### Hausdorff measure on o-minimal structures Version 3.6

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November 12, 2018

#### Abstract

We introduce the Hausdorff measure for definable sets in an ominimal structure, and prove the Cauchy-Crofton and co-area formulae for the o-minimal Hausdorff measure. We also prove that every definable set can be partitioned into "basic rectifiable sets", and that the Whitney arc property holds for basic rectifiable sets.

*Keywords*: O-minimality, Hausdorff measure, Whitney arc property, Cauchy-Crofton, coarea.

MSC 2010: Primary: 03C64; Secondary: 28A75.

#### 1 Introduction

Let K be an o-minimal structure expanding a field. We introduce, for every  $e \in \mathbb{N}$ , the *e*-dimensional Hausdorff measure for definable sets, which is the generalization of the usual Hausdorff measure for real sets [Morgan88]. We also show that every definable set can be partitioned into "basic *e*-rectifiable sets" (§3). Moreover, we generalize some well known result from geometric measure theory, such as the Cauchy-Crofton formula (which computes the Hausdorff measure of a set as the average number of points of intersection with hyperplanes of complementary dimension) and the co-area formula (a generalization of Fubini's theorem), to the o-minimal context.

The measure defined in [BO04] is the starting point for our construction of the Hausdorff measure. A theorem of [BP98] allows us to prove that integration using the Berarducci-Otero measure satisfies properties analogous to the ones for integration over the reals (for example, the change of variable formula). If K is sufficiently saturated, the Berarducci-Otero measure of a bounded definable set X is  $\mathcal{L}_{\mathbb{R}}(\mathrm{st}(X))$ , where  $\mathcal{L}_{\mathbb{R}}$  is the Lebesgue measure and st is the standard-part map. However, the naive definition of Hausdorff measure given by

$$\mathcal{H}^{e}(X) := \mathcal{H}^{e}_{\mathbb{R}}(\mathrm{st}(X)) \tag{1}$$

does not work (because the resulting "measure" is not additive: see Example 5.8). The correct definition for the *e*-dimensional Hausdorff measure is defining it first for basic *e*-rectifiable sets via (1), and then extending it to definable sets by using a partition into basic *e*-rectifiable pieces. Such a partition is obtained by using partitions into  $M_n$ -cells ([K92], [P08], [VR06]), a consequence of which is the Whitney arc property for basic *e*-rectifiable sets (§4).

#### 2 Lebesgue measure on o-minimal structures

The definitions of measure theory are taken from [Halmos50].

Let  $\mathbb{R} := \mathbb{R} \cup \{\pm \infty\}$  be the extended real line. Let K be a  $\aleph_1$ -saturated o-minimal structure, expanding a field. Let  $\mathring{K}$  be the set of finite elements of K. Let st :  $K^n \to \mathbb{R}^n$  be the function mapping  $\bar{x}$  to the *n*-tuple of standard parts of the components of  $\bar{x}$ .

For every  $n \in \mathbb{N}$ , let  $\mathcal{L}_{\mathbb{R}}^n$  be the *n*-dimensional Lebesgue measure (on  $\mathbb{R}^n$ ). If *n* is clear from context we drop the superscript. Let  $\mathcal{L}_1^n$  be the o-minimal measure on  $\mathring{K}^n$  defined in [BO04]. More precisely,  $\mathcal{L}_1^n$  is a measure on the  $\sigma$ -ring  $R_n$  generated by the definable subsets of  $\mathring{K}^n$ ; thus,  $(\mathring{K}^n, R_n, \mathcal{L}_1^n)$  is a measure space. Moreover, since  $\mathring{K}^n \in R_n$ ,  $R_n$  is actually a  $\sigma$ -algebra.

Notice that  $\mathcal{L}_1^n$  can be extended in a natural way to a measure  $\mathcal{L}_2^n$  on the  $\sigma$ -ring  $\mathcal{B}_n$  generated by the definable subsets of  $K^n$  of finite diameter. Finally, we denote by  $\mathcal{L}^n$  the completion of  $\mathcal{L}_2^n$ , and if n is clear from context we drop the superscript. Notice that the  $\sigma$ -ring  $\mathcal{B}_n$  is not a  $\sigma$ -algebra.

