they can be put together.

- e. Is obvious. □

For  $A_1, \dots, A_k$  linear maps, and  $\varphi(\bar{x}, p_1, \dots, p_k)$  a formula, we set

$$(A_1, \dots, A_k)^* \varphi = \bar{A}^* \varphi \stackrel{\text{def}}{=} \varphi(\bar{x}, A_1(q_1), \dots, A_k(q_k))$$

and

$$(A_1, \dots, A_k)_* \varphi = \bar{A}_* \varphi \stackrel{\text{def}}{=} \varphi(\bar{x}, A_1^{-1}(q_1), \dots, A_k^{-1}(q_k))$$

Note that these formulas will depend on the additional variables of the  $A_i$ .<sup>1</sup>

We now go back to the full  $\tilde{\mathfrak{L}}$ , and the next step is to analyse the quantifier free formulas in the theory. The main lemma we need is:

<sup>1</sup> Strictly speaking, the operators  $A_i$  do not actually exist in the language, but the formulas exist (and are quantifier free).

**Lemma 4.** *Any quantifier free formula  $\varphi$  is equivalent to a quantifier free formula  $\psi$ , in which for every term of the form  $D(t_1, \dots, t_n)$ , each  $t_i$  is either a module variable or a module constant<sup>2</sup>*

*Proof.* The claim follows from the fact that for any linear map  $A$ , we have  $D(A\bar{v}) = A^{-1}(D(\bar{v}))$ : Indeed, both sides are equal to the space of all  $\bar{x}$  such that  $\bar{x}A\bar{v} = 0$ . Now, any quantifier free formula  $\varphi$  has the form

$$\varphi'(\bar{x}, D(A_1\bar{v}), \dots, D(A_k\bar{v}), \bar{p})$$

so by the above equality,  $\varphi$  is equivalent to

$$(\bar{A}_*\varphi')(\bar{x}, D(\bar{v}), \dots, D(\bar{v}), \bar{p})$$

□

We can now prove quantifier elimination:

*Proof of proposition 2.* Let  $\varphi(\bar{x}, \bar{p}, \bar{v})$  be some quantifier free formula. We need to find a quantifier free formula equivalent to  $\exists A\varphi$ , where  $A$  is one of  $x_0$ ,  $p_0$  or  $v_0$ . Now, by lemma 4, there is some formula  $\varphi'(\bar{x}, \bar{p}, q)$  such that  $\varphi$  is equivalent to  $\varphi'(\bar{x}, \bar{p}, D(\bar{v}))$ . Hence, for the cases that  $A$  is either  $x_0$  or  $p_0$ ,  $\exists A\varphi$  is equivalent to  $\exists A\varphi'(\bar{x}, \bar{p}, D(\bar{v}))$ , and  $\exists A\varphi'$  is equivalent to a quantifier free formula by lemma 3. Thus the only case left is  $\exists v_0\varphi$ .

Let  $\varphi'_0$  be the formula

$$\varphi'(\bar{x}, \bar{p}, q) \wedge \exists \bar{y}((1, \bar{y}) \in q)$$

(i.e., the set of  $q$  satisfying  $\varphi$  whose projection to the first coordinate is not 0), and let  $\varphi'_1 = \varphi' \wedge \neg\varphi'_0$ . Since existential quantifiers commute with disjunction, it is enough to prove for each of these cases separately.

Assume first that  $\varphi' = \varphi'_0$ . Then  $\exists v_0\varphi$  is equivalent to

$$\exists \bar{y}\varphi'(\bar{x}, \bar{p}, D(-\sum_{i>0} y_i v_i, v_1, \dots, v_n))$$

In the case  $\varphi = \varphi'_1$ ,  $\varphi$  says that  $v_0$  is independent of the other vectors, and therefore  $D(\bar{v})$  coincides with  $i(D(v_1, \dots, v_n))$ , where  $i : K^n \mapsto K^{n+1}$  is the inclusion as the last  $n$  coordinates. Hence  $\exists v_0\varphi$  is equivalent to

$$\varphi'(\bar{x}, \bar{p}, i(D(v_1, \dots, v_n))) \wedge (\langle v_1, \dots, v_n \rangle \neq M)$$

Since the theory determines the dimension of  $M$ , the statement that the  $v_i$  span  $M$  depends only on the dimensions of  $D$  applied to subsets of the  $v_i$ , hence is quantifier free. □

The next goal is to analyse the quantifier free types. Since we don't use quantifier elimination here, we will be able to use this to give a second proof of quantifier elimination. Then, because of quantifier elimination, this will give information about the spaces of types, and eventually  $\omega$ -stability will be shown.

