JUMP INVERSIONS INSIDE EFFECTIVELY CLOSED SETS AND APPLICATIONS TO RANDOMNESS

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Abstract. We study inversions of the jump operator on Π^0_1 classes, combined with certain basis theorems. These jump inversions have implications for the study of the jump operator on the random degrees—for various notions of randomness. For example, we characterize the jumps of the weakly 2-random sets which are not 2-random, and the jumps of the weakly 1-random relative to $\mathbf{0}'$ sets which are not 2-random. Both of the classes coincide with the degrees above $\mathbf{0}'$ which are not $\mathbf{0}'$-dominated. A further application is the complete solution of [Nie09, Problem 3.6.9]: one direction of van Lambalgen's theorem holds for weak 2-randomness, while the other fails.

Finally we discuss various techniques for coding information into incomplete randoms. Using these techniques we give a negative answer to [Nie09, Problem 8.2.14]: not all weakly 2-random sets are array computable. In fact, given any oracle X, there is a weakly 2-random which is not array computable relative to X. This contrasts with the fact that all 2-random sets are array computable.

1. Introduction

One of the fundamental notions of algorithmic information theory is the notion of a Martin-Löf random real.[1] Here we recall that a computable sequence of Σ^0_1 classes $\{V_e \mid e \in \omega\}$ with (measure) $\mu(V_e) \leq 2^{-e}$ for all e, is called a Martin-Löf test and a real A passes this test if $A \notin \cap_e V_e$. The real is Martin-Löf random iff it passes all such tests. Recall that we say that A is n-random iff it passes all such tests where now the V_e can be Σ^0_n classes in place of Σ^0_1 classes.

However many would argue that the most natural concept of algorithmic randomness is the notion of a *weakly 2-random* real. A weakly 2-random real is one that avoids all null Π^0_2 classes; whereas Martin-Löf randomness only asks that a real avoid all null Π^0_2 classes where, as defined above, we have some effective convergence criteria for the nullness. That is, we now ask that A passes all *generalized Martin-Löf tests*, where this is a computable sequence of Σ^0_1 classes $\{V_e : e \in \omega\}$ with $\mu(V_e) \to 0$. The first mention in print of weak 2-randomness can be found in Gaifman and Snir [GS82], but the class of such reals can be traced earlier, such as in Solovay's notes, where it is shown that weakly 2-random reals are not Δ^0_2, an observation of Martin.

One good reason we might regard weakly 2-random reals as a natural concept of randomness is that they cannot have very high computational power (as we would expect of a random real) in the sense that they are not of PA degree and,

Supported by the Marsden Foundation of New Zealand. We would like to thank Adam Day and Andrew Fitzgerald for some discussions about constructions of weakly 2-random sets. We are also grateful to the referees for many corrections.

[1]Hereby "real" we mean a member of Cantor space. We equip this with the standard uniform measure, $\mu([\sigma]) = 2^{-|\sigma|}$, where $[\sigma] = \{\sigma\alpha \mid \alpha \in 2^{\omega}\}$, for any $\sigma \in 2^{<\omega}$.

in particular, they cannot code the halting problem. Somehow a lot of the more typical behavior of random reals seems to begin at weak 2-randomness. Moreover, they seem to share properties with the Martin-Löf random reals in a way that we do not yet understand. For instance, Downey, Nies, Weber and Yu [DNWY06] proved that the class of the reals low for this randomness concept coincide with the class of reals low for Martin-Löf randomness. Thus, this class seems highly worthy of being studied both for its own sake and for the insight it gives into other better understood classes such as the Martin-Löf random reals and the 2-random reals.

On the other hand, little is really known about weakly 2-random reals, perhaps because Martin-Löf randomness *suffices* for many applications. This seems also because we lack techniques for dealing with the class of weakly 2-random reals.

This paper was motivated by two natural questions (soon to be discussed) in algorithmic randomness concerning weakly 2-random reals. Both questions are concerned with the computational power of such reals. However, before we discuss those questions, we wish to mention the larger goals of the present paper.

The larger programme of the present paper concerns basis theorems for Π^0_1 classes combined with various degree theoretical restrictions. The study of Π^0_1 classes has a long and rich history. These natural effectively closed sets code a wide variety of algorithmic behavior such as degrees of theories, ideals in rings, etc (see, for instance, Cenzer and Remmel [CR98]), as well as things like reverse mathematics, and, as was first realized by Kučera (e.g. [Kuč85, Kuč86]), algorithmic randomness. Notable here are the various *basis theorems* such as the Kreisel Basis Theorem which asserts that every nonempty Π^0_1 class has a member of computably enumerable degree, and the famous Low Basis Theorem of Jockusch and Soare [JS72] which asserts that every Π^0_1 class has a member of low degree, and from the same paper, the Jockusch-Soare hyperimmune-free basis theorem which asserts that every nonempty Π^0_1 class has a member of hyperimmune-free degree[2]. Perhaps less well known are extensions of these basis results in the situation that the class is somehow *fatter*. The point being that a class might be finite or have all members being computable, in which case little else can be said. As an illustration, a Π^0_1 class is called *special* iff it has no computable members. One can use the method of the Low Basis Theorem to show that for any degree $\mathbf{a} \geq \mathbf{0}'$, if $\mathcal{P}$ is a special Π^0_1 class then $\mathcal{P}$ has a member whose jump is $\mathbf{a}$. Moreover, in the case where the Π^0_1 class has positive measure, as suggested by Kučera [Kuč85] and proven by Downey and Miller [DM06], it has Δ^0_2 members of all Σ^0_2 degree above $\mathbf{0}'$.

As we mentioned earlier, the original impetus for the present paper was consideration of two open problems concerning weakly 2-random reals from Nies [Nie09].

Problem 3.6.9 in [Nie09]. To what extent does van Lambalgen's Theorem hold for weak 2-randomness?

[2]Recall that a real A has hyperimmune free degree iff for all functions $f \leq_T A$, there is a computable function g such that g dominates f, meaning that $f(x) \leq g(x)$ for all x. It has become fashionable by some authors, notably Soare and Nies to refer to these as *computably dominated*, or $\mathbf{0}$-*dominated*. This lends itself to a (partially) relativized version: a real (or degree) A is $\mathbf{b}$-dominated or $\mathbf{b}$-hyperimmune free iff for functions $f \leq_T A$ are dominated by $\mathbf{b}$-computable functions. The opposite will be $\mathbf{b}$-hyperimmune. We will use both notations choosing the one which best emphasizes the point we are trying to make.

Problem 8.2.14 in [Nie09]. Does every weakly 2-random have array computable degree?

Van Lambalgen's Theorem is one of the fundamental and important results in algorithmic randomness. It asserts that $A \oplus B$ is n-random iff A is n-random and B is n-random relative to A (and hence A is n-random relative to B). In particular, $A \oplus B$ is Martin-Löf random iff A is B-random and B is A-random. This result is known to have myriad implications since it neatly ties the notion of relative randomness to the well-understood join operator.

Given the overall importance of van Lambalgen's Theorem, it is natural to ask whether it holds for other notions of algorithmic randomness besides Martin-Löf randomness and higher level versions. At the time of writing the present paper, it was known that van Lambalgen's Theorem failed for notions of randomness *weaker* than Martin-Löf such as Schnorr and computable randomness (see [MMN+06, Yu07]). The reader might recall that Schnorr randomness is obtained by asking that $\mu(V_e) = 2^{-e}$ in the definition of Martin-Löf randomness, and that computable randomness is a variation using martingales. It is not really important for this paper what they are, but that they give weaker notions: every Martin-Löf random real is computably random and every computably random real is Schnorr random with all inclusions proper (see [DH10, Section 6.1] or [Nie09, Chapter 7] for details). It seemed reasonable that the reason that van Lambalgen's Theorem fails for these concepts was their *weakness*. In Problem 3.6.9 in [Nie09], Nies articulates this intuition by asking whether either direction of van Lambalgen's Theorem holds if we replace 1-randomness with weak 2-randomness.

In Section 2.3 we answer [Nie09, Problem 3.6.9] by showing that one direction of van Lambalgen's theorem holds for weak 2-randomness while the other fails. As we mentioned earlier, our result turned out to be a corollary of a more general programme concerning theorems about the jump operator on Π^0_1 classes. These theorems are given in Sections 2 and 3. The work presented there pointed to a somewhat unexplored yet very interesting area, and a number of further questions about the jumps of members of Π^0_1 classes with additional properties. One such property that we study (in connection with jump inversion within a Π^0_1 class) is 'forming a minimal pair with $\mathbf{0}'$'. The salient point here is that Downey, Nies, Weber and Yu [DNWY06] proved that *a 1-random real is weakly 2-random iff it forms a minimal pair with* $\mathbf{0}'$. (The hard direction of this result follows from an ingenious theorem of Hirschfeldt and independently Miller, on Σ^0_3 null classes. Here we refer to Downey and Hirschfeldt [DH10, Theorem 6.2.11] and Nies [Nie09, Theorem 5.3.16]). Applying this result in relativized form, if van Lambalgen's Theorem *were to hold* then if $A \oplus B$ is weakly 2-random we would need that A is B-random (by the original van Lambalgen's Theorem) and forms a minimal pair with B' over B. Now of course A and B must both be weakly 2-random, but it is the weak 2-B-randomness which seems problematical.

The solution that suggests itself is the following. We need to build B and A but additionally control the jump and also avoid cones. However nothing so far seems known about such questions. Moreover, nothing is really known about the cases where the classes are special and the classes are of positive measure as above, and this is the gist of our new programme. For instance, two prototypical results we will prove are the following.

Theorem. *Given a special non-empty Π^0_1 class P and $C \geq_T \emptyset''$ there is $A \in P$ which forms a minimal pair with $\emptyset'$ and $A' \equiv_T A \oplus \emptyset' \equiv_T C$.*

Theorem. *Given $C \geq_T \emptyset''$ and a Π^0_1 class P of positive measure, there is a path $A \oplus B$ of P which forms a minimal pair with $\emptyset'$, such that $A' \equiv_T A \oplus \emptyset' \equiv_T C$ and $B' \equiv_T B \oplus \emptyset' \equiv_T C$.*

The latter result easily allows for a proof of the failure of van Lambalgen's Theorem. In subsequent work we look at weakening the hypothesis that $C \geq_T \emptyset''$ using weaker oracle hypotheses. As an example, we establish the following.

Theorem. *If $C \geq_T \emptyset'$ is $\mathbf{0}'$-hyperimmune and P is a special non-empty Π^0_1 class, then there is $A \in P$ which forms a minimal pair with $\emptyset'$ and $A' \equiv_T A \oplus \emptyset' \equiv_T C$.*

Of course the result we would like to have proven is that if $\mathbf{a} > \mathbf{0}'$ then $\mathbf{a}$ is the jump of some weakly 2-random set. Unfortunately this attractive conjecture is not true. However, we are able to completely classify the jumps of weakly 2-random reals which are not 2-random.

Theorem. *A degree above $\mathbf{0}'$ is the jump of a weakly 2-random but not 2-random degree iff it is $\mathbf{0}'$-hyperimmune.*

The complete classification of the jumps of weakly 2-random degrees remains open, but we show that it must be a subclass of the $\mathbf{0}'$-hyperimmune degrees together with those that are $\mathbf{0}'$-dominated and relative to $\emptyset'$, both diagonally noncomputable and not PA.

In the introductory Section 1.1, we recall and give short motivational proofs of the main known jump-inversion theorems inside Π^0_1 classes and present an overview of our contributions. In Sections 2 and 3 we discuss this direction of research and the implications it has to the study of the jumps of the random degrees (for various randomness notions).

The second question answered by our methods, [Nie09, Problem 8.2.14], concerns the fine delineation of the computational power of weakly 2-random reals. From Downey, Jockusch and Stob [DJS96], we recall that a degree $\mathbf{a}$ is called *array computable* if there is a function $f \leq_{\mathrm{wtt}} \mathbf{0}'$ which dominates all functions computable from $\mathbf{a}$. Array non-computability is known to code a certain computational power. For example, by Downey, Jockusch and Stob [DJS96], we know that every array non-computable degree bounds a 1-generic degree, and below it we can embed any finite lattice in the degrees preserving least and greatest element. Moreover, recent work by Downey and Greenberg [DG08] shows that every array non-computable degree has positive effective packing dimension, and hence there are many constructions which such degrees have *enough* computational power to carry out.