**Remark 2.1** ([BO04, Thm. 4.3]). If  $C \subset \mathring{K}^n$  is definable, then  $\mathcal{L}^n(C)$  is the Lebesgue measure of  $\operatorname{st}(C)$ .

**Definition 2.2.** For  $A \subseteq K^n$  and  $f: K^n \to K^m$  we define  $\operatorname{st}(f): A \to \overline{\mathbb{R}}^m$  by  $\operatorname{st}(f)(x) = \operatorname{st}(f(x))$ .

**Remark 2.3.** If  $A \subseteq \check{K}^n$  and  $f : A \to K$  are definable, then st(f) is an  $\mathcal{L}^n$ -measurable function.

**Definition 2.4.** Let  $A \subseteq \mathring{K}^n$  and  $f : A \to K$  be definable. If st(f) is  $\mathcal{L}^n$ -integrable we will denote its integral by

$$\int_{A} f \, \mathrm{d}\mathcal{L}^{n}; \quad \int_{A} f(x) \, \mathrm{d}x; \quad \int_{A} f(x) \, \mathrm{d}\mathcal{L}^{n}(x) \quad \text{or} \quad \int_{A} f.$$

**Remark 2.5.** If  $A \subseteq \mathring{K}^n$  and  $f : A \to \mathring{K}$  are definable, then st(f) is  $\mathcal{L}$ -integrable.

Let  $\mathbb{R}_K$  be the structure on  $\mathbb{R}$  generated by the sets of the form  $\mathrm{st}(U)$ , where U varies among the definable subsets of  $K^n$ . By [BP98],  $\mathbb{R}_K$  is ominimal.

**Remark 2.6.** Let  $U \subseteq \mathring{K}^n$  be definable. Then,  $\dim(\operatorname{st}(U)) \leq \dim(U)$ .

*Proof.* Let dim(U) = d. After a cell decomposition, we can assume that U is the graph of a definable continuous function  $f: V \to \mathring{K}^{n-d}$ , with  $V \subset \mathring{K}^d$  open cell. We can then conclude by applying the method in [HPP08, Lemma 10.3].

**Definition 2.7.** A function f is Lipschitz if there is  $C \in \mathring{K}$  such that, for all  $x, y \in \text{dom}(f)$ , we have |f(x) - f(y)| < C|x - y| (notice the condition on C being finite). An invertible function f is bi-Lipschitz if both f and  $f^{-1}$  are Lipschitz.

**Remark 2.8.** Let  $U \subset \mathring{K}^n$  and  $f: U \to \mathring{K}$  be definable, with  $f \ge 0$ . Then,

$$\int_{U} f \, \mathrm{d}\mathcal{L}^{n} = \mathcal{L}^{n+1} \big( \{ \langle \bar{x}, y \rangle \in U \times K : 0 \le y \le f(\bar{x}) \} \big).$$

**Lemma 2.9** (Change of variables). Let  $U, V \subseteq \mathring{K}^n$  be open and definable, and let  $A \subseteq U$  be definable. Let  $f : U \to V$  be definable and bi-Lipschitz and  $g: V \to \mathring{K}$  be definable, then

$$\int_{f(A)} g = \int_{A} |\det \mathbf{D}f| \ g \circ f$$

Before proving the above lemma, we need some preliminary definitions and results.

**Lemma 2.10.** Let  $U \subset \mathring{K}^n$  be open and let  $f : U \to \mathring{K}$  be definable. Then there is a  $\mathbb{R}_K$ -definable function  $\overline{f} : C \to \mathbb{R}$ , where  $C \subset \operatorname{st}(U)$  is an open set, such that

- i)  $E := (\operatorname{st}(U) \setminus C) \cup (C \cap \operatorname{st}(K^n \setminus U))$  is  $\mathcal{L}^n_{\mathbb{R}}$ -negligible (and, therefore,  $\operatorname{st}^{-1}(E)$  is  $\mathcal{L}^n$ -negligible).
- ii) f and  $\overline{f}$  are  $\mathcal{C}^1$  on  $U \setminus \operatorname{st}^{-1}(E)$  and C, respectively.
- iii) For every  $x \in U$  with  $\operatorname{st}(x) \in C$  we have  $\operatorname{st}(f(x)) = \overline{f}(\operatorname{st}(x))$ . Moreover,  $\operatorname{D} f$  is finite and  $\operatorname{D}(\overline{f})(\operatorname{st} x) = \operatorname{st}(\operatorname{D} f(x))$ .