To prove quantifier elimination, we will use the following criterion (cf [3]):

**Criterion 5.** *A theory  $T$  eliminates quantifiers if for any model  $M$  and any  $A \subseteq M$ , any quantifier free 1-type over  $A$  is also a type (i.e. consistent) with respect to any extension of  $T_A$  (where  $T_A$  is the theory obtained by adding to  $T$  all quantifier free sentences over  $A$  that hold in  $M$ .)*

<sup>2</sup> We assume that the base set is a substructure

We begin by analysing the substructures of a model of  $\tilde{\mathfrak{T}}$ , and first, as before, we consider only the restriction to the sorts  $K$  and  $G^i$ . For this restricted theory we will assume elimination of quantifiers (as proved in lemma 3.) Let  $A$  be a substructure of a model of this theory. Then  $K(A)$  is an integral domain, whose fraction field we denote by  $L$ ,  $M(A)$  is a vector space over  $L$ , and  $G^i(A)$  contains all the subspaces of  $L^i$ .

**Claim 6.** *There is a unique minimal extension  $B$  of  $A$  such that  $K(B)$  is a field, and for each  $i$ ,  $\pi_i : K(B)^{i^2} \rightarrow G(B)^i$  is onto.*

*Proof.* First, we may assume that  $K(A)$  is a field by passing to the fraction field. Consider the subset  $\mathbb{P}(A)$  of  $G(A)$  consisting of the one-dimensional subspaces. A point of this subset corresponds to a line in some affine space  $K^i$ . For  $\pi_i$  to be onto, this line should have a point in  $B$ . This will happen if and only if the unique point on this line whose  $k$ -th coordinate is 1 has its other coordinates in  $B$ . Such points correspond to intersection of this line with the standard cover of  $\mathbb{P}$ . This cover corresponds to some elements (over 0) of  $G$ , and the intersections are encoded in the structure of  $G$ . We thus get a finite set of points in affine space, one for each such intersection. Using the standard projections, we get a finite set of points in  $\mathbb{A}^1$ . The type of these points as field elements is well defined, since both the field  $K(A)$  and the field operations can be viewed as part of the structure  $G$ . We thus get a field extension  $K(B)$ , which, by construction, contains a point in each element of  $\mathbb{P}(A)$ , and is obviously minimal with this property.

It remains to show that  $K(B)$  contains a basis for any other element of  $G(A)$  as well. Consider the elements  $G_k^n(A)$  in  $G(A)$  corresponding to  $k$  dimensional subspaces of  $K^n$ . This set has a natural embedding (over  $\mathbb{Z}$ ) into  $\mathbb{P}(\bigwedge^k K^n)$ , corresponding to the natural map  $(K^n)^k \rightarrow \bigwedge^k K^n$ . For a given point of  $G_k^n(A)$ , its image in the above projective space contains, by the definition of  $B$ , a point of  $K(B)$ . Thus the problem reduces to showing that, for any field  $L$ , any point of  $\bigwedge^k L^n$  has a pre-image in  $(L^n)^k$  under the natural map. However, the pre-image set is a  $GL_k$ -torsor (the action of  $GL_k$  corresponds to changing the vector space basis), and any such torsor has an  $L$  point (see, e.g., [2], Lemma 4.10.)  $\square$

Next, we extend the statement to the sort  $M$ :

**Claim 7.** *Let  $A$  be a substructure, and let  $B$  be as promised by claim 6. Then there is a unique minimal vector space  $V$  over  $K(B)$  such that  $(B, V)$  is an extension of  $A$  as a substructure.*