On the other hand, it is well known (and easy to see) that every 2-random degree is array computable. In fact, there is a randomness notion significantly weaker than 2-randomness which implies array computability. The original almost everywhere dominating function in [Kur81] is a function $\leq_{\mathrm{wtt}} \emptyset'$ which dominates Φ^X for all Turing functionals and all X which do not belong to a certain null set S. The set S is in fact what we now call a *Demuth test*[3]. Hence all Demuth random sets are array computable. For the definition, see [Nie09, Definition 3.6.24], or [DH10, Section 6.6.1].

[3] It is not important for the present paper what a Demuth test is, but simply that it gives rise to a notion of randomness which is weaker than 2-randomness.

Randomness notions that are stronger than 1-randomness tend to have certain computational weaknesses. Also incomplete 1-random sets (here, and henceforth, "incomplete" refers to those not above $\mathbf{0}'$) are considered to be quite 'weak' from a computational point of view. For example, Frank Stephan [Ste06] proved that the only 1-random reals that can compute a complete extension of Peano arithmetic are those above $\mathbf{0}'$. Hence all weakly 2-random reals are not sufficiently computationally powerful to be able to compute a $\{0,1\}$ valued fixed point free function.

The difficulty of coding information into incomplete 1-random sets had already been discussed in the seminal work of Kučera [Kuč85]. This difficulty has been associated with a number of central problems in randomness since (for example see [MN06, Questions 4.6–4.8]).

In this context, [Nie09, Problem 8.2.14] is particularly interesting since weakly 2-random sets form a natural class of incomplete random sets. Can they be sufficiently computationally powerful to be array noncomputable? In Section 5, we show that the answer is (remarkably) yes. Again we do this by proving a much stronger statement: *for every function f, we can code enough information into a weakly 2-random X such that some function $g \leq_T X$ is not dominated by f.* The method we use is based on the classic methods of Kučera for coding into random sets, but has novel features which are likely to have further applications.

In Section 1.2 we give a brief introduction to coding methods for (incomplete) random sets. This is not only helpful in making this paper reasonably self-contained, but the main methods that we sketch (all due to Kučera) have not been adequately discussed in the literature (nor are they widely known). For example, the jump inversion theorem for Δ^0_2 random sets (Theorem 1.2 below) was only stated in [Kuč85] (not proved) and the first proof appeared in Downey and Miller [DM06]. This theorem combined with upper cone avoidance (Theorem 1.4 below) was also mentioned in [Kuč85] but no hint of the proof has appeared in the literature. We briefly discuss these arguments in Section 1.2 in order to prepare the reader for the proof of Section 5.

1.1. **Jump inversions inside Π^0_1 classes.** As we mentioned above, a special Π^0_1 class is one that contains no computable paths. The basic jump inversion theorem for such classes is the following. A proof can be found in [DH10, Theorem 1.18.16], and is a slight modification of the Low Basis Theorem with a little coding thrown in. The hypothesis that the Π^0_1 class is special is used to show that the induction hypothesis can be continued after coding.

Theorem 1.1 (Folklore)**.** *The range of the jump operator on any special Π^0_1 class is the upper cone of degrees above $\mathbf{0}'$.*

By the Shoenfield jump Inversion theorem [Sho59], the range of the jump operator on the Δ^0_2 sets is the class of Σ^0_2 degrees above $\mathbf{0}'$. However the obvious analog of Shoenfield's Theorem does not hold restricted to special Π^0_1 classes. Indeed, there is a special Π^0_1 class consisting of GL_1 paths [Cen99][4]. The following theorem states that it is enough to require that the Π^0_1 class has positive measure.

Theorem 1.2 (Kučera [Kuč85], and Downey and Miller [DM06])**.** *The range of the jump operator on the Δ^0_2 paths of any Π^0_1 class of positive measure is the class of Σ^0_2 degrees above $\mathbf{0}'$.*

[4]For example, take a perfect thin Π^0_1 class

In Section 2 we study the range of the jump operator on the paths of a special Π^0_1 class whose degrees form a minimal pair with $\mathbf{0}'$. In Section 2.1 we show that given a Π^0_1 class P without computable paths, every degree above $\mathbf{0}''$ is the jump of a member of P which forms a minimal pair with $\mathbf{0}'$. In Section 2.2 we present a version of this result for Π^0_1 classes of positive measure. In that case we are able to construct a path $A \oplus B$ in the class, while controlling the jumps of both A and B. This last result is used in Section 2.3 in order to answer [Nie09, Problem 3.6.9]: one direction of van Lambalgen's theorem holds for weak 2-randomness while the other fails.

In Section 3 we generalize the above jump inversion theorems by replacing the condition 'above $\mathbf{0}''$' with the much weaker 'above $\mathbf{0}'$ and $\mathbf{0}'$-hyperimmune'.

In Section 4, we combine the results of Section 3 with various results from the literature in order to study the jumps of random sets—for various randomness notions. We ultimately seek a full characterization of these classes and we do get one such for the class of weakly 2-random sets which are not 2-random. All the other cases are interesting open questions. The randomness notions that we consider are displayed in Table 1 along with the symbols used to denote them.[5]

Martin-Löf randomness	ML
weak randomness relative to $\mathbf{0}'$	$\mathsf{Kurtz}[\emptyset']$
weak 2-randomness	$\mathsf{W2R}$
2-randomness	$\mathsf{ML}[\emptyset']$

TABLE 1. Randomness notions and the symbols used to denote them.

Here, the reader should recall that a set is *weakly 1-random* or *Kurtz random* if it is not a member of any null Π^0_1 class, or equivalently is in every Σ^0_1 class of measure 1. The reader should remember that whilst 2-randomness is 1-randomness relative to $\emptyset'$, weak 2-randomness is *not* weak 1-randomness relative to $\emptyset'$. Certainly weak 2-randomness implies weak 1-randomness relative to $\emptyset'$, but every 2-generic is weakly 1-random relative to $\emptyset'$, but surely not weakly 2-random (see Downey and Hirschfeldt [DH10]).

As we explain in this section, it seems useful to study the jumps of all these classes simultaneously. Indeed, one class gives information about the others. The main results of Section 4 are displayed in Table 2. Here DOM denotes the class of $\mathbf{0}$-dominated degrees and $\mathsf{DOM}[\emptyset']$ is the class of $\mathbf{0}'$-dominated degrees. Recall that, given an effective enumeration $\{\varphi_e\}$ of all partial computable functions, a function f is diagonally non-computable if $f(e)$ is different than $\varphi_e(e)$ for all e such that $\varphi_e(e)\downarrow$. A degree is diagonally non-computable if it computes a diagonally non-computable function. The collection of these degrees is denoted by DNC. This notion relativizes naturally to any oracle X, giving the class of degrees $\mathsf{DNC}[X]$. Finally PA is the collection of degrees that compute a complete extension of Peano arithmetic. These are known to coincide with the degrees that compute a diagonally non-computable function with binary values.

We use $\mathcal{D}$ to denote the set of all degrees. Given a class $\mathcal{X}$ of sets/degrees we denote the collection of the jumps of these the sets/degrees by $\mathcal{X}'$. By the Friedberg

[5] We borrow this notation from [BMN].

completeness criterion [Fri57], $\mathcal{D}'$ is the sets of degrees which are greater or equal to $\mathbf{0}'$. Note that (a), (b) of Table 2 are the known Theorems 1.1, 1.2.

	Class $\mathcal{X}$	**Jump class $\mathcal{X}'$**
(a)	ML	$\mathcal{X}' = \mathcal{D}'$
(b)	$\mathsf{ML} \cap \Delta^0_2$	$\mathcal{X}' = \mathcal{D}' \cap \Sigma^0_2$
(c)	$\mathsf{W2R}, \mathsf{Kurtz}[\emptyset'], \mathsf{ML}[\emptyset']$	$\mathcal{X}' \cap \mathsf{DOM}[\emptyset'] \subseteq \mathsf{DNC}[\emptyset'] - \mathsf{PA}[\emptyset']$
(d)	$\mathsf{W2R}, \mathsf{Kurtz}[\emptyset']$	$\mathcal{D}' - \mathsf{DOM}[\emptyset'] \subset \mathcal{X}'$
(e)	$\mathsf{W2R} - \mathsf{ML}[\emptyset'], \mathsf{Kurtz}[\emptyset'] - \mathsf{ML}[\emptyset']$	$\mathcal{X}' = \mathcal{D}' - \mathsf{DOM}[\emptyset']$
(f)	$\mathsf{ML}[\emptyset'']$	$\mathcal{X}' \subset \mathcal{D}' - \mathsf{DOM}[\emptyset']$

TABLE 2. Classes of the jumps of sets that possess various randomness properties.

1.2. **Coding into random sets.** Coding into Martin-Löf random sets was first demonstrated in the Kučera-Gács theorem [Gác86, Kuč85], which says that every set is computable from a random set. According to this method, the *bits* of the given set A are coded into *segments* of the random sequence X, that are determined by a computable function f. Let P be a Π^0_1 class containing only random paths, for example a member of the standard universal Martin-Löf test. Recall that the nth member of this test is $\cup_{i>e+n+2} V^e_i$ where V^e_i is the ith member of the eth Martin-Löf test according to an effective enumeration of all Martin-Löf tests. Typically, $X \upharpoonright f(n+1)$ will be the leftmost or rightmost path (according to whether $A(n+1) = 0$ or $A(n+1) = 1$ respectively) of P extending $X \upharpoonright f(n)$. The following basic lemma ensures that for every path $Y \in P$ and $n \in \mathbb{N}$ there exist at least two P-extendible nodes of length $f(n+1)$ with common prefix $Y \upharpoonright f(n)$.

Lemma 1.3 (Kučera [Kuč85])**.** *Let P be the complement of a member of the standard universal Martin-Löf test. Also let (P_e) be an effective enumeration of all Π^0_1 classes. There exists a computable function g of two arguments, such that for all P-extendible strings σ and all $e \in \mathbb{N}$, if $P \cap P_e \cap [\sigma] \neq \emptyset$ there exist at least two $P \cap P_e$-extendible strings of length $g(|\sigma|, e)$ with common prefix σ.*

Another formulation of Lemma 1.3 is in terms of a lower bound on the measure of $P \cap P_e \cap [\sigma]$, in case this is non-empty. In other words *there exists a computable function g of two arguments, such that for all P-extendible strings σ and all $e \in \mathbb{N}$, if $P \cap P_e \cap [\sigma] \neq \emptyset$ then $\mu(P \cap P_e \cap [\sigma]) > 2^{-g(|\sigma|,e)}$.*[6]

Already from [Kuč85] it was demonstrated that this kind of coding can be used in a forcing construction with Π^0_1 classes, e.g. for the construction of incomplete Martin-Löf random sets. One such result in [Kuč85] is the existence of an incomplete

[6]The original version of Lemma 1.3 that appears in [Kuč85] refers to a particular class P which is specially constructed. However it is not hard to see that this holds for the complement of any member of the standard Martin-Löf test. Indeed, by the recursion theorem we can embed into it any desirable Σ^0_1 class of suitably small measure.

Δ^0_2 high random set A. Let us recall and discuss the ideas used in the proof of this theorem (especially since only a sketch is provided in [Kuč85]).

1.2.1. *High random strictly below* $\mathbf{0}'$. To make A high we need to code $\emptyset''$ into A' via the familiar Kučera-Gács coding. The construction will be an oracle argument relative to $\emptyset'$. Suppose that P is the complement of a member of the standard universal Martin-Löf test, and g is the computable function of Lemma 1.3. Notice that P contains only Martin-Löf random sets. Later we will define a computable function f (based on g) which will be the basis of our coding (in the sense that the segments of A from bit $f(i)$ to $f(i+1)$ will be used to encode various events). The levels of P from bit $f(\langle e,i\rangle)$ to bit $f(\langle e,i\rangle+1)$, $i\in\mathbb{N}$ will be devoted to coding whether $e\in\emptyset''$. This is the e-th "column" and the strategy is a'thickness' method which works as follows.[7] At the beginning of stage s of the construction $A\restriction f(s-1)$ will be already be defined (and P-extendible). By the end of stage s, $A\restriction f(s)$ will be defined. As long as $0\not\in\emptyset''$ (with respect to an enumeration of $\emptyset''$ relative to $\emptyset'$) we keep on defining A such that in the 0-th column we never take the rightmost path (with respect to P). If and when this happens at some stage s, we choose the least number $m>s$ in $\mathbb{N}^{[0]}$ and at stage $m+1$ we define $A\restriction f(m+1)$ to be the *rightmost* node in P of length $f(m+1)$ which extends $A\restriction f(m)$. Now to decide '$0\in\emptyset''$?' we just have to ask the following $\Sigma^0_1(A)$ question: 'is there some $m\in\mathbb{N}^{[0]}$ such that $A\restriction f(m+1)$ is the *rightmost* node in P of length $f(m+1)$ which extends $A\restriction f(m)$?'. If yes, then $0\in\emptyset''$, otherwise $0\not\in\emptyset''$. The only property of f that we used, is that for every P-extendible string of length $f(n)$ there are at least two P-extendible extensions of it of length $f(n+1)$. This property can easily be established via g.