iv)

$$\int_{U} f = \int_{C} \overline{f}.$$

Proof. By cell decomposition, we may assume that f is a function of class  $C^1$ , and that U is an open cell. Since  $\dim(\Gamma(f)) = n$ , we have, by Remark 2.6,  $\dim(\operatorname{st}(\Gamma(f)) \leq n$ . By cell decomposition, there is an  $\mathbb{R}_K$ -definable, closed, negligible set  $E \subset \operatorname{st}(U)$ , and definable functions  $g_k : \operatorname{st}(U) \setminus E \to \mathbb{R}$  of class  $C^1$  for  $k = 1, \ldots, r$  such that  $\operatorname{st}(\Gamma(f)) \cap ((\operatorname{st}(U) \setminus E) \times \mathbb{R})$  is the union of the graphs of the functions  $g_i$ . We claim that r = 1:

In fact, if  $g_1$ ,  $g_2$  are two different such functions, and say  $g_1 < g_2$ , then for some  $x \in \operatorname{st}(U)$  we have  $\langle x, g_1(x) \rangle$ ,  $\langle x, g_2(x) \rangle \in \operatorname{st}(\Gamma(f))$ . Since f is continuous,  $\{\langle x, y \rangle : y \in (g_1(x), g_2(x))\} \subset \operatorname{st}(\Gamma(f))$ . On the other hand,  $\{\langle x, y \rangle : \langle x, y \rangle \in \operatorname{st}(\Gamma(f))\}$  is the finite set  $\{\langle x, g_1(x) \rangle, \ldots, \langle x, g_r(x) \rangle\}$ , absurd.

By [HPP08, Theorem 10.4], after enlarging E by a negligible set, we obtain i).

Let  $\overline{f} := g_1$ . ii) holds, and for every  $x \in U$  with  $\operatorname{st}(x) \in C$  we have  $\operatorname{st}(f(x)) = \overline{f}(\operatorname{st}(x))$ . The equality of the integrals in iv) follows from Remark 2.8. To obtain the second part of iii) we will enlarge E by a negligible set. For  $i = 1, \ldots, n$  let

$$E_i := \operatorname{st}\left(\left\{x \in U : \frac{\partial f}{\partial x_i}(x) \notin \mathring{K}\right\}\right).$$

By [BP98],  $E_i$  is  $\mathbb{R}_K$ -definable. If dim $(E_i) = n$ , then  $E_i$  contains an open ball. This contradicts Lemma 2.5 of [BO04] by which every definable, one variable function into  $\mathring{K}$  has finite derivative except on  $\mathrm{st}^{-1}(A)$ , for a finite set A. It follows that each set  $E_i$  is negligible and therefore, after enlarging E, we may assume that  $\mathrm{D}(f)$  is finite on  $U \setminus \mathrm{st}^{-1}(E)$ .

It remains to prove  $D(f)(\operatorname{st} x) = \operatorname{st}(Df(x))$ . As before, we will enlarge E by a negligible set. Let  $V := \{x \in \mathbb{R}^n : D(\overline{f})(x) \neq \overline{Df}(x)\}$ . The set V is  $\mathbb{R}_K$ -definable. If V is non-negligible, then it contains an open ball and therefore w.l.o.g. we may assume that V is an open ball centered at 0. We may also assume f(0) = 0. After substracting from f a linear function, we can assume that  $\frac{\partial f}{\partial x_i}(0) = 0$  and  $\frac{\partial \overline{f}}{\partial x_i}(0) = 3\epsilon > 0$  for some index  $i = 1, \ldots, n$ . Therefore, on a smaller neighborhood of 0, we have  $\frac{\partial f}{\partial x_i} < \epsilon$  and  $\frac{\partial \overline{f}}{\partial x_i} > 2\epsilon$ . Thus, for x along the  $x_i$  axis,  $|f(x)| < |x|\epsilon$  and  $\overline{f}(x) \ge 2|x|\epsilon$  contradicting the first part of iii), namely,  $\operatorname{st}(f(x)) = \overline{f}(x)$ . We conclude that V is negligible. Let E' be a negligible set such that away from  $\operatorname{st}^{-1}(E')$  the equality  $\operatorname{st}(Df(x)) = \overline{Df}(\operatorname{st} x)$  holds. Then away from  $\operatorname{st}^{-1}(V \cup E')$  we

have  $\operatorname{st}(\operatorname{D} f(x)) = \overline{\operatorname{D} f}(\operatorname{st}(x)) = \overline{\operatorname{D} f}(\operatorname{st}(x))$  as wanted. By cell decomposition, E can be further enlarged so that C is open.