*Proof.* Let

$$V = K(B) \otimes_{K(A)} M(A) / \langle \sum x_i \otimes v_i \mid \bar{x} \in D(\bar{v}) \rangle$$

Since  $D(\sum x_i^1 \otimes v_i^1, \dots, \sum x_i^m \otimes v_i^m)$  is determined by  $D(\bar{v}^1, \dots, \bar{v}^m)$ , this already defines a structure. It is obvious that this is what we want.  $\square$

Let us say that  $A$  is a *good substructure* if  $K(A)$  is a field and  $G^i(A)$  is the set of subspaces of  $K(A)^i$  (in other words, it is a model of  $\mathfrak{T}$ ). Then the above claims say that any substructure has a unique minimal extension to a good substructure (in other words, definably closed structures are good.)

**Claim 8.** *Let  $A$  be a good substructure,  $B$  an extension of  $A$ . Then  $M(B)$  contains  $K(B) \otimes_{K(A)} M(A)$ .*

*Proof.* Since  $M(B)$  is a vector space over  $K(B)$  containing  $M(A)$ , there is a canonical map  $i : K(B) \otimes_{K(A)} M(A) \rightarrow M(B)$ . Assume that  $\sum x_j \otimes v_j$  goes to 0 in  $M(B)$  ( $v_j \in M(A)$ ). Then  $\bar{x} \in D(\bar{v})$ , but according to the assumption,  $D(\bar{v})$  has a basis with coordinates in  $K(A)$ , so  $\sum x_j \otimes v_j$  is 0 already in  $K(B) \otimes_{K(A)} M(A)$ .  $\square$

This implies that for good substructures, statements regarding the vector space are unambiguous: In general, for example, the statement “ $v_1, \dots, v_n$  are linearly independent” might mean either that it is independent over the field part of the structure, or that  $D(\bar{v}) = 0$ . For good substructures, this is the same.

We can now give a

*Second proof of quantifier elimination.* We use criterion 5. Let  $A$  be a substructure. Any model of  $\tilde{\mathfrak{T}}$  containing  $A$  will also contain the substructure given by claim 7, and  $K$  will contain an algebraic closure of  $K(A)$ , so by claim 8 we may assume that

- $K(A)$  is an algebraically closed field.
- each  $\pi$  is onto.
- $M(A)$  is a vector space over  $K(A)$ .

Let  $\mathfrak{T}_1$  be a theory extending  $\tilde{\mathfrak{T}}_A$ , and let  $\mathfrak{p}(v)$  be a quantifier free 1-type over  $A$  with respect to  $\tilde{\mathfrak{T}}$ , in the module sort (quantifiers on other sorts are eliminated as before). Consider the set of formulas  $D(v, v_1, \dots, v_n) \neq 0$  with  $v_i \in M(A)$ , satisfying  $D(v_1, \dots, v_n) = 0$ . Assume first that there is no such formula in  $\mathfrak{p}$ . Then (since  $\mathfrak{p}$  is consistent), the vector space is either  $\infty$ -dimensional, or of dimension greater than the dimension of  $M(A)$ . In any case, there is a model of  $\mathfrak{T}_1$  which has a member outside the space generated by  $M(A)$ . Any such member will satisfy the type.

Now assume, conversely, that there are formulas as above, and assume that  $n$  is minimal. Then for any  $u_i$  with  $D(v, \bar{u}) \neq 0$ , the space  $V$  spanned by  $v_1, \dots, v_n$  is contained in the space spanned by  $\bar{u}$  (Otherwise, the intersection of these spaces is properly contained in  $V$ , and any basis of it is a contradiction to the minimality of  $n$ ). We claim that the set