The main conflict appears when we also try to satisfy the following requirements

$$Q_e : \exists z\ [\Phi^A_e(z)\uparrow\ \vee\ \Phi^A_e(z)\downarrow\neq\emptyset'(z)].$$

It is not hard to see that for the satisfaction of these incompleteness requirements we may need to force with additional Π^0_1 classes in the construction. If (T_i) are the Π^0_1 classes that have been introduced in the construction (for the satisfaction of Q_e, $e\in\mathbb{N}$) we have $A\in P\cap(\cap_i T_i)$. The problem is that when we introduce some Π^0_1 class T at some stage s, the function f may no longer give a bound on the 'splittings' in $P\cap T$, thus potentially forcing us to give the wrong answer to $0\in\emptyset''$. To ensure that this never happens, we insist that the construction and f have the following property: if T is introduced at stage s, then above $A\restriction f(s-1)$ in $P\cap T$ the function f bounds splittings. In other words, for all $t\geq s$ and every $P\cap T$-extendible string of length $f(t-1)$ which extends $A\restriction f(s-1)$, there are at least two $P\cap T$-extendible nodes of length $f(t)$ extending $A\restriction f(t-1)$. It is a routine to verify that under such a condition the strategy for $A'\geq_T\emptyset''$ would succeed as before.

It remains therefore to show how we can meet this condition. The first observation is that we can compute the indices of the Π^0_1 classes that might be used for the satisfaction of the Q_e, $e\in\mathbb{N}$. In particular, given an index i of a Π^0_1 class P_i

[7] The reader should draw a parallel with the plain constructions of a high incomplete set: the Δ^0_2 and the c.e. case. Columns and thickness requirements were essential in those classic arguments.

and $e \in \mathbb{N}$ we can compute an index of a Π^0_1 class T such that one of the following holds:[8]

- either Q_e is satisfied by all $A \in P_i$ or
- $P_i \cap T \neq \emptyset$ and Q_e is satisfied by all $A \in P_i \cap T$.

For future reference, let h be a computable function such that $h(i, e)$ is an index of $P \cap T$, as above.

Second, we can define f in such a way that it takes into account every possibility as far as the introduction of new classes is concerned (from some level on). In other words, let D_0 contain an index of P, and inductively let $D_{n+1} = D_n \cup \{h(i,n) \mid i \in D_n\}$. Also, let $f(0) = 0$ and $f(n+1)$ be the maximum of all $g(f(n), i)$ for $i \in D_n$, where g is from Lemma 1.3. According to the definition of f, the coding into A' will not be affected as long as we act for Q_e after $A \upharpoonright f(e)$ has been defined. In other words, at stage s if T is the current Π^0_1 condition (the intersection of all Π^0_1 conditions that are currently active) and $n \in \mathbb{N}$, every T-extendible node above $A \upharpoonright f(s)$ of length $f(n)$ has (at least) two T-extendible extensions of length $f(n+1)$. In the global construction there is also a finite injury of the Q_e requirements due to the coding requirements of higher priority.

This argument is very similar to the jump inversion theorem for Δ^0_2 random sets, which was stated without proof in [Kuč85] and proved in [DM06]. This result states that for every set C which is c.e. in $\emptyset'$ and above $\emptyset'$ there is a Δ^0_2 random X such that $X' \equiv_T C$. It was also noted in [Kuč85] that both of these arguments are compatible with upper cone avoidance. Namely that the constructed set A (in the case of the incomplete high random) or X (in the jump inversion) can be chosen to be not Turing above any given non-computable Δ^0_2 set. Since no proof of this was given, we discuss the argument in the following.

1.2.2. *Sacks preservation with Kučera-Gács coding.* In this section we show how to combine the construction which uses Kučera-Gács coding, with cone avoidance. This method can be combined with the argument detailed in [DM06] in order to show the following.

Theorem 1.4 (Kučera [Kuč85]). *Given any non-computable $D \leq_T \emptyset'$ and some $C \geq \emptyset'$ which is c.e. in $\emptyset'$, there is a random $A \leq_T \emptyset'$ such that $A' \equiv_T C$ and $D \not\leq_T A$.*

To demonstrate how to avoid upper cones, we discuss the following extension of the construction of a high incomplete random of Section 1.2.1.

Theorem 1.5 (Kučera [Kuč85]). *Given any non-computable $D \leq_T \emptyset'$, there exists a random $A \leq_T \emptyset'$ such that $A' \equiv_T \emptyset''$ and $D \not\leq_T A$.*

Now instead of mere incompleteness requirements we have the following.

$$Q_e : \exists z\ [\Phi^A_e(z)\uparrow\ \vee\ \Phi^A_e(z)\downarrow \neq D(z)].$$

These requirements correspond to a Π^0_1 condition, much like the incompleteness requirements of Section 1.2.1 did. The difference here is that now we cannot compute that Π^0_1 condition, given the index e of the requirement. Therefore, we cannot define f accordingly as before, so that the introduction of new Π^0_1 classes in the

[8] for example, since $\emptyset'$ is complete, we can choose some witness $z \in \mathbb{N}$ and $T = \{X \mid \Phi^X_e(z)\uparrow \vee\ \Phi^X_e(z) = 1\}$, in such a way that if and when $P_i \cap T = \emptyset$, z is enumerated into $\emptyset'$.

construction does not affect the coding. In particular, if at stage s of the construction we satisfy Q_0 by restricting A into $P \cap T$ for a Π^0_1 class T, it is possible that for some $n > s$, $n \in \mathbb{N}^{[0]}$ this restriction forces $A \restriction f(n+1)$ to be the rightmost path of P extending $A \restriction f(n)$. If $0 \notin \emptyset''$, this makes the coding of $\emptyset''$ into A' invalid.

The solution is to find some Π^0_1 class T for the satisfaction of Q_e which respects the higher priority coding, i.e. the coding of $\emptyset'' \restriction e$ into A'. We do this through a Π^0_1 version of the Sacks preservation strategy from the theory of the c.e. Turing degrees (e.g. see [Soa87]). Notice that we can define f in such a way that it takes into account all Π^0_1 classes (from some point on). If we let $f(n) = \max\{g(i,j) \mid i,j \leq n\}$, then for every Π^0_1 class T there exists some level $n \in \mathbb{N}$ such that for all $T \cap P$-extendible strings of length $f(i)$, $i > n$ there are at least two $T \cap P$-extendible strings of length $f(i+1)$ extending them. Suppose that we have currently defined $A \restriction f(s) = \tau$, currently restricted in a Π^0_1 class T and we have committed to the belief that $\emptyset'' \restriction n = \sigma$ (for some string σ). Now let $[T]^\sigma_\tau$ be the Π^0_1 class which consists of $T \cap [\tau]$, apart from the paths which extend T-extendible strings ρ of length $f(n+1)$, $n \in \mathbb{N}^{[i]}$, $i \leq |\sigma|$ which

- are the leftmost extending $\rho \restriction f(n)$ and $\sigma(i) = 1$
- are the rightmost extending $\rho \restriction f(n)$ and $\sigma(i) = 0$.

Notice that committing to any Π^0_1 subclass of $[T]^\sigma_\tau$ is in accordance to the current coding of $\emptyset'' \restriction |\sigma|$ into A'. Moreover, due to the universal way that f was defined, it allows any kind of coding *from some higher level on*. To find a subclass of $[T]^\sigma_\tau$ which satisfies Q_e we only have to find some t such that

$$\{X \mid \Phi^X_e \restriction t \uparrow \ \vee \ \Phi^X_e \restriction t \downarrow \neq D \restriction t\} \cap [T]^\sigma_\tau \neq \emptyset.$$

Such a $t \in \mathbb{N}$ exists by a classic result in [JS72]. Now the construction can proceed as discussed in Section 1.2.1, using this modification.[9]

2. Jump Inversion in Π^0_1 classes and forming minimal pairs with $\emptyset'$

Our interest on the special topic of inverting the jump with degrees that form a minimal pair with $\mathbf{0}'$ began with our study of [Nie09, Problem 3.6.9]. As mentioned in the introduction, in order to see the relevance between these two problems, recall that

- The weakly 2-random sets are the 1-random sets whose degree forms a minimal pair with $\mathbf{0}'$.
- Two sets with Turing equivalent jumps cannot be relatively weakly 2-random.

The first fact is due to Downey, Nies, Weber and Yu [DNWY06], the crucial step being established by a theorem of Hirschfeldt and Miller (see [Nie09, Theorem 5.3.16] or [DH10, Theorem 6.2.11]). The second follows from the definition of weak 2-randomness. We do not have to look far to see that such inversions are always possible outside of Π^0_1 classes.

Theorem 2.1 (Implicit in [Pos81])**.** *If* $\mathbf{c} > \mathbf{0}'$ *then there is* $\mathbf{a}$ *such that* $\mathbf{a}' = \mathbf{a} \cup \mathbf{0}' = \mathbf{c}$ *and* $\mathbf{a} \cap \mathbf{0}' = \mathbf{0}$.

[9]The readers that are familiar with the construction of a high c.e. incomplete set (see [Soa87]) should notice the following analogy. The special class $[T]^\sigma_\tau$ in which we look for solutions, corresponds to the 'correct' computations with respect to certain 'thickness' requirement in the c.e. construction. The 'thickness' requirements in the c.e. argument are entirely analogous to our coding of $\emptyset''$ into A', only that the latter is expressed in terms of Kučera-Gács coding.

Proof. We sketch this proof as we will use (some of) the techniques within Π^0_1 classes soon.

First suppose that $\mathbf{c}$ is $\mathbf{0}'$-dominated. In particular, it is not c.e. in $\mathbf{0}'$. By the jump inversion with minimal degrees [Coo73] there is a minimal degree $\mathbf{m}$ such that $\mathbf{m}' = \mathbf{m} \vee \mathbf{0}' = \mathbf{c}$. Since $\mathbf{m}'$ is not c.e. in $\mathbf{0}'$, the degree $\mathbf{m}$ is not below $\mathbf{0}'$. Therefore, it forms a minimal pair with $\mathbf{0}'$.

Now suppose that $\mathbf{c}$ is $\mathbf{0}'$-hyperimmune. Then there is a function $f \equiv_T \mathbf{c}$ which is not dominated by any $\mathbf{0}'$-computable function. We can use f in a standard way as a search bound for a stage where $\Phi^{\emptyset'}_e(n)$ is defined (for some $e \in \mathbb{N}$ corresponding to a requirement). Given a Turing functional Φ_j, the oracle $\mathbf{0}'$ can determine if splittings exist (i.e. strings σ, τ and $n \in \mathbb{N}$ such that $\Phi^{\sigma}_j(n) \neq \Phi^{\tau}_j(n)$), and if they do exist it can find them. The minimal pair strategies are combined with forcing the jump of the constructed set $\mathbf{a}$ and the values of f are simultaneously coded into $\mathbf{a} \cup \mathbf{0}'$. The domination property of f implies the satisfaction of all requirements in this f-computable construction. Indeed, on the assumption that a least requirement with index e remains unsatisfied, the $\mathbf{0}'$-computable function bounding the computation time of $\Phi^{\emptyset'}_e$ dominates f. The condition $\mathbf{a}' \leq \mathbf{c}$ follows by the forcing of the jump, while $\mathbf{a} \cup \mathbf{0}'$ can recover bit by bit the construction and the values of f.

The details of this construction are omitted, as it is quite standard. Moreover it is very similar to the main construction of [Pos81] only that instead of forcing $\mathbf{a}'$ the author only makes $\mathbf{a} \cup \mathbf{0}' = \mathbf{c}$. The construction sketched above is easier than the one in [Pos81] because we only complement with $\mathbf{0}'$ which is very useful as an oracle, whereas Posner has a more complex complementation condition. $\square$

It is interesting that the proof is non-uniform, the cases depending on whether $\mathbf{c}$ is $\mathbf{0}'$-dominated. This distinction is also vital in the jump inversions within Π^0_1 classes. However in this case when $\mathbf{c}$ is $\mathbf{0}'$-dominated the situation is more complex as we show in Section 4.