**Remark 2.11.** If  $f^{-1}(A)$  is negligible whenever A is, then, outside a negligible closed set,  $\overline{(f \circ g)} = \overline{f} \circ \overline{g}$ .

Proof of Lemma 2.9. The fact that f is bi-Lipschitz implies that  $\overline{f}$  is injective (since it is also bi-Lipschitz).

Claim 1. Let  $C \subset \operatorname{st}(V)$  be Lebesgue measurable. Then,

$$\mathcal{L}^{n}(C) = \int_{(\operatorname{st} f)^{-1}(C)} \operatorname{st}(|\det \mathrm{D}f|).$$

In fact, by the change of variables formula (on the reals!) and Lemma 2.10,

$$\mathcal{L}^{n}(C) = \int_{\overline{f}^{-1}(C)} |\det \mathbf{D}\overline{f}|) = \int_{(\operatorname{st} f)^{-1}(C)} \operatorname{st}(|\det \mathbf{D}f|).$$

Claim 2. Let  $h: V \to \overline{\mathbb{R}}$  be an integrable function. Then,

$$\int_{V} h = \int_{U} \operatorname{st}(|\det \mathrm{D}f|) \ h \circ f.$$

Claim 1 implies that the statement is true if h is a simple function. By continuity, the statement is true for any integrable function h.

In particular, we can apply Claim 2 to the function

$$h: x \mapsto \begin{cases} \operatorname{st}(g(x)) & \text{if } x \in f(A), \\ 0 & \text{otherwise,} \end{cases}$$

and obtain the conclusion.

**Lemma 2.12** (Fubini's theorem).  $\mathcal{L}^{n+m}$  is the completion of the product measure  $\mathcal{L}^n \times \mathcal{L}^m$ . Therefore, if D is the interval  $[0,1] \subset K$  and given  $f: D^{n+m} \to D$  definable,

$$\int_{D^{n+m}} f(x,y) \, \mathrm{d}\mathcal{L}^{n+m}(x,y) = \iint_{D^m D^n} f(x,y) \, \mathrm{d}\mathcal{L}^m(x) \, \mathrm{d}\mathcal{L}^n(y)$$

*Proof.* Follows from the definition of  $\mathcal{L}^n$  in [BO04].

#### 2.1 Measure on semialgebraic sets

**Definition 2.13.** We say that  $E \subseteq K^n$  is  $\emptyset$ -semialgebraic if E is definable without parameters in the language of pure fields. If  $E \subseteq K^n$  is  $\emptyset$ -semialgebraic we denote the subset of  $\mathbb{R}^n$  defined by the same formula that defines E by  $E_{\mathbb{R}}$ .

**Remark 2.14.** Let  $E \subseteq \mathring{K}^n$  be  $\emptyset$ -semialgebraic. Then,  $\operatorname{st}(E) = \overline{E_{\mathbb{R}}}$ .

Let  $E \subseteq K^n$  be closed and  $\emptyset$ -semialgebraic submanifold. Working in local charts, from [BO04] one can easily define a measure  $\mathcal{L}^E$  on the  $\sigma$ -ring generated by the definable subsets of E of bounded diameter. We will denote in the same way the completion of  $\mathcal{L}^E$ . Notice that  $\mathcal{L}^{K^n} = \mathcal{L}^n$ .

**Remark 2.15.** Let *E* be a closed,  $\emptyset$ -semialgebraic submanifold of  $K^n$  of dimension *e*, *F* := st(*E*), and *C*  $\subseteq$  *E* be definable and bounded. Then,  $\mathcal{L}^E(C) = \mathcal{L}^F_{\mathbb{R}}(\mathrm{st}(C))$ , where  $\mathcal{L}^F_{\mathbb{R}}$  is the *e*-dimensional Hausdorff measure on *F*.