$$\{P_\varphi(D(v, \bar{v})) \in \mathfrak{p}\}$$

determines the type. Let  $\psi(D(v, \bar{u}_1), \dots, D(v, \bar{u}_k))$  be a formula in  $\mathfrak{p}$ . We first note, that if  $\bar{w}_1$  spans the same subspace as  $\bar{u}_1$ , then  $\psi$  is equivalent to some formula  $\psi'(D(v, \bar{w}_1), \dots, D(v, \bar{u}_k))$ : by assumption, there is some matrix  $U$  (over  $K(A)$ !) such that  $(v, \bar{u}_1) = U(v, \bar{w}_1)$ . Hence the equivalence follows from lemma 4. In particular, we may assume that the first  $n$  vectors in each  $\bar{u}_i$  coincide with  $\bar{v}$ , and that each  $\bar{u}_i$  is linearly independent. But then, letting  $i_m : K^n \hookrightarrow K^{l_m}$  be the inclusion of the first  $n$  coordinates (where  $l_m$  is the length of  $\bar{u}_m$ ), it is clear that  $(i_1, \dots, i_k)^* \psi$  is the formula we seek.

Let  $\mathfrak{q} = \{P_\varphi(p) : P_\varphi(D(v, \bar{v})) \in \mathfrak{p}\}$ . Since ACF eliminates quantifiers, there is a model of  $\mathfrak{T}_1$  in which  $\mathfrak{q}$  has a realisation,  $q$ . Since  $\bar{v}$  is independent,  $q$  will be of dimension either 1 or 0. If it's 1, let  $x_1, \dots, x_n$  be the unique tuple with  $(1, \bar{x}) \in q$ . Then  $v = -\sum x_i v_i$  satisfies  $\mathfrak{p}$ . If  $q = 0$ , then the dimension of  $M$  must be more than  $n$  (otherwise  $\mathfrak{p}$  would be inconsistent, since the dimension is given already in  $\mathfrak{T}$ ). Then any  $v$  independent of  $\bar{v}$  satisfies  $\mathfrak{p}$ .  $\square$

Let's record the result in the proof as a separate claim:

**Claim 9** (description of the vector space types). *A 1-type  $\mathfrak{p}(v)$  over a good structure  $A$  is determined by either a sequence  $v_1, \dots, v_n \in M(A)$  of minimal length such that*

$$D(v, v_1, \dots, v_n) = 0$$

*is in  $\mathfrak{p}$ , together with the type  $\mathfrak{q}$  (in the field and Grassmanian sorts) such that*

$$\mathfrak{p} = \mathfrak{q}(D(v, v_1, \dots, v_n))$$

*or by the fact that there is no such sequence (In other words, it is determined by the minimal subspace to which  $v$  belongs, together with the minimal field over which it happens).*

*Remark 10.* In the proof we dealt only with quantifier free types, but now we know that this is all there is.

Recall that a theory is  $\omega$ -stable if the set of types over any countable set is countable. As a corollary of the description of types we get

**Corollary 11.**  $\tilde{\mathfrak{T}}$  is  $\omega$ -stable.

*Proof.* This follows by counting the types, using the above claim and  $\omega$ -stability of  $ACF$ .  $\square$

We note that in the case that  $M$  is finite-dimensional, this corollary already follows from the  $\omega$ -stability of  $ACF$ , since, after adding a basis,  $M$  is interpretable in the field. However, the quantifier elimination result holds without adding any parameters.

### 3. THE GENERAL CASE

Unlike the case  $n = 0$ , for  $n > 0$ ,  $\mathfrak{T}_I$  is far from being complete (unless  $I$  is maximal), even if the field is algebraically closed. Nevertheless, quantifier elimination in the field (and  $G$ ) variables follows automatically from the case  $n = 0$ . For the full quantifier elimination, we consider an extended theory  $\tilde{\mathfrak{T}}$  whose models satisfy the following property: Given a model  $N$ , let  $\mathbb{A}$  be the algebra  $K(N)[T_1, \dots, T_n]/\langle I \rangle$ . Then any (finite) set of conditions:

$$(6) \quad f_i v = v_i$$

$$(7) \quad g_j v \notin U_j$$

where  $f_i, g_i$  are in  $\mathbb{A}$ ,  $v_i \in M(N)$  are module elements, and  $U_i$  are finite dimensional subspaces of  $M(N)$ , has a solution  $v$ , provided that:

- If

$$(8) \quad \sum t_i f_i = 0$$

then

$$(9) \quad \sum t_i v_i = 0$$

for any  $t_i \in \mathbb{A}$ .