In the next sections, we study this type of jump inversion within a given special Π^0_1 class. When the measure of the given class is positive we can prove something stronger (see Section 2.2).

2.1. **Special Π^0_1 classes.** The following result and its proof contain the basis of the other arguments in the rest of Section 2.

Theorem 2.2. *Given a special non-empty Π^0_1 class P and $C \geq_T \emptyset''$ there is $A \in P$ which forms a minimal pair with $\emptyset'$ and $A' \equiv_T A \oplus \emptyset' \equiv_T C$.*

Proof. We force with Π^0_1 classes, by defining a sequence $P \supseteq P_1 \supseteq \dots$. Class P_i will be defined at stage $2i+1$ and all of these steps are for the sake of making $A \in \cap_i P_i$ a minimal pair with $\emptyset'$. In the same steps we will force the jump so that $A' \leq_T A \oplus \emptyset'$. At stage $2i$ we will code the ith bit of C into A, by fixing (defining) a certain segment σ_i of A. We will have, $P_i \subseteq [\sigma_i]$ and $\sigma_i \subseteq \sigma_{i+1}$.

At stage $2e$ let σ be the least node such that both $\sigma * 0$ and $\sigma * 1$ are P_{e-1}-extendible. Let $\sigma_e = \sigma * i$ for $i = C(e)$. At stage $2e+1$ let σ be the least node such that both $\sigma * 0$ and $\sigma * 1$ are $P_{e-1} \cap [\sigma_e]$-extendible. Check if for some z we have $\Phi^{\emptyset'}_e(z) \downarrow = r$ and

$$P_{e-1} \cap [\sigma * 1] \cap \{X \mid \Phi^X_e(z) \neq r \ \vee \ \Phi^X_e(z) \uparrow\} \neq \emptyset. \tag{2.1}$$

If there is such z, consider the least one and set P_e^* equal to the Π^0_1 class of (2.1). Otherwise, let $P_e^* := P_{e-1} \cap [\sigma * 0]$. Finally, let P_e be $P_e^* \cap \{X \mid \Phi_e^X(e) \uparrow\}$ if this non-empty, or P_e^* otherwise.

Let $A = \cup_e \sigma_e$. Clearly $A \in \cap_e P_e$ and the construction is computable in $C \oplus \emptyset'' \equiv_T C$. Therefore, $A \oplus \emptyset' \leq_T A' \leq_T C$ since we force the jump in the odd stages. To show that A forms a minimal pair with $\emptyset'$, assume that the functions Φ_e^A, $\Phi_e^{\emptyset'}$ are total and equal.[10] Then the construction did not act for (2.1), which means that $\Phi_e^Z(t)$ is defined for all $Z \in P_{e-1} \cap [\sigma_e * 1]$, $t \in \mathbb{N}$ and the values agree with those of $\Phi_e^{\emptyset'}$. Since $P_{e-1} \cap [\sigma_e * 1]$ is a Π^0_1 class, $\Phi_e^{\emptyset'}$ must be computable.

Finally it remains to show that $C \leq_T A \oplus \emptyset'$. We show how $A \oplus \emptyset'$ can uncover the construction (and along with that, C). Suppose inductively that the construction and $C \upharpoonright e$ has been uncovered up to step $2e$. Using $\emptyset'$ and P_{e-1} we can compute σ of step $2e$. Then $C(e) = i$ where i is such that $\sigma * i \subset A$. Also, $\sigma_e = \sigma * i$. In the same way, $\emptyset'$ can compute σ of step $2e+1$. If $\sigma * 0 \subset A$, the construction must have set $P_e^* := P_{e-1} \cap [\sigma * 0]$. Otherwise, if $\sigma * 1 \subset A$, the construction sets P_e^* equal to the class of (2.1). The notice that z was chosen the least with the property (2.1), so it is computable from $\emptyset'$.[11] Finally, the index of P_e is easily computable from P_e^*, given $\emptyset'$. This concludes the induction and the proof. $\square$

2.2. **Π^0_1 classes of positive measure.** The following is a jump-inversion theorem for Π^0_1 classes of positive measure. An application of it shows that one direction of van Lambalgen's theorem does not hold for weak 2-random sets.

Theorem 2.3. *Given $C \geq_T \emptyset''$ and a Π^0_1 class P of positive measure, there is a path $A \oplus B$ of P which forms a minimal pair with $\emptyset'$, such that $A' \equiv_T A \oplus \emptyset' \equiv_T C$ and $B' \equiv_T B \oplus \emptyset' \equiv_T C$.*

Proof. Without loss of generality, let P be a Π^0_1 class consisting entirely of 1-random paths. We are going to use the fact that if node σ is Q-extendible for some Π^0_1 class $Q \subseteq P$, then the measure of $[\sigma] \cap Q$ is positive. We construct $A \oplus B$ as a path of P.

As in the proof of Theorem 2.2 we will deal with making $A \oplus B$ a minimal pair with $\emptyset'$ on odd stages, while we code C into A', B'. Stage $2e$ will define σ_e and stage $2e+1$ will define P_e.

At stage $2e$ let τ, ρ be the least[12] P_{e-1}-extendible strings which differ both on an odd position and an even position. Notice that since P_{e-1} is perfect, τ, ρ exist and have equal lengths. Without loss of generality assume that τ is (lexicographically) smaller than ρ. Let $\sigma_e = \tau$ if $C(e) = 0$ and $\sigma_e = \rho$ if $C(e) = 1$.

At stage $2e+1$ let τ, ρ be the least $P_{e-1} \cap [\sigma_e]$-extendible strings which differ both on an odd position and an even position. Check if for some z we have $\Phi_e^{\emptyset'}(z) \downarrow = r$

$$P_{e-1} \cap [\rho] \cap \{X \mid \Phi_e^X(z) \neq r \ \vee \ \Phi_e^X(z) \uparrow\} \neq \emptyset. \tag{2.2}$$

[10]Without loss of generality we can assume that the index e of the Turing functionals is the same for both functions, see footnote 19 on page 22.

[11]Here we use the standard convention that if $\Phi^{\emptyset'}(x) \downarrow$ then $\Phi^{\emptyset'}(y) \downarrow$ for all $y < x$.

[12]The ordering on pairs of strings that is used here is based on the ordering on strings which is first by length and then lexicographically. We let $(\tau, \rho) \prec (\tau', \rho')$ if either τ is less than τ' or they are equal and ρ is less than τ.

If there is such z, consider the least one and set P_e^* equal to the Π_1^0 class of (2.2). Otherwise, let $P_e^* := P_{e-1} \cap [\tau]$. Finally, notice that given a Π_1^0 class Q, the classes

$$[Q]_0 = \{X \mid \exists Y\ (X \oplus Y \in Q)\} \quad \text{and} \quad [Q]_1 = \{Y \mid \exists X\ (X \oplus Y \in Q)\}$$

are also Π_1^0 by compactness. This last step consists of two actions which ensure that $A' \leq_T A \oplus \emptyset'$ and $B' \leq_T B \oplus \emptyset'$ respectively. If $[P_e^*]_0 \cap \{X \mid \Phi_e^X(e)\uparrow\} \neq \emptyset$ we let

$$T = \{X \oplus Y \in P_e^* \mid \Phi_e^X(e)\uparrow\}. \tag{2.3}$$

Otherwise we let $T = P_e^*$. Symmetrically, if $[T]_1 \cap \{Y \mid \Phi_e^Y(e)\uparrow\} \neq \emptyset$ we let $P_e = \{X \oplus Y \in T \mid \Phi_e^Y(e)\uparrow\}$. Otherwise we let $P_e = T$.

Verification. First notice that the construction is computable in $C \oplus \emptyset''$, hence computable in C by the hypothesis $C \geq_T \emptyset''$. Hence $A \oplus B \leq_T C$, where $A \oplus B = \cup_e \sigma_e$. Also, we show that $A' \leq_T C$ and $B' \leq_T C$. Given the query '$e \in A'$?' we can run the construction up to the end of step $2e+1$ using C as an oracle. If T was defined according to (2.3) in step $2e+1$, we can conclude that $e \notin A'$. Otherwise, $e \in A'$. A symmetric argument shows that $B' \leq_T C$. Hence $A \oplus \emptyset' \leq_T A' \leq_T C$ and $B \oplus \emptyset' \leq_T B' \leq_T C$.

Second, to show that $A \oplus B$ forms a minimal pair with $\emptyset'$, assume that the functions $\Phi_e^{A\oplus B}$, $\Phi_e^{\emptyset'}$ are total and equal.[13] Then the construction did not act for (2.2), which means that $\Phi_e^Z(t)$ is defined for all $Z \in P_{e-1} \cap [\rho]$, $t \in \mathbb{N}$ and the values agree with those of $\Phi_e^{\emptyset'}$. Since $P_{e-1} \cap [\rho]$ is a Π_1^0 class, $\Phi_e^{\emptyset'}$ must be computable.

It remains to show that each of $A \oplus \emptyset'$, $B \oplus \emptyset'$ can compute C and, in fact, recover the entire construction. Suppose inductively that with the use of $A \oplus \emptyset'$ we have recovered the construction up to step $2e$, as well as $C \upharpoonright e$. Hence we have computed σ_{e-1} and (an index of) P_{e-1}. Using $\emptyset'$ and P_{e-1} we can compute the strings τ, ρ that where used in step $2e$ (where τ is less than ρ). By the choice of these strings, exactly one of them agrees with A on the even positions[14]. If τ agrees with A, then $C(e) = 0$; otherwise $C(e) = 1$. Moreover, σ_e equals τ, if A agrees with τ (on the even positions), and ρ otherwise.

To recover step $2e+1$, we use $\emptyset'$, P_{e-1} and σ_e to determine the strings τ, ρ that were used in this step. Again, exactly one of them agrees with A on the even positions. Although we cannot decide (2.2) directly (it is a two-quantifier question) we know that it holds iff A agrees with ρ on the even positions. In this way we can compute an index of P_e^*. Using P_e^* and $\emptyset'$ we can compute T. Similarly, using T and $\emptyset'$ we can compute P_e. This completes the induction.

In a symmetric way, it follows that $C \leq_T B \oplus \emptyset'$. This concludes the proof. $\square$

2.3. **Failure of van Lambalgen's theorem for weak 2-randomness.** For an application of Theorem 2.3 to the question of whether van Lambalgen's theorem holds for weak 2-randomness we need the following definition.

Definition 2.1. A set A forms a minimal pair with C over B if there is no $X >_T B$ such that $X \leq_T A \oplus B$ and $X \leq_T C$.

[13] Without loss of generality we can assume that the index e of the Turing functionals is the same for both functions, see footnote 19 on page 22.

[14] That is, the $2i$-th digit of the string is the same as the ith digit of A, for all i such that $2i < |\tau| = |\rho|$.

Recall that A is weakly 2-random relative to B if it does not belong to any $\Pi^0_2(B)$ null class; equivalently, if it does not belong to any $\Sigma^0_3(B)$ null class. The following was proved in [DNWY06] for the special case when $D = \emptyset$.

Proposition 2.4. *A set C is weakly 2-random relative to D iff it is Martin-Löf random relative to D and it forms a minimal pair with D' over D.*

Proof. If C is weakly 2-random relative to D then it clearly is Martin-Löf random relative to D. Furthermore, it forms a minimal pair with D' over D. Indeed, suppose that it does not. Then there is $X >_T D$ such that $X \leq_T C \oplus D$ and $X \leq_T D'$. Hence C belongs to the class $\{Y \mid X \leq_T Y \oplus D\}$. Since $X \leq_T D'$, this is a $\Sigma^0_3(D)$ class. On the other hand, since $D <_T X$ it is null by a theorem of Stillwell [Sti72]. This contradicts the fact that C is weakly 2-random relative to D.

For the other direction assume that C is not weakly 2-random relative to D and it is Martin-Löf random relative to D. Hence it belongs to a null $\Pi^0_2(D)$ class Q. A theorem of Hirschfeldt/Miller (see [Nie09, Theorem 5.3.16], and [DH10, Theorem 6.2.11]) says that given a Π^0_2 null class (in fact Σ^0_3 null class) there is a noncomputable c.e. simple set which is computed by all Martin-Löf random members of the class. Relativizing this argument with respect to oracle D, we have that given a $\Pi^0_2(D)$ null class there exists a D-c.e. set X such that $D <_T X$ and $X \leq_T Y \oplus D$ for all members Y of the class which are Martin-Löf random relative to D. Applying this to D and the null $\Pi^0_2(D)$ class Q, we get that $X \leq_T C \oplus D$ for some set $X \leq_T D'$ such that $D <_T X$. This shows that C is not a minimal pair with D' over D, which concludes the argument. □

The following theorem answers [Nie09, Problem 3.6.9]. It shows that one direction of van Lambalgen's theorem does not hold for weak 2-randomness, while the other does.