One could also take the above remark as the definition of  $\mathcal{L}^E$  on  $E \cap \mathring{K}^n$ .

#### **3** Rectifiable partitions

Theorem 3.8 shows that every definable set  $A \subset \mathring{K}^n$  has a partition into definable sets which are  $M_n$ -cells after an orthonormal change of coordinates (where  $M_n \in \mathbb{Q}$  depends only on n). In [P08], the author shows that a permutation of the coordinates suffices. The proof of 3.8 follows closely that of [K92]. The partition in 3.8 is then used in Corollary 3.11 to show that definable sets have a rectifiable partition.

**Definition 3.1.** Let  $L: V \to W$  be a linear map between normed K-vector spaces. The norm of L is given by

$$||L|| := \sup_{|v|=1} |L(v)|.$$

For V, W in the Grassmannian of *e*-dimensional linear subspaces of  $K^n$ , namely  $\mathcal{G}_e(K^n)$ , let  $\pi_V$  and  $\pi_W \in \operatorname{End}_K(K^n)$  be the orthogonal projections onto V and W respectively. In this way we have a canonical embedding  $\mathcal{G}_e(K^n) \subset \operatorname{End}_K(K^n)$ . The **distance function** on the Grassmannian is given by the inclusion above:

$$\delta(V,W) := \|\pi_V - \pi_W\|.$$

For P in  $\mathcal{G}_1(K^n)$  and  $X \in \mathcal{G}_k(K^n)$ , define

$$\delta(P, X) := |v - \pi_X(v)|,$$

where  $\pi_X$  is the orthogonal projection onto X, and v is a generator of P of norm 1. Note that  $\delta(P, X) = 0$  if and only if  $P \subset X$ ,  $0 \leq \delta(P, X) \leq 1$  and  $\delta(P, X) = 1$  if and only if  $P \perp X$ . Note also that  $\delta(P, X)$  is the definable analogous of the sine of the angle between P and X.

**Lemma 3.2.** Let  $n \in \mathbb{N}_{>0}$ . Then there exists an  $\epsilon_n \in \mathbb{Q}_{>0}$ ,  $\epsilon_n < 1$ , such that for any  $X_1, \ldots, X_{2n} \in \mathcal{G}_{n-1}(K^n)$ , there is a line  $P \in \mathcal{G}_1(K^n)$  such that whenever  $Y_1, \ldots, Y_{2n} \in \mathcal{G}_{n-1}(K^n)$  and

$$\delta(X_i, Y_i) < \epsilon_n, \quad i = 1, \dots, 2n, \quad then$$
  
$$\delta(P, Y_i) > \epsilon_n, \quad i = 1, \dots, 2n.$$

Proof. For  $\epsilon > 0$  define  $S_i(\epsilon) = \{v \in S^{n-1} : |v - \pi_{X_i}(v)| \le 2\epsilon\}$ . If  $K = \mathbb{R}$ , let  $\epsilon_n \in \mathbb{Q}_{>0}$  be small enough so that  $2n \operatorname{Vol}(S_1(\epsilon_n)) < \operatorname{Vol}(S^{n-1})$ , where Vol is the measure  $\mathcal{L}^{S^{n-1}}$  defined in §2.1. Then

$$\operatorname{Vol}(\bigcup_{i=1}^{2n} S_i(\epsilon_n)) \le 2n \operatorname{Vol}(S_1(\epsilon_n)) < \operatorname{Vol}(S^{n-1})$$

and therefore

$$\bigcup_{i=1}^{2n} S_i(\epsilon_n) \neq S^{n-1}.$$
 (2)

The same  $\epsilon_n$  will necessarily satisfy (2) for any field K containing  $\mathbb{R}$ .

Now, we choose

$$v \in S^{n-1} - \bigcup_{i=1}^{2n} S_i(\epsilon_n)$$

and let  $P := \langle v \rangle$ . Then

$$\delta(P, Y_i) = |v - \pi_{Y_i} v| \ge |v - \pi_{X_i} v| - |\pi_{X_i} v - \pi_{Y_i} v| > \epsilon_n.$$

**Definition 3.3.** Let  $\epsilon > 0$ . A definable embedded submanifold M of  $K^n$  is  $\epsilon$ -flat if for each  $x, y \in M$  we have  $\delta(TM_x, TM_y) < \epsilon$ , where  $TM_x$  denotes the tangent space to M at x.