- No  $g_i$  is in the ideal generated by the  $f_i$ .

Note, that these conditions are necessary for a solution to exist.

Since, as they are written, these conditions involve quantifying over all elements of  $\mathbb{A}$ , it is not clear that this is a first order condition. Thus we need to show that such a theory  $\tilde{\mathfrak{T}}$  indeed exists, that it eliminates quantifiers, and that any model of  $\mathfrak{T}_I$  can be embedded in a model of this kind.

The fact that the above condition is actually first order, follows from the following theorem of [4] (by the degree of a polynomial we mean the *total* degree):

**Fact 12.** *Let  $A$  be the polynomial algebra in  $n$  variables over an arbitrary field,  $d$  a fixed degree. There is a degree  $e$  depending only on  $n$  and  $d$  (and not on the base field), such that for any  $p_1, \dots, p_m \in A$  of degree at most  $d$ :*

- (1) *For any  $f \in A$  of degree at most  $d$ , if  $f$  is in the ideal generated by the  $p_i$ , then  $f = \sum h_i p_i$  for  $h_i$  of degree at most  $e$ .*
- (2) *The module of tuples  $(s_1, \dots, s_m)$  such that  $\sum s_i p_i = 0$  is generated by tuples of elements of degree at most  $e$ .*

*More generally, the same results hold when  $A$  is replaced by  $A^k$ . Here, the degree of  $(t_1, \dots, t_k) \in A^k$  is the maximum of the degrees, and  $e$  depends also on  $k$ .*

*Remark 13.*

- (1) For the polynomial algebra, the set of polynomials of a given degree forms, in a natural way, a definable set. For the more general algebra  $\mathbb{A}$  we may define the degree of an element  $r$  to be the minimal degree of a pre-image of  $r$  in  $K[T_1, \dots, T_n]$ . A priori, it is not clear that the set of elements of a given degree in  $\mathbb{A}$  is again definable, since an element is represented in more than one way as a polynomial. However, since (according to fact 12) membership in  $I$  is a first order property (of the coefficients), we have formulas whose free variables represent an element of  $\mathbb{A}$  of a given degree. Alternatively, for the purpose of describing members of  $\mathbb{A}$  we may assume that  $\mathbb{A}$  is actually the polynomial algebra, since  $I$  only appears as a condition on the modules.
- (2) Elements of  $K^m$  will usually be considered as coefficients of polynomials in the  $T_i$ . This means that we fix an order on the monomials in the  $T_i$ , and for  $\bar{x} \in K^m$ ,  $x_i$  is the coefficient of the  $i$ -th monomial. Multiplication of polynomials induces an operation  $*$ :  $K^m \times K^l \rightarrow K^N$ .

The same is true for elements of the  $G^i$ : if two such elements  $p, q$  corresponds to vector spaces  $V_p$  and  $V_q$  of polynomials in the  $T_i$ ,  $p*q$  corresponds to the image of  $V_p \otimes V_q$  in the polynomial algebra.

Sometimes, instead of thinking of a tuple as a polynomial, we think of it as a tuple of polynomials (it will be clear from the context.) In that case, multiplication (by a polynomial or one vector space) is done term-wise.

- (3) Here is an instance of the above notation: Let  $J$  be an ideal in  $\mathbb{A}$ . Then  $J$  is finitely generated; let  $p$  be the vector space generated by a finite set of generators. It follows from fact 12, that given a degree  $d$  there is a degree  $e$  such that, setting  $q = K^e$ , the set of elements of degree  $d$  in  $J$  is precisely the set of elements of degree  $d$  in  $q * p$ .
- (4) Some more notation:  $D_m(v, \bar{v})$  will denote  $D(T^{\bar{i}}v, \dots, v, \bar{v})$ , where the  $\dots$  stands for all monomials of total degree at most  $m$  (with the prescribed order).
- (5) Recall from the case  $n = 0$  that over a good substructure, the type of  $D(v, \bar{v})$  determines the type of  $D(v, \bar{u})$  whenever both are not 0 and  $D(\bar{v}) = D(\bar{u}) =$

0. The passage to a good substructure is done precisely as in the previous case.

The fact that the condition on the  $v_i$  is first order now follows from the second item of fact 12, since it is enough to state the conditions on the  $f_i$  for generators of the tuples  $(t_i)$ . The fact that the condition on the  $g_i$  is first order follows from the first item of fact 12.