Corollary 2.1. *There is a weakly 2-random real $A \oplus B$ such that A is not weakly 2-random relative to B (and B is not weakly 2-random relative to A). Thus, van-Lambalgen's theorem does not hold for weak 2-randomness. However if A is weakly 2-random relative to B and B is weakly 2-random then $A \oplus B$ is weakly 2-random.*

Proof. For the first claim we apply Theorem 2.3 on a Π^0_1 class P consisting entirely of 1-random reals, and let $C = \emptyset''$. We have that $A \oplus B$ is 1-random and forms a minimal pair with $\emptyset'$. Therefore, $A \oplus B$ is weakly 2-random. On the other hand, $A \leq_T B'$ so A belongs to a $\Pi^0_2(B)$ null class and hence it is not weakly 2-random relative to B. Similarly, $B \leq_T A'$ and hence B is not weakly 2-random relative to A.

For the second claim, assume that A is weakly 2-random relative to B and B is weakly 2-random. Then A is Martin-Löf random relative to B and B is Martin-Löf random. By van Lambalgen's Theorem, $A \oplus B$ is Martin-Löf random. According to Proposition 2.4 (for $C = A \oplus B$ and $D = \emptyset$) it remains to show that $A \oplus B$ forms a minimal pair with $\emptyset'$. For a contradiction let us assume that this is not the case, i.e. there exists a non-computable set $X \leq_T \emptyset'$ such that $X \leq_T A \oplus B$. Since B is weakly 2-random, $X \not\leq_T B$. Hence $B <_T X \oplus B$. Also, $X \oplus B \leq_T A \oplus B$ and $X \oplus B \leq_T B'$. Hence A does not form a minimal pair with B' over B. According to Proposition 2.4, this is a contradiction. □

3. More general jump inversion results

The arguments of Section 2 can be extended (with reasonable effort) in order to yield more general results. Although those arguments had constructions which used the full power of $\mathbf{0}''$, it turns out we can run them more efficiently using only a small fragment of the strength of this oracle. In other words, we can replace the condition $C \geq_T \emptyset''$ in Theorems 2.2 and 2.3 with the much weaker requirement that C is $\mathbf{0}'$-hyperimmune (and computes $\mathbf{0}'$). Thus, in particular, they will apply to any degree $\mathbf{a} > \mathbf{0}'$ which is computably enumerable relative to $\mathbf{0}'$ or with $\mathbf{0}' < \mathbf{a} \leq \mathbf{0}''$. As we see, for various randomness notions, these generalizations have interesting consequences in the study of the jumps of the random degrees. We discuss these consequences in Section 4.

It is important to determine exactly what properties of the oracle are needed in order to run the arguments of Section 2, especially if one is interested in characterizing the associated degree classes. For such arguments, where sufficiently long searches are needed, one cannot require less than *non*-$\mathbf{0}'$-domination. However, there may be other reasons why a certain jump inversion is possible inside a Π^0_1 class. Interesting examples can be given by random sets (see Section 4). Thus we remark that the theorems in this section are not fully general. Many interesting cases occur when the given degree that we are trying to invert is $\mathbf{0}'$-dominated.

We begin with a generalization of Theorem 2.2.

Theorem 3.1. *If $C \geq_T \emptyset'$ is not $\emptyset'$-dominated and P is a special non-empty Π^0_1 class, then there is $A \in P$ which forms a minimal pair with $\emptyset'$ and $A' \equiv_T A \oplus \emptyset' \equiv_T C$.*

Proof. We force with Π^0_1 classes and permit below C in order to define a sequence (P_s) of Π^0_1 classes such that the unique real in $\cap_s P_s$ meets the conditions of the theorem. By the hypothesis there exists an increasing function $f \leq_T C$ which is not dominated by any $\emptyset'$-computable function. To make the degree of A a minimal pair with the degree of $\emptyset'$, it suffices to satisfy the following conditions.[15]

$$Q_e: \ \Phi_e^{\emptyset'} \text{ is total and non-computable } \Rightarrow \Phi_e^{\emptyset'} \neq \Phi_e^A$$

During the construction, requirements Q_e will be in the state of either *satisfied* or *unsatisfied*, with the latter being the default. We say that Q_e *requires attention* at stage $s+1$ if it is unsatisfied, $e \leq s$ and there exists $x < f(s)$ such that

$$\begin{aligned} &\Phi_{e,f(s+1)}^{\emptyset'\restriction f(s)}(x)\downarrow \quad \text{and} \\ &P_s \cap [T_{0^{e+2}1}] \cap \{Y \mid \Phi_e^Y(x)\uparrow \ \vee \ \Phi_e^Y(x)\downarrow \neq \Phi_e^{\emptyset'}(x)\} \neq \emptyset \end{aligned} \tag{3.1}$$

where T is the tree (as a function from $2^{<\omega}$ to $2^{<\omega}$) corresponding to the perfect class P_s, and $T_{0^{e+2}1}$ is its value on $0^{e+2}1$. We also have the following coding requirements.

$$R_e: \ \text{code } C(e) \text{ into } A \oplus \emptyset'.$$

We say that R_e *requires attention* at stage s if $e \leq s$ and $C(e)$ has not been coded into A in the stages prior to s. Finally we have the following requirements which force the jump, so that $A' \leq_T C$.

$$L_e: \ \text{Decide if } \Phi_e^A(e)\downarrow$$

[15] See footnote 19 on page 22.

Here to 'decide' means to determine the answer to this question in the construction (once and for all). Indeed, if this happens for all $e \in \mathbb{N}$ we get $A' \leq_T C$ since the construction is computable in C. Requirement L_e will be satisfied at the end of stage e, after some actions for the other requirements have been performed.

We fix the following priority list for conditions R_e, Q_e: $R_0 < Q_0 < R_1 < \dots$. Let $P_0 = P$.

Construction. At stage $s+1$, perform the following steps:
(a) Consider the highest priority R or Q requirement M which requires attention. Also, let T be the perfect binary tree that corresponds to the perfect Π^0_1 class P_s. If $M = R_e$, let P^* be $P_s \cap [T_1]$ or $P_s \cap [T_{01}]$ according to whether $C(e)$ is 1 or 0, and say that the eth digit of C has been 'coded'. If $M = Q_e$, let P^* be the Π^0_1 class in the second part of (3.1), for the least x satisfying (3.1). Declare Q_e to be satisfied.
(b) Let P_{s+1} be $P^* \cap \{X \mid \Phi^X_e(e) \uparrow\}$ if this is not empty, and P^* otherwise.

Verification. The construction is computable in C and $\cap_e P_e \neq \emptyset$ by induction. Let A be the unique set in $\cap_e P_e$. Since we force the jump at step (b) of every stage, we have $A \oplus \emptyset' \leq_T A' \leq_T C$. First we show that $C \leq_T A \oplus \emptyset'$.

We show how $A \oplus \emptyset'$ can uncover the construction (and along with that, C). Suppose inductively that $A \oplus \emptyset'$ has reproduced the construction up to stage $s-1$, i.e. it has computed P_{s-1}. Notice that $\emptyset'$ can compute the tree T of stage s, i.e. the tree corresponding to the perfect Π^0_1 class P_{s-1}. According to the construction, A must be prefixed by exactly one of T_1, T_{01}, or $T_{0^{e+2}1}$, for some $e \in \mathbb{N}$. Let $A \oplus \emptyset'$ calculate which of these cases holds. Given this information, according to step (a) of the construction it can also calculate P^* of step s. Finally, using $\emptyset'$ we can calculate P_s according to step (b) of the construction. To compute C from $A \oplus \emptyset'$ it suffices to show that every digit of C is 'coded' during the construction, since $A \oplus \emptyset'$ can reproduce the construction. For the latter it suffices to show that every R, Q condition requires attention finitely often. Indeed, in that case all digits of C will be coded during the construction, since each R condition requires attention as long as it is not satisfied. If some Q condition requires attention at some stage, it will either be satisfied, or some higher priority condition will receive attention. Since there are only finitely many conditions of higher priority than Q, an inductive argument shows that eventually it will stop requiring attention. The same holds for R. Thus $C \leq_T A \oplus \emptyset'$.

It remains to show that Q_n is satisfied for all $n \in \mathbb{N}$. For a contradiction, suppose that e is the least number such that Q_e is not satisfied. Then $\Phi^{\emptyset'}_e$ is total and not computable. Let s_0 be a stage after which no condition of higher priority than Q_e requires attention. It follows that there is no stage $s \geq s_0$ where Q_e requires attention. Indeed if it did, the construction would act on it and would satisfy it. We are going to derive a contradiction by defining a function $g \leq_T \emptyset'$ which dominates f. Since $\Phi^{\emptyset'}_e$ is non-computable, given any Π^0_1 class $S \neq \emptyset$ there exists a least $\ell \in \mathbb{N}$ such that for some $x < \ell$,

$$(3.2) \qquad \begin{array}{l} \Phi^{\emptyset' \restriction \ell}_e(x)[\ell] \downarrow \quad \text{and} \\ S \cap \{Y \mid \Phi^Y_e(x) \uparrow \ \vee \ \Phi^Y_e(x) \downarrow \neq \Phi^{\emptyset'}_e(x)\} \neq \emptyset. \end{array}$$

Also, such ℓ can be computed by $\emptyset'$. We define recursively in $\emptyset'$ a sequence (J_s) of finite sets of Π^0_1 classes such that $P_s \in J_s$ and a function g such that $g(s) > f(s)$ for all $s \geq s_0$. Let J_{s_0} consist of P_{s_0} and let $g(s_0)$ be the least number that is greater than each such ℓ of (3.2), where $S = P_{s_0} \cap [T_{0^{e+2}1}]$ and T is the perfect tree representing P_{s_0}. For $s \geq s_0$ let J_{s+1} consist of the intersections of classes in J_s with all boolean combinations of the Π^0_1 classes $[T_{0^{k+2}1}]$, $[T_1], [T_{01}]$, $\{Y \mid \Phi^Y_t(x) \uparrow \ \vee \ \Phi^Y_t(x) \downarrow \neq j\}$ and $\{X \mid \Phi^X_i(i) \uparrow\}$ where T ranges over the perfect trees representing classes in J_s, $j \leq 1$, $x \leq g(s)$ and $i, k, t \leq s$. Also let $g(s+1)$ be the least number that is greater than each ℓ of (3.2), where $S = Q \cap [T_{0^{e+2}1}]$ and Q ranges over all classes in J_s and is represented by the perfect tree T (if $S = \emptyset$ let $\ell = 0$).

Clearly $g \leq_T \emptyset'$. Also, $P_{s_0} \in J_{s_0}$ trivially and since Q_e does not require attention at s_0, $g(s_0) > f(s_0 + 1)$. Inductively assume that $P_s \in J_s$ and $g(s) > f(s+1)$. By the latter clause, the construction and the definition of J_{s+1} we have that $P_{s+1} \in J_{s+1}$. On the other hand since Q_e is not satisfied at $s+2$, by the definition of $g(s+1)$ we have that $f(s+2) < g(s+1)$. This finishes the induction and shows $g(s) > f(s+1)$ for all $s \geq s_0$, which is a contradiction. □

Theorem 2.3 can be generalized in a similar way.

Theorem 3.2. *Given $C \geq_T \emptyset'$ which is $\mathbf{0}'$-hyperimmune and a Π^0_1 class P of positive measure, there is a path $A \oplus B$ of P which forms a minimal pair with $\emptyset'$, such that $A' \equiv_T A \oplus \emptyset' \equiv_T C$ and $B' \equiv_T B \oplus \emptyset' \equiv_T C$.*

The proof of this theorem is based on the original argument for the proof of Theorem 2.3, along with the ideas elaborated in the proof of Theorem 3.1 for making the construction effective in C. Since no new ideas are involved (other than the ones we have elaborated) we leave the details of the proof to the reader.

Theorem 3.1 will be used in Section 4 for characterizing the jumps of the weakly 2-random sets which are not 2-randoms. For this application we will need the following.