**Lemma 3.4.** Let  $A \subset K^n$  be a definable submanifold of dimension e and  $e \in \mathbb{R}_{>0}$ . Then there is a cell decomposition  $A = \bigcup_{i=0}^k A_i$  of A such that for every i we have either  $\dim(A_i) < \dim(A)$  or  $A_i$  is an  $\epsilon$ -flat submanifold of  $K^n$ .

*Proof.* Cover  $\mathcal{G}_e(K^n)$  by a finite number of balls  $B_i$  of radius  $\epsilon/2$ ; and consider the Gauss map  $G : A \to \mathcal{G}_e(K^n)$  taking an element a of A to  $TA_a$ . Take a cell decomposition of  $K^e$  compatible with A and partitioning each  $G^{-1}(B_i)$ . Then the e-dimensional cells contained in A are  $\epsilon$ -flat.

**Lemma 3.5.** Let  $\epsilon \in \mathbb{Q}_{>0}$ , and let  $A \subset \mathring{K}^n$  be an open definable set. Then there are open, pairwise disjoint cells  $A_1, \ldots, A_p \subset A$  such that

- (i)  $\dim(A \bigcup A_i) < n$ .
- (ii) For each *i*, there are definable, pairwise disjoint sets  $B_1, \ldots, B_k$  (with k depending on *i*) such that
  - (a)  $k \leq 2n;$
  - (b) each  $B_j$  is a definable subset of  $\partial A_i$  and an  $\epsilon$ -flat, (n-1)-dimensional,  $\mathcal{C}^1$ -submanifold of  $K^n$ ;
  - (c) dim $(\partial A_i \bigcup_{j=1}^k B_j) < n-1.$

*Proof.* By induction on n. The lemma is clear for n = 1. Assume that n > 1 and the lemma holds for smaller values of n.

Take a cell decomposition of  $\overline{A}$  compatible with A into  $\mathcal{C}^1$ -cells. Let Cbe an open cell in this decomposition; it suffices to prove the lemma for C. Note that  $C = (f, g)_X$ , where X is an open cell in  $K^{n-1}$  and f, g are definable  $\mathcal{C}^1$ -functions on X. Take finite covers of  $\Gamma(f)$  and  $\Gamma(g)$  by open, definable sets  $U_i$  and  $V_j$ , respectively, such that each  $U_i \cap \Gamma(f)$  and each  $V_j \cap \Gamma(g)$  is  $\epsilon$ -flat (to do this, take a finite cover of the Grassmannian by  $\epsilon$ -balls and pull it back via the Gauss maps for  $\Gamma(f)$  and  $\Gamma(g)$ ). The collection of all sets  $\pi(U_i) \cap \pi(V_j)$  is an open cover  $\mathcal{O}$  of X. By the cell decomposition theorem, there is a  $\mathcal{C}^1$ -cell decomposition of X partitioning each set in  $\mathcal{O}$ . Let S be an open cell in this decomposition, and let  $C_0 := (f, g)_S$ . It suffices to prove the lemma for  $C_0$ . By the inductive hypothesis, we can find  $A'_1, \ldots, A'_p \subset S$ and  $B'_1, \ldots, B'_k \subset \partial A'_i$  satisfying the conditions (i) and (ii) above (with nreplaced by n - 1). Define

$$A_i := (f, g)_{A'_i}, \qquad i = 1, \dots, p.$$

Then dim $(C_0 - \bigcup_{i=1}^p A_i) < n$ . For j = 1, ..., k, the set  $(B'_j \times K) \cap \partial A_i$  is definable. Take a  $\mathcal{C}^1$ -cell decomposition of this set, and let  $B_j$  be the union of the (n-1)-dimensional cells in this decomposition (note that  $B_j$  may be empty). Then  $B_j$  is an  $\epsilon$ -flat  $\mathcal{C}^1$ -submanifold of  $K^n$  and

$$\dim\left(\left(\left(B'_{j}\times K\right)\cap\partial A_{i}\right)-B_{j}\right)< n-1.$$