Using the last point of remark 13, we may obtain a description of the types:

**Claim 14** (Description of types, general case). *For any quantifier free 1-type  $\mathfrak{p}(v)$ , either there are  $m$  and  $\bar{v}$  such that  $\mathfrak{p}$  is determined by the formulas in it of the form  $\varphi(D_m(v, \bar{v}))$  (where  $\varphi$  does not involve any module stuff), or  $\mathfrak{p}$  is the unique quantifier free type determined by the set of formulas  $D_m(v, \bar{v}) = 0$  for all  $m$  and  $\bar{v}$ .*

*Proof.* Let  $N$  be a model realising  $\mathfrak{p}$ ,  $v \in M(N)$  a realisation. Since we are working over a good substructure  $N_0$ , we may view  $K(N) \otimes_{K(N_0)} M(N_0)$  as a sub  $\mathbb{A}$ -module of  $M(N)$ . Let  $J$  be the ideal in  $\mathbb{A}$  of elements  $f$  such that  $fv \in K(N) \otimes_{K(N_0)} M(N_0)$ . If this ideal is 0, we are in the second case. Otherwise, let  $f_1, \dots, f_n$  generate  $J$ , and let  $\bar{v}_i \in M(N_0)$ , for  $i$  between 1 and  $n$ , be bases for the minimal  $K(N_0)$  subspace containing  $f_i v$ . We set  $m$  to be the maximum of the degrees of the  $f_i$  and  $\bar{v} = (\bar{v}_1, \dots, \bar{v}_n)$ . A different choice of  $N$  and  $v$  will result choosing  $f_i$  of the same form, with coefficients satisfying the same type over  $K(N_0)$ . Thus  $m$  and  $\bar{v}$  do not depend on the choice of  $N$  and  $v$ .

Let  $V_i$  be the vector space spanned by  $\bar{v}_i$ . Let  $g \in \mathbb{A}$  be such that  $\mathfrak{p}$  says that  $D(gv, u_1, \dots, u_k) \neq 0$  for some module elements  $u_i$ . Then  $g = \sum h_i f_i$  for some  $h_i \in \mathbb{A}$ . Since the base is a structure, applying the operators  $T_i$  to elements of  $V_i$  is well defined. If  $\psi(v) = \varphi(D(gv, \bar{u}))$  is a formulas in  $\mathfrak{p}$ , consider the definable set

$$\{(w_1, \dots, w_n) \in V_1 \oplus \dots \oplus V_n \mid \varphi(D(\sum h_i w_i, u_1, \dots, u_k))\}$$

(The spaces  $V_i$  are represented by tuples of field elements, so this is a subset of the  $G$  sorts.) Since we are over a good substructure, the type of this space is determined by the base. Also,  $v$  satisfies  $\psi$  if and only if  $(f_1 v, \dots, f_n v)$  belongs to this set. But this is determined by the type of  $D_m(v, \bar{v})$  (over the base  $N_0$ )  $\square$

As in the case  $n = 0$ , the result we seek easily follows from this:

**Theorem 15.** *Let  $\tilde{\mathfrak{T}}$  be the theory extending  $\mathfrak{T}$ , and stating, in addition, that for any  $f_1, \dots, f_m, g_1, \dots, g_k \in \mathbb{A}$  and any  $v_1, \dots, v_m, \bar{u} \in M$  such that, for any  $i$ ,  $g_i$  is not in the ideal generated by  $f_1, \dots, f_m$ , and such that for any  $t_1, \dots, t_m \in \mathbb{A}$ ,  $\sum t_i f_i = 0$  implies  $\sum t_i v_i = 0$ , the formula*

$$\bigwedge_i f_i x = v_i \wedge \bigwedge_i D(g_i x, \bar{u}) = 0$$

*has a solution  $x$ .*

*Then  $\tilde{\mathfrak{T}}$  eliminates quantifiers.*

*Proof.* Using criterion 5, and the above claim, we need to show that given a good substructure  $M_0$ , and a quantifier free type  $\mathfrak{p}$  over  $K(M_0)$  and  $G(M_0)$ , we may satisfy  $\mathfrak{p}(D_m(v, \bar{v}))$  in any theory extending  $\mathfrak{T}_{M_0}$ .