Proposition 3.3. *The constructed sets A, B in the proofs of Theorems 3.1, 3.2 can be made so as to be not 2-random.*

To show Proposition 3.3, consider a universal Solovay test for 2-randomness: a $\mathbf{0}'$-computable sequence of strings (ρ_s) such that for all sets X,

(3.3) $\quad X$ is not 2-random iff there exist infinitely many s such that $\rho_s \subset X$

and $\sum_i 2^{-|\rho_i|}$ is finite. Now recall that every (non-empty) Π^0_1 class contains a set which is not 2-random. Hence for every Π^0_1 class P there exists s such that $P \cap [\rho_s] \neq \emptyset$. Moreover such s can be found effectively in $\emptyset'$. Since the constructions of Theorems 3.1, 3.2 are forcing arguments with Π^0_1 classes, it is a routine to add a step which forces the sets in the current Π^0_1 classes to be prefixed by some ρ_s. If this is done in every stage, the constructions will produce sets that are not 2-random, according to (3.3). Moreover the verifications will not be affected as $\emptyset'$ is always given in those arguments.

4. Consequences for the jumps of random sets

The results of Section 3.1 have certain consequences on the jumps of random sets—for various notions of randomness. In this section we discuss these consequences. Additionally, we will combine them with other results in the literature, and hence form a better picture of jump inversion with random sets. Some of the results are summarized in Table 2. Note that the only new complete characterization that we obtain is for the jumps of the weakly 2-random sets which are not 2-random, and the weakly 1-random relative to $\emptyset'$ which are not 2-random. The other cases are interesting open questions, although we do contribute some information about them.

Corollary 4.1. *Every $\mathbf{0}'$-hyperimmune degree above $\mathbf{0}'$ is the jump of a weakly 2-random.*

Proof. Since a 1-random is weakly 2-random iff it forms a minimal pair with $\mathbf{0}'$, the corollary follows by applying Theorem 3.1 to a Π^0_1 class consisting entirely of 1-randoms. □

Theorem 4.1. *The jumps of almost all sets (i.e. all but a class of measure 0) are $\mathbf{0}'$-hyperimmune. In fact, the jump of any 3-random is $\mathbf{0}'$-hyperimmune.*

Proof. By [Mar0s] (also see [DH10] for a presentation of the proof) all 2-random sets are hyperimmune. If we relativize Martin's result to $\mathbf{0}'$ we get the following: if X is 3-random, then $X \oplus \emptyset'$ is $\mathbf{0}'$-hyperimmune. Then it suffices to note that by counting the quantifiers in the proofs of [Sac63, Kau91] every 3-random (in fact, every 2-random) is GL_1. □

We note that, by a relativization of the above argument, Theorem 4.1 holds throughout the jump hierarchy. In particular, if X is n-random then $X^{(n-2)}$ is $\mathbf{0}^{(n-2)}$-hyperimmune.

The following theorem shows that Corollary 4.1 does not provide a characterization of the jumps of weakly 2-random sets.

Theorem 4.2. *There is a 2-random whose jump is $\mathbf{0}'$-dominated.*

Proof. Recall that every 2-random is GL_1 (see [Kau91]). Therefore it suffices to show that there is a 2-random X such that $X \oplus \emptyset'$ is $\mathbf{0}'$-dominated. By the Hyperimmune-free Basis Theorem of [JS72] every Π^0_1 class contains a set of $\mathbf{0}$-dominated degree. If we relativize this argument to $\mathbf{0}'$ we get that every $\Pi^0_1[\emptyset']$ class contains a set X such that $X \oplus \emptyset'$ is $\mathbf{0}'$-dominated. Now apply the relativized theorem to a $\Pi^0_1[\emptyset']$ class which consists entirely of 2-random sets. □

We note that, by a relativization of the above argument, Theorem 4.2 holds throughout the jump hierarchy. In particular, for each $n > 1$ there is an n-random whose jump is $\mathbf{0}^{(n)}$-dominated.

In trying to characterize the jumps of weakly 2-random sets, it seems that it is useful to study the jumps of 2-randoms and the jumps of weakly 1-randoms relative to $\emptyset'$. Recall that a set is weakly 1-random relative to $\emptyset'$ if it does not belong to any null $\Pi^0_1[\emptyset']$ class. As noted earlier, 2-randomness implies weak 2-randomness, which in turn implies weak 1-randomness relative to $\emptyset'$ but none of these implications can be reversed. Some more subtle connections between 2-randomness, weak 2-randomness, weak 1-randomness relative to $\emptyset'$ and 1-randomness are given in [BMN].

As mentioned in the introduction, it seemed possible that each degree strictly above $\mathbf{0}'$ is the degree of the jump of a weakly 2-random. This is not true.

Theorem 4.3. *If a degree is $\mathbf{0}'$-dominated and not diagonally non-computable relative to $\mathbf{0}'$ then it is not the jump of a weakly 2-random. In fact, it is not the jump of any degree weakly 1-random relative to $\mathbf{0}'$.*

Proof. Let C satisfy the hypotheses of the theorem. By a result in [SY06] (relativized to $\mathbf{0}'$) the set $C \oplus \emptyset'$ does not compute any weakly 1-random relative to $\mathbf{0}'$ (in fact it is low with respect to this notion). $\square$

So perhaps the degrees above $\mathbf{0}'$ which are either $\mathbf{0}'$-hyperimmune or $\mathbf{0}'$-DNC characterize the jumps of weakly 2-random degrees. Alas, this is also not the case.

Theorem 4.4. *If a degree is PA relative to $\mathbf{0}'$ and is not above $\mathbf{0}''$, then it is not the jump of a 2-random.*

Proof. By [Ste06] all 1-random PA degrees bound $\mathbf{0}'$. The relativization of this argument to $\mathbf{0}'$ gives if A is 2-random and $A \oplus \emptyset'$ is PA relative to $\mathbf{0}'$, then $\emptyset'' \leq_T A \oplus \emptyset'$. However 2-random degrees are GL_1 [Sac63] so their join with $\mathbf{0}'$ equals their jump. $\square$

The following gives some more information about the jumps of 2-random sets.

Theorem 4.5. *Every degree $\geq \mathbf{0}''$ is the jump of a 2-random degree.*

Proof. Since 2-random degrees are GL_1 [Sac63] it suffices to show that given $\mathbf{c} \geq \mathbf{0}''$ and a $\Pi^0_1[\emptyset']$ class P which contains only 2-random sets, there is a path through P of degree $\mathbf{x}$ such that $\mathbf{x} \vee \mathbf{0}' = \mathbf{c}'$. The latter follows by a direct relativization (with respect to $\mathbf{0}'$) of Kučera's result in [Kuč85] that every degree $\geq \mathbf{0}'$ contains a 1-random set. $\square$

The following theorem shows that although the jumps of some weakly 2-random sets are $\mathbf{0}'$-dominated, this does not happen for those that are not already 2-random.

Theorem 4.6. *The jump of a set which is weakly 1-random relative to $\mathbf{0}'$ but not 2-random is $\mathbf{0}'$-hyperimmune.*

Proof. Recall that by [NST05], inside the class of $\mathbf{0}$-dominated sets, the notions of weak 1-randomness and Martin-Löf randomness coincide. If we relativize this argument to $\mathbf{0}'$ we get the following: if $C \oplus \emptyset'$ is $\mathbf{0}'$-dominated and C is weakly 1-random relative to $\mathbf{0}'$, then C is 2-random. The result follows since $C \oplus \emptyset' \leq_T C'$. $\square$

If we combine Theorem 4.6 with Proposition 3.3 we get the following characterization.

Corollary 4.2. *Given a degree $\mathbf{c} \geq \mathbf{0}'$ the following are equivalent:*

(a) $\mathbf{c}$ *contains the jump of a weakly 2-random which is not 2-random*
(b) $\mathbf{c}$ *contains the jump of a weakly 1-random relative to $\mathbf{0}'$ which is not 2-random*
(c) $\mathbf{c}$ *is $\mathbf{0}'$-hyperimmune.*

By Theorems 4.3, 4.4, 4.6 we get that the jump of a weakly 2-random set falls into one of the following cases:

- $\mathbf{0}'$-hyperimmune
- diagonally non-computable relative to $\emptyset'$ but not PA relative to $\emptyset'$.

Which of the $\mathbf{0}'$-dominated degrees above $\mathbf{0}'$ are jumps of weakly 2-random sets? We discuss this question and other related issues.

The PA degrees are exactly the ones that compute a diagonally non-computable function with binary values. On the other hand, if a diagonally non-computable degree is hyperimmune-free, then, in particular, the diagonally non-computable function that it computes should have values that are bounded by a computable function. These statements relativize to $\mathbf{0}'$. Hence in the $\mathbf{0}'$-dominated degrees, the ones that are jumps of weakly 2-randoms need to compute a diagonally non-computable function whose values grow moderately fast. For a characterization of the jumps of weakly 2-random sets one needs to study this narrow class of DNR degrees, which is determined by the growth of the DNR function that they compute.

The work of Jockusch and Stephan in [JS93, JS97] on the jumps of cohesive sets is very relevant to this topic. For example, Jockusch and Stephan [JS93, Theorem 3.5 and Corollary 3.7] show that the jump of a hyperimmune-free degree cannot be diagonally non-computable relative to $\mathbf{0}'$.

Our problems are related to the problem of characterizing the jumps of the hyperimmune-free degrees, which appears in [JS93]. It should be noted that the following present certain similarities:

(i) the construction of a hyperimmune-free degree inside a Π^0_1 class
(ii) the construction of a weakly 2-random degree
(iii) the construction within a Π^0_1 class of a degree forming a minimal pair with $\mathbf{0}'$.

Since a set is weakly 2-random iff it is 1-random and it forms a minimal pair with $\mathbf{0}'$, the relation between (ii) ad (iii) is almost obvious, as in order to get (ii) one can apply (iii) on a Π^0_1 class containing only 1-randoms. Of course from (ii) we automatically get (iii) since every weakly 2-random forms a minimal pair with $\mathbf{0}'$.

If we apply (i) to a Π^0_1 class of 1-randoms we get (ii). However the direct construction of a weakly 2-random set is more flexible than (i) mainly because measure plays an important role (we only need to avoid Π^0_2 classes of measure 0). This flexibility is reflected by the fact that the class of weakly 2-random sets has measure 1 while the hyperimmune-free degrees have measure 0 by [Mar0s].

The comparison between (i) and (iii) is very instructive, especially if one is interested in characterizing the jumps of the hyperimmune-free degrees. To recall (i) we refer to the classic Hyperimmune-Free Basis Theorem of [JS72], while (iii) is essentially contained in the proof of Theorem 2.2 of Section 2.1. The two constructions look remarkably similar, being naturally Δ^0_3 forcing arguments with Π^0_1 classes and relying on the same compactness argument for the satisfaction of the requirements.

However there must be a difference since with respect to (iii) we are able to prove strong statements like Theorem 3.1 while (i) is not compatible in this way. Indeed, by Martin's characterization [Mar66] of highness no hyperimmune-free degree is high ([JS93, JS97] contain stronger results as mentioned above). The difference is exactly the following: when we find that a Turing functional Φ (corresponding to a requirement) is defined on all paths of a Π^0_1 class P, in (i) we are forced to commit to P (for the satisfaction of the requirement). In (ii) however Φ gives the same function on all oracles in P, so the requirement is satisfied even if we do not commit to it (indeed, in that case Φ^X is computable for all $X \in P$). This flexibility allows us to code complicated facts into the set we are building, and was a key ingredient in the proofs of Theorems 2.2 and 2.3.

5. Domination and weak 2-randomness

The standard construction of an almost everywhere dominating function [Kur81] (also see [Nie09, Proposition 5.6.28]) gives a function $f \leq_{tt} \emptyset'$ which dominates Φ^X_e for all $e \in \mathbb{N}$ and all X which do not belong to a certain null set. The latter set is of the form $\{Y \mid \exists^\infty n\ (Y \in V_{g(n)})\}$ where (V_i) is an effective enumeration of all Σ^0_1 classes and $g \leq_{tt} \emptyset'$ such that $\mu(V_{g(i)}) < 2^{-i}$ for all $i \in \mathbb{N}$. In current terminology (e.g. [DH10, Section 6.6.1] and [Nie09, Section 3.6]), f dominates Φ_e^X for all $e \in \mathbb{N}$ and all X which are outside a certain Demuth test. This shows that there is an ω-c.e. function which dominates all functions that are computable from any Demuth (hence any 2-random) set. In other words, all Demuth random (hence all 2-random) sets are array computable. Since weak 2-randomness is in some ways an apparently mild strengthening of Martin-Löf randomness, Nies [Nie09, Problem 8.2.14] asked if all weakly 2-random reals are array computable. Theorem 5.1 not only answers this question in the negative, but it also shows that there is no function that dominates all functions computable from any weakly 2-random set.