Since  $\mathfrak{p}$  is a type in the  $G$  sorts over  $K(M_0)$ , it follows from section 2 that  $\mathfrak{p}$  is consistent. Let  $p$  satisfy  $\mathfrak{p}$ . Again, by the case  $n = 0$ , we may assume that  $p$  is in  $M_0$ , and we may extend the field so that  $p$  corresponds to some subspace of

$K^l$ . This means that satisfying  $\mathfrak{p}(D_m(v, \bar{v}))$  amounts to satisfying conditions of the form

$$\begin{aligned} f v &= \sum x_i v_i \\ g v &\notin \langle v_j \rangle \end{aligned}$$

Where  $f$  is an element of  $\mathbb{A}$  and  $x_i \in K$ . Since  $\mathfrak{p}$  was consistent to start with, the conditions appearing in the axioms are satisfied for any set of conditions like this that appears in  $\mathfrak{p}$ . Hence, the axioms imply that these equations have a solution.  $\square$

**Corollary 16.**  $\tilde{\mathfrak{T}}$  is  $\omega$ -stable

*Proof.* from the theorem, by counting the types.  $\square$

*Remark 17.* Following through the proofs, one sees that they work just as well with the commutativity assumption on the generators replaced by some other axioms, provided that the resulting algebra is (left) Noetherian, and the class of modules satisfying the solvability conditions is first order. In particular, using the more general version of fact 12, we see that the same result holds for algebras finite over their centre (where the field is contained in the centre.)

The last thing is to prove that any module over  $\mathbb{A}$  (considered as a model of  $\mathfrak{T}$ ) can be embedded in a model of  $\tilde{\mathfrak{T}}$ . First note that the axioms can be split into two parts:

- (1) There is a solution for any finite set of equations  $f_i v = u_i$ , provided that if  $\sum t_i f_i = 0$  then  $\sum t_i u_i = 0$ .
- (2) There is a solution for any finite set of formulas  $f_j v = 0$ ,  $g_i v \notin U_i$  (where  $U_i$  is a finite dimensional vectors space), provided that no  $g_i$  is in the ideal generated by the  $f_j$ .

This is true since a solution of a general set of equations of the type considered is the sum of a solution of the corresponding equations of the first kind, and of the second kind.

We claim:

**Claim 18.** Let  $\mathbb{A}$  be a Noetherian ring. An  $\mathbb{A}$  module  $M$  satisfies condition (1) above (for  $f_i, t_i \in \mathbb{A}$  and  $v, u_i \in M$ ) if and only if  $M$  is injective.

*Proof.* Let  $M$  be an injective  $\mathbb{A}$ -module, let  $U \subseteq M$  be the submodule generated by the  $u_i$ , and let  $V = (U \oplus \mathbb{A}) / \langle f_i - u_i \rangle$ . The condition

$$\sum t_i f_i = 0 \implies \sum t_i u_i = 0$$

is equivalent to the map  $U \rightarrow V$  being injective. Therefore, the inclusion map of  $U$  in  $M$  extends to  $V$ , and the image of  $1 \in V$  in  $M$  is a solution.

Conversely, by a result of Baer (cf. [1]), it is enough to check the condition of injectivity for the inclusion of an ideal  $I$  in  $\mathbb{A}$ . Since  $\mathbb{A}$  is Noetherian,  $I$  is finitely generated, say by  $f_i$ . Let  $u_i$  be the images of  $f_i$  in  $M$ . Then  $f_i, u_i$  satisfy the assumption of (1), so there is some  $v \in M$  such that  $f_i v = u_i$  for all  $i$ . Now, the map from  $\mathbb{A}$  to  $M$  that takes 1 to  $v$  extends the given map.  $\square$

Regarding the other condition, consider the module  $M = \prod M_I$ , where  $M_I = (\mathbb{A}/I)^\omega$ , for  $I$  an arbitrary ideal of  $\mathbb{A}$  (so the product is over all ideals.) We claim

that any module containing  $M$  satisfies the second condition. To see this, we first note that it is enough to show that  $M$  itself satisfies the condition. Indeed, given arbitrary finite dimensional vector spaces  $U_i$  in a module containing  $M$ , any solution in  $M$  to the problem, with  $U_i$  replaced by  $U_i \cap M$  will solve the original problem.

For  $M$  itself, let  $I$  be the ideal generated by the  $f_j$ . For the same reason as before, it is enough to find a solution in  $M_I$  (note that the condition is non-trivial only if  $I$  is a proper ideal.)

Now, any element of  $M_I$  is a solution to the equations. Thus we only need to satisfy the inequalities. Since the  $g_i$  are not in  $I$ , they are non-zero in each  $\mathbb{A}/I$ . Hence, almost all of the unit vectors in  $M_I$  satisfy the inequalities. This solves the problem.

We now can prove:

**Claim 19.** *Any module over  $\mathbb{A}$  embeds into a model of  $\tilde{\mathfrak{T}}$ .*

*Proof.* Let  $N$  be any module. Then  $N \oplus M$  can be embedded into some injective module  $I$  (where  $M$  is the module constructed above.) Then  $I$  contains  $N$ , satisfies the first condition since it is injective, and satisfies the second condition since it contains  $M$ .  $\square$

Finally, combining theorem 15 and claim 19, we get:

**Corollary 20.** *The theory  $\tilde{\mathfrak{T}}$  is the model completion of the theory  $\mathfrak{T} = \mathfrak{T}_I$ .*

Most definability results coming from algebra (such as fibre dimension) are concerned with *finitely generated* modules. The following example shows the theory of a finitely generated module is far from having quantifier elimination. Therefore, such definability results can not be derived directly by considering the theory of the module, but should probably be obtained by interpreting the module in our theory  $\tilde{\mathfrak{T}}$ .

**Example 21.** Let  $\mathbb{A} = K[T]$ , the polynomial algebra in one variable over a field  $K$  of characteristic 0, and consider  $M = \mathbb{A}$  as a module over itself. We will show that the semi-ring of natural numbers can be interpreted in this theory.

For elements  $v$  of  $M$  and  $x$  of  $K$ , denote by  $v(x) = 0$  the formula  $\exists u((T-x)u = v)$ . Now consider the formula:

$$\begin{aligned} \phi(v, y) = \\ v \neq 0 \wedge v(0) = 0 \wedge \\ \forall x(v(x) = 0 \implies (x = y \vee v(x+1) = 0)) \end{aligned}$$

We claim that for any  $y$ , the fibre  $\phi(v, y)$  is non-empty (in  $M$ ) if and only if  $y$  is a natural number. Indeed, assume that  $y$  is not in  $\mathbb{Z}$ , and that  $v$  satisfies  $\phi(v, y)$ . Then  $v$  is a non-zero element that is divisible by  $T + n$  for any natural  $n$ . There is no such element in  $K[T]$  (here we use that the characteristic is 0.) Conversely, when  $y$  is natural, the element  $T(T-1) \dots (T-y)$  satisfies the formula.

The conclusion is that  $\exists v \in M(\phi(v, y))$  defines the set of natural numbers. The ring operations are automatically defined, since this formula actually defines the copy of the natural numbers contained in  $K$ .

This example holds more generally: If  $\mathbb{A}$  is any finitely generated algebra over a field  $K$  of characteristic 0, and  $M$  is any finitely generated module over  $\mathbb{A}$  of infinite

dimension over  $K$ , there is dominant map of  $\text{spec}(\mathbb{A})$  to the affine line, that makes  $M$  into a  $K[T]$  module, which, after a localisation becomes free (and in particular, torsion free.) The fact that  $M$  is infinite dimensional means that the support of  $M$  has dimension at least 1, so that this map can be chosen so that the resulting free module is non-zero. Now we may repeat the above example to interpret (all but finitely many of) the natural numbers.

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