For the proof of Theorem 5.1 we use the following standard apparatus for Kučera-Gács coding. Let P be the complement of a member of the standard universal Martin-Löf test. Notice that P contains only Martin-Löf random sets. By Lemma 1.3 there is a computable function h such that for each $\sigma \in 2^{<\omega}$ and each (index[16] of a) Π^0_1 class Q, if $[\sigma] \cap P \cap Q \neq \emptyset$ then the latter class has measure $> 2^{-h(|\sigma|,Q)}$. It follows that $h(|\sigma|, Q) > |\sigma|$ and that there are two distinct extensions of σ of length $h(|\sigma|, Q)$ which are extendible in $P \cap Q$. Given a Π^0_1 class $Q \subseteq P$ we let $T[Q]$ be the tree[17] that is used in the standard Kučera-Gács coding via h. Formally, we define $T[Q](\emptyset) = \emptyset$ and supposing that level n of $T[Q]$ has been defined we define level $n+1$ as follows. Given a string σ of length n let

(5.1) $T[Q](\sigma * i) = T[Q](\sigma) * \tau_i$, where τ_0, τ_1 are the leftmost/rightmost Q-extendible strings of length $h(|T[Q](\sigma)|, Q)$ respectively.

Clearly the length ℓ_n of level n of $T[Q]$ is defined by $\ell_{n+1} = h(\ell_n, Q)$ (and $\ell_0 = 0$); hence it is a computable function on n. Moreover, given any string σ on some level of $T[Q]$ we can effectively find (the unique) $\tau \in 2^{<\omega}$ such that $T[Q](\tau) = \sigma$.[18] Indeed, a computable function p for this task can be defined recursively as follows.

$$p[Q](\sigma) = \begin{cases} p[Q](\sigma \restriction \ell_n) * 0 & \text{if } |\sigma| = \ell_{n+1} \text{ and there exists a stage such that all strings extending } \sigma \restriction \ell_n \text{ that have length } \ell_0 \text{ and are to the left of } \sigma \text{ are not } Q\text{-extendible.} \\ p[Q](\sigma \restriction \ell_n) * 1 & \text{if the previous condition holds with `to the left' replaced by `to the right'.} \\ \uparrow & \text{if } |\sigma| \text{ is not } \ell_{n+1} \text{ for some } n. \end{cases} \tag{5.2}$$

It is clear that given a string σ on some level of $T[Q]$, we have that $p[Q](\sigma)\downarrow$ and $T[Q](p[Q](\sigma)) = \sigma$. The behaviour of $p[Q]$ on strings which do not belong on some level of $T[Q]$ is not important.

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[16] Here and in the following when a Π^0_1 class seems to be an argument in a function, we actually mean an index (a presentation) of it. The presentation of the class that we are referring to will be clear from the context.

[17] That is, as a function from $2^{<\omega}$ to $2^{<\omega}$ which respects compatibility and incompatibility relations of strings. Level n of a tree T is the collection of strings $T(\sigma)$ for all strings σ of length n.

[18] This tree gives the reduction of any set to a random set in P. See the original argument in [Kuč85].

Theorem 5.1. *Given any function $f : \mathbb{N} \to \mathbb{N}$ there exists a weakly 2-random set X and a function $g \leq_T X$ which is not dominated by f.*

In order to obtain a suitable function g as in the statement of Theorem 5.1, we need to code part of f into some $X \in P$ while ensuring that X forms a minimal pair with $\emptyset'$. For the latter we need to meet the following conditions:[19]

$$N_e : \ [\Phi_e^X, \Phi_e^{\emptyset'} \text{ are total } \wedge \ \Phi_e^X = \Phi_e^{\emptyset'}] \Rightarrow \Phi_e^X \text{ is computable.}$$

The argument will be forcing with Π^0_1 classes. The effectiveness of the construction is not an issue of concern here. The oracle $f \oplus \emptyset''$ suffices in order to perform the construction of X. However we must ensure that a function g as described in Theorem 5.1 can be *effectively* extracted from X.

In order to code information to a member X of P we will use Kučera-Gács coding, as this was described in Section 1.2.1. However note that this coding is in general incompatible with forcing with Π^0_1 conditions, which is required for the satisfaction of N_e. For example, it is clear that if $f = \emptyset'$ it is not possible to code f into X while ensuring that $\emptyset', X$ form a minimal pair[20]. Hence we need to use a more flexible coding method, which is compatible with the satisfaction of N_e.

The construction will proceed in stages s at which a decreasing sequence (P_s) of subclasses of P, and an increasing sequence of initial segments $X \upharpoonright t_s$ are defined. In particular, at stages $s \in 4\mathbb{N} \cup (4\mathbb{N}+1)$ we define t_s, at stages $s \in 4\mathbb{N}+2$ we define P_s and at stages $s \in 4\mathbb{N}+3$ we define t_s (again). Stages $4e, 4e+1$ correspond to coding information into X which is relevant to the satisfaction of N_e. In stage $4e+2$ the actual satisfaction of N_e takes place, by possibly restricting P_{4e} to a smaller class P_{4e+2}. Finally at stage $4e+3$, the value of f on some argument is coded into X, for the purpose of computing a function g as prescribed in Theorem 5.1. The value of t_s (in stages s in $4\mathbb{N}$, $4\mathbb{N}+1$ or $4\mathbb{N}+3$) will be effectively obtained from an index of P_s and $X \upharpoonright t_{s-1}$, via the computable function h that was described at the beginning of Section 5. (as in the standard Kučera-Gács coding). Morevover, $X \upharpoonright t_s$ will be the leftmost or rightmost P_s-extendible extension of $X \upharpoonright t_{s-1}$ of length t_s.

With a forcing argument in mind, suppose that we are given a Π^0_1 class Q and we wish to restrict it so that all paths X in the new class satisfy N_e. Then it is crucial to decide whether (5.3) holds.

$$\forall n, t \ \forall X \in Q \ \exists s \ [s > t \ \wedge \ \Phi_e^X(n)[s] \downarrow = \Phi_e^{\emptyset'}(n)[s] \downarrow]. \tag{5.3}$$[21]

If (5.3) holds, then Φ_e is total on Q and its range on Q consists of a single set. This set is computable by compactness. Hence N_e is satisfied for all $X \in Q$. So if (5.3) holds we do not need to restrict Q.

On the other hand if (5.3) does not hold, there exists $\langle n, t\rangle$ such that either $\Phi_e^{\emptyset'}(n) \uparrow$ or

$$\{Z \in Q \mid \Phi_e^Z(n) \uparrow \ \vee \Phi_e^Z(n) \downarrow \neq \Phi_e^{\emptyset'}(n)[s]\} \neq \emptyset \quad \text{ for all } s > t.$$

[19]Notice that the requirements N_e are sufficient. Without loss of generality we can assume that $0 \notin \emptyset'$ and that all paths through P start with 1. Now given any $i, j \in \mathbb{N}$ it is easy to see that there is $e \in \mathbb{N}$ such that $\Phi_i^X = \Phi_e^X$ and $\Phi_j^{\emptyset'} = \Phi_e^{\emptyset'}$ for each $X \in P$.

[20]It is instructive to think of the technical reasons for this impossibility.

[21]Here $\Phi_e^{\emptyset'}(n)[s] \downarrow$ is the computable predicate which says that $\Phi_e(n)$ at stage s is defined on oracle $\emptyset'[s]$. By compactness (5.3) is a Π^0_2 sentence. Therefore it is decidable in $\emptyset''$. Notice that it does not imply that $\Phi_e^{\emptyset'}$ is total.

In the latter case we have

$$\{Z \in Q \mid \Phi_e^Z(n)\uparrow \ \vee \ \Phi_e^Z(n)\downarrow \neq \Phi_e^{\emptyset'}(n)\downarrow\} \neq \emptyset. \tag{5.4}$$

Notice that if (5.3) does not hold, the least $\langle n, t\rangle$ witnessing the negation $\neg$(5.3) of (5.3) is $\emptyset'$-computable from Q, e. If $\Phi_e^{\emptyset'}(n)\uparrow$ requirement N_e is satisfied as $\Phi_e^{\emptyset'}$ is partial . If $\Phi_e^{\emptyset'}(n)\downarrow$, to satisfy N_e we need to intersect Q with the class in (5.4).[22] For reference in the construction, let d be a computable function such that for all Π_1^0 classes Q and all $e \in \mathbb{N}$ we either have (5.3) or $\lim_s d(Q, e, s)$ exists and

$$\lim_s d(Q, e, s) = \text{the least } \langle n, t\rangle \text{ such that } \neg(5.3) \text{ holds.} \tag{5.5}$$

Also let m_d be the modulus function of d for these limits, namely

$$m_d(Q, e) = \text{ the least } k \text{ such that } d(Q, e, t) = d(Q, e, k) \text{ for all } t \geq k \tag{5.6}$$

for those $e \in \mathbb{N}$ such that $\neg$(5.3) holds and $m_d(Q, e)\uparrow$ for those $e \in \mathbb{N}$ such that (5.3) holds. To conclude, there are the following possibilities that we distinguish in the construction below:

(i) (5.3) holds
(ii) $\neg$(5.3) holds and $\Phi_e^{\emptyset'}(n)\uparrow$
(iii) $\neg$(5.3) holds and $\Phi_e^{\emptyset'}(n)\downarrow$

where n is the first coordinate of $\lim_s d(Q, e, s)$. In case (ii) we will call N_e *potentially needy* as there is a possibility that we need to force with a suitable class. In case (iii) we will call N_e *needy* as we certainly need to intersect with a suitable class in order to satisfy of N_e.

5.1. Construction of X.

Stage $s = 4e$. (Code information about how is N_e going to be satisfied.)
Code into X the information whether (5.3) holds for Q equal to $P_{s-1} \cap T[P_{s-1}](11)$:

$$X \upharpoonright t_s = \begin{cases} T[P_{s-1}](0); & \text{if (5.3) holds (for the above class)} \\ T[P_{s-1}](1); & \text{otherwise.} \end{cases}$$

where $t_s = |T[P_{s-1}](0)|$. If the second clause above was used, say that N_e is *potentially needy*. Let $P_s = P_{s-1}$.

Stage $s = 4e + 1$. (Code more information about how is N_e going to be satisfied.)
If in the previous stage we found that (5.3) did not hold, code into X the information whether $\Phi_e^{\emptyset'}(n)\downarrow$, where $\langle n, t\rangle$ is the least number witnessing $\neg$(5.3) for Q equal to $P_{s-1} \cap T[P_{s-1}](11)$:

$$X \upharpoonright t_s = \begin{cases} T[P_{s-1}](10); & \text{if } \Phi_e^{\emptyset'}(n)\uparrow \\ T[P_{s-1}](11); & \text{if } \Phi_e^{\emptyset'}(n)\downarrow \end{cases}$$

where $t_s = |T[P_{s-1}](10)|$. Also, if the second clause was used, say that N_e is *needy*.

If in the previous stage we found that (5.3) holds, let

$$t_s = |T[P_{s-1}](00)| \quad \text{and} \quad X \upharpoonright t_s = T[P_{s-1}](00)$$

and say that N_e is *not needy*. In any case let $P_s = P_{s-1} \cap [X \upharpoonright t_s]$.

[22] Notice that the argument presented here is valid even with a modified (5.3), where in place of Q we have any non-empty Π_1^0 subclass of Q. We will use this in the construction below.

Stage $s = 4e + 2$. (*Satisfy* N_e.)
If N_e is *needy*, let n be as in step $4e+1$ and let P_s be the class in (5.4) with Q equal to P_{s-1}. Otherwise, if N_e is *not needy*, let P_s be P_{s-1}. In any case, $t_s = t_{s-1}$.

Stage $s = 4e + 3$. (*Code a value of* f *on some argument* s_e.)
Let

$$s_e = \begin{cases} \text{The largest of} \\ \star\ s_{e-1}+1,\ m_d(P_{4e+1}, e) \text{ (if this is defined)} \\ \star \text{ the first stage where } p[P_{4j+1}](X \upharpoonright t_{4j+3}) \downarrow, \text{ for each } \\ \quad j < e \\ \star \text{ the modulus of convergence of } \Phi_j^{\emptyset'}(n)[m] \text{ in case } N_j \text{ is } \\ \quad \textit{needy}, \text{ for each } j \le e. \end{cases} \tag{5.7}$$

where p is the decoding function of (5.2). Code $f(s_e)$ into X: let

$$t_s = |T[P_{s-1}](0^{f(s_e)}1)| \qquad \text{and} \qquad X \upharpoonright t_s = T[P_{s-1}](0^{f(s_e)}1). \tag{5.8}$$

Define $P_s = P_{s-1} \cap [X \upharpoonright t_s]$.

5.2. **Verification of the construction of X.** According to the construction and the standard Kučera-Gács machinery (Lemma 1.3, the function h and the tree in (5.1)) the set X and sequences (P_s), (t_s) are totally defined in such a way that $P \supseteq P_0 \supseteq P_1 \supseteq \dots$ and $X \in \cap_t P_t$. Moreover by the discussion made before the construction and the action in Step $4e + 2$ of the construction, requirement N_e is satisfied for all $e \in \mathbb{N}$. Therefore the degree of X forms a minimal pair with the degree of $\emptyset'$. It remains to show that some function $g \le_T X$ is not dominated by f.

By the definition of X we have

$$X \upharpoonright t_s \in \{T[P_{s-1}](k) \mid k = 0, 1\} \quad \text{for all } s \in 4\mathbb{N}. \tag{5.9}$$

$$X \upharpoonright t_s \in \{T[P_{s-1}](\sigma) \mid \sigma = 00, 10, 11\} \quad \text{for all } s \in 4\mathbb{N}+1. \tag{5.10}$$

$$X \upharpoonright t_s \in \{T[P_{s-1}](0^k 1) \mid k \in \mathbb{N}\} \quad \text{for all } s \in 4\mathbb{N}+3. \tag{5.11}$$

Using the decoder function p of (5.2) we can recover what values are coded in X. That is, we can find the unique string σ such that $X \upharpoonright t_s = T[P_{s-1}](\sigma)$. The difficulty however is that we cannot do this uniformly in the stages s. The reason is that in stages $s \in 4\mathbb{N} + 2$, the class P_s is defined in a non-effective way. The choice for P_s was not coded into X. However once we know the information coded into X in stage $s - 1 \in 4\mathbb{N} + 1$, the index of P_s can be computably approximated via the function d of (5.5).

5.2.1. *Extracting g from X.* For each e we give a procedure G_e which attempts to reconstruct steps $4e, 4e + 1, 4e + 2, 4e + 3$ of the construction. In other words, it produces approximations $P_k[s], t_k[s]$ of the parameters P_k, t_k, $k \in \{4e, 4e + 1, 4e + 2, 4e + 3\}$ of the construction when it is run at stage s. The family of procedures $(G_e)_{e\in\mathbb{N}}$ is interconnected, in the sense that a procedure may call another one. Moreover the actions of G_e depend on the stage that it is called. We denote by $G_e[m]$ the version of procedure G_e at stage m. These algorithms work with oracle X and define the function $g \le_T X$ that is required for the proof of the theorem. Notice that by the definition of (s_e) in (5.7) there is and X-computable double sequence $s_e[m]$ such that $s_e[m] \le s_e[m + 1]$ and $s_e = \lim_m s_e[m]$ for each $e \in \mathbb{N}$.

Let $P_{-1}[s] = P$ for all $s \in \mathbb{N}$. We start running $G_0[0]$ and follow the procedures as detailed below.

Procedure $\mathbf{G_e[m]}$. If $s_i[m] = m$ for some $i < e$, go to $G_i[m]$ for the least such i. Let $t_{4e}[m] = h(1, P_{4e-1}))[m-1]$, $t_{4e+1}[m] = h(2, P_{4e-1})[m-1]$ and check if one of the following holds:

(a) $d(P_{4k+1}[m-1], k, m) \neq d(P_{4k+1}[m], k, m)$ for some $k < e$ such that N_k is *potentially needy*; or $\Phi_k^{\emptyset'}(n)[m] \uparrow$ where n is the first coordinate of $d(P_{4k+1}[m-1], k, m-1)$, for some $k < e$ such that N_k is *needy* .[23]

(b) $p[P_{4e-1}](X \upharpoonright t_{4e})[m] \downarrow$, $p[P_{4e-1}](X \upharpoonright t_{4e+1})[m] \downarrow$.

If none of these holds, go to $G_e[m+1]$. If (a) holds, go to procedure $G_k[m+1]$ (for the least k such that (a) is true). If (b) holds (and (a) does not hold) let $P_{4e}[m] = P_{4e-1}[m]$, $P_{4e+1}[m] = P_{4e}[m] \cap [X \upharpoonright t_{4e+1}[m]]$ and do the following:

(c) If $p[P_{4e-1}](X \upharpoonright t_{4e})[m] = 0$ or $p[P_{4e}](X \upharpoonright t_{4e+1})[m] = 10$ let $P_{4e+2}[m] = P_{4e+1}[m]$ and in the second case say that N_e is *potentially needy.*

(d) Otherwise (if $p[P_{4e}](X \upharpoonright t_{4e+1})[m] = 11$) let $P_{4e+2}[m]$ be the intersection of $P_{4e+1}[m]$ with the class of (5.4) where n is taken to be the first coordinate of $d(P_{4e+1}[m], e, m)$ and in place of $\Phi_e^{\emptyset'}(n) \downarrow$ we have the approximation $\Phi_e^{\emptyset'}(n)[m]$. Say that N_e is *needy.*

Let $t_{4e+2}[m] = t_{4e+1}[m]$. Check if $p[P_{4e+2}](X \upharpoonright z)[m] \downarrow = 0^q 1$ for some $q, z < m$. If not, go to $G_e[m+1]$. Otherwise let $t_{4e+3}[m] = h(P_{4e+2}[m], q+1)$ and $P_{4e+3}[m] = (P_{4e+2} \cap [X \upharpoonright t_{4e+3}])[m]$. Also let m_*^e be the largest stage $\leq m$ such that during the stages in $[m_\star^e, m]$ the only procedure that ran was G_e. Define $g(m_\star^e) = q$ and $g(x) = 0$ for all arguments $x < m_\star^e$ on which g is not yet defined; and go to $G_{e+1}[m+1]$.

5.2.2. *Verification of procedures and g.* First of all, notice that a procedure $G_e[m]$ can define g for at most the first $m_\star^e$ arguments, where $m_\star^e$ is the largest stage $\leq m$ such that during the stages in $[m_\star^e, m]$ the only procedure that ran was G_e. Also, if $G_e[m]$ runs, during the stages in $[m_\star^e, m)$ no definition of g is given. Therefore there is no conflict in the definition of g. In other words, when some $g(m)$ is defined, it has been previously undefined.

We are going to show the following claims, which conclude the proof of Theorem 5.1. Recall the definition of s_e from (5.7). For each e let r_e be the maximum stage $m \geq s_e$ such that $m_\star^e = s_e$. Notice that for each e there is $r_e \geq s_e$ such that procedure $G_e[m]$ runs continuously for $m \in [s_e, r_e]$. The fact that G_e will run at s_e follows by the first instruction in the family of $G_i[m]$ procedures and the fact that $s_e[m]$ tends monotonically to s_e.

(I) When G_e runs in the interval of stages $[s_e, r_e]$, it reproduces steps $4e, 4e+1, 4e+2, 4e+3$ of the construction of X. In other words, $P_k[r_e] = P_k$, $t_k[r_e] = t_k$ for all $k \in \{4e, 4e+1, 4e+2, 4e+3\}$.

(II) For each e procedure $G_e[m]$ does not run for any $m > r_e$.

(III) The function g is total and $g(s_e) = f(s_e)$ for all $e \in \mathbb{N}$.

[23]We assume the standard *hat-trick* for the functionals $\Phi_e^{\emptyset'}$. That is, Φ_e is modified (without loss of generality) in such a way that, if a certain computation $\Phi_e^{\emptyset'}[s-1] \downarrow$ no longer exists at stage s, we have $\Phi_e^{\emptyset'}[s] \uparrow$. In other words, any subsequent computation is delayed by at most one stage.

Notice that (III) implies that in the last run of G_e (at stage r_e), this procedure will run completely. In other words, it will reach the final instruction (the definition of g).

Proof of (I),(II). By induction. For $e = 0$ the procedure will define $t_i[s_0] = t_i$, $P_i[s_0] = P_i$ for all $i = 0, 1, 2$. Indeed, if $m_d(P_1, 0) \uparrow$ this is immediate. Otherwise it follows from the fact that at this stage d has reached a limit on arguments P_1 and 0. Notice that it will also declare N_0 *potentially needy*, *needy* or not, in full accordance with stages $0, 1$ of the construction of X.

Also, it will stay in the last part of the instructions, waiting for a stage $m \geq s_0$ with $p[P_2](X \upharpoonright z)[m] \downarrow= 0^q1$ for some $q, z < m$. By the construction of X, such a stage $r_0 \geq s_0$ (large enough so that the relevant computations halt) will be found. Then it will define $t_3[r_0] = t_3$ and $P_3[r_0] = P_3$. This shows Claim (I) for $e = 0$. Claim (II) follows from the definition of s_0 and in particular the fact that $d(P_1, 0, s)$ has reached a limit at $s = s_0$ in the case that the limit exists (i.e. N_e has been declared *potentially needy*). So procedure G_0 will never run after it finishes, after this stage (i.e. after r_0). This finishes the base of the induction.

Now suppose that Claims (I),(II) hold for all $e < j$ and stages s_e have been defined for $e < j$. Recall the definition of s_j in (5.7). As before, at s_j procedure G_j will run and will define $t_i[s_j] = t_i$, $P_i[s_j] = P_i$ for all $i = 4j, 4j+1, 4j+2$. Indeed, by the induction hypothesis we have that $P_{4j-1}[s_j] = P_{4j-1}$ so $G_j[s_j]$ will run exactly as in steps $4j, 4j+1, 4j+2$, of the construction of X. Now if $m_d(P_{4j+1}, j) \uparrow$ clearly it will also run as step $4j+3$ of the construction. If $m_d(P_{4j+1}, j) \downarrow$, since d has reached a limit on arguments P_{4n+1} and n, for $n \leq j$, it will also run as step $4j+3$ of the construction. Notice that it will also declare N_j *potentially needy*, *needy* or not, in full accordance with stages $4j$, $4j+1$ of the construction of X.

Also, it will stay in the last part of the instructions, waiting for a stage $m \geq s_j$ with $p[P_{4j+2}](X \upharpoonright z)[m] \downarrow= 0^q1$ for some $q, z < m$. By the construction of X, such a stage $r_j \geq s_j$ (large enough so that the relevant computations halt) will be found. Then it will define $t_{4j+3}[r_j] = t_{4j+3}$ and $P_{4j+3}[r_j] = P_{4j+3}$. This shows Claim (I) for $e = j$. Claim (II) follows from the definition of s_j and in particular the fact that $d(P_{4j+1}, 0, s)$ has reached a limit at $s = s_j$ in the case that the limit exists (i.e. N_j has been declared *potentially needy*). So procedure G_j will never run after it finishes, after this stage (i.e. after r_j).

Proof of (III). By Claim (I) and the routines $G_e[m]$ we have that g is total. Moreover, by the routine $G_e[t]$ in the interval $t \in [s_e, r_e]$ (along with the definition (5.2) of the decoding function p and the definition of s_j in (5.7)) we get that $g(s_e) = f(s_e)$. Indeed, notice that $r_j < s_{j+1} \leq r_{j+1}$ by the conditions on the definition of s_e in (5.7). Since $s_e < s_{e+1}$ for all $e \in \mathbb{N}$, this finishes the proof of the Theorem 5.1.

5.3. **Application to Problem 8.2.14 of Nies.** Now we can give a positive answer to the second problem of Nies [Nie09] that was discussed in Section 1.

Corollary 5.1. *There are weakly 2-random reals that are array non-computable.*

Proof. It follows from Theorem 5.1 by taking f to be a function which dominates all functions which are truth-table reducible to $\emptyset'$. $\square$

Theorem 5.1 actually shows that for any oracle X, there are weakly 2-random sets which are not array computable relative to X. We also remark that the proof of Theorem 5.1 shows something apparently stronger than its statement. Namely, for every function f there is a weakly 2-random X which computes a function g, which in turn agrees with f on infinitely many values. Another consequence of the proof is that $X \leq_T f \oplus \emptyset''$.

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