Define  $B_{k+1} := \Gamma(f|A'_i)$  and  $B_{k+2} := \Gamma(g|A'_i)$ ; by construction these are  $\epsilon$ -flat. It is routine to see that  $\partial A_i \subset B_{k+1} \cup B_{k+2} \cup (\partial A'_i \times K)$ . Thus

$$\partial A_i - \bigcup_{j=1}^{k+2} B_j \subset \left( (\partial A'_i \times K) \cap \partial A_i \right) - \bigcup_{j=1}^k B_j$$
  
=  $\left( \bigcup_{j=1}^k ((B'_j \times K) \cap \partial A_i) \cup E \right) - \bigcup_{j=1}^k B_j$   
 $\subset \bigcup_{j=1}^k (((B'_j \times K) \cap \partial A_i) - B_j) \cup E,$ 

where E is a definable set with  $\dim(E) < n - 1$ . Therefore  $\dim(\partial A_i - \bigcup_{j=1}^{k+2} B_j) < n - 1$ . Since  $k \le 2(n - 1)$ , we get  $k + 2 \le 2n$  and the lemma is proved.

**Definition 3.6.** Let  $U \subseteq K^n$  be open and let  $f : U \to K^m$  be definable. Given  $0 < M \in K$ , we say that f is an M-function if  $|Df| \leq M$ . We say that f has finite derivative if |Df| is finite.

Notice that, by  $\omega$ -saturation of K, if f is definable and has finite derivative, then it is an M-function for some finite M.

Let  $M \in K_{>0}$ . An *M*-cell is a  $\mathcal{C}^1$ -cell where the  $\mathcal{C}^1$  functions that define the cell are *M*-functions. More precisely:

**Definition 3.7.** Let  $(i_1, \ldots, i_m)$  be a sequence of zeros and ones, and  $M \in K_{>0}$ . An  $(i_1, \ldots, i_m)$ -*M*-cell is a subset of  $K^m$  defined inductively as follows:

- (i) A (0)-*M*-cell is a point  $\{r\} \subset K$ , a (1)-*M*-cell is an interval  $(a, b) \subset K$ , where  $a, b \in K$ .
- (ii) An  $(i_1, \ldots, i_m, 0)$ -*M*-cell is the graph  $\Gamma(f)$  of a definable *M*-function  $f : X \to K$  of class  $\mathcal{C}^1$ , where X is an  $(i_1, \ldots, i_m)$ -*M*-cell; an  $(i_1, \ldots, i_m, 1)$ -*M*-cell is a set

$$(f,g)_X := \{ (x,r) \in X \times K : f(x) < r < g(x) \},\$$

where X is an  $(i_1, \ldots, i_m)$ -M-cell and  $f, g : X \to K$  are definable M-functions of class  $\mathcal{C}^1$  on X such that for all  $x \in X$ , f(x) < g(x).

**Theorem 3.8.** Let  $A \subset \check{K}^n$  be definable. Then there are definable, pairwise disjoint sets  $A_i$ , i = 1, ..., s, such that  $A = \bigcup_i A_i$  and for each  $A_i$ , there is a change of coordinates  $\sigma_i \in O_n(K)$  such that  $\sigma_i(A_i)$  is an  $M_n$ -cell, where  $M_n \in \mathbb{Q}_{>0}$  is a constant depending only on n.

*Proof.* We will make use of the following fact:

Let  $\epsilon \in [0, 1]$ ,  $P \in \mathcal{G}_1(K^n)$ ,  $X \in \mathcal{G}_k(K^n)$  and and  $w \in X$  be a unit vector. Suppose  $\delta(P, X) > \epsilon$ . If  $\pi_P(w) \ge 1/2$ , where  $\pi_P$  is the orthogonal projection onto P, then

$$|\pi_P(w) - w| \ge |\pi_P(w) - \pi_X(\pi_P(w))| > |\pi_P(w)| \epsilon \ge 1/2\epsilon.$$

If  $\pi_P(w) < 1/2$ , then  $|w| \le |\pi_P(w)| + |\pi_p(w) - w| \le 1/2 + |\pi_p(w) - w|$ . In either case, we have

$$|\pi_P(w) - w| \ge \frac{1}{2}\epsilon.$$
(3)

We prove the theorem by induction on n; for n = 1 the theorem is clear. We assume that n > 1 and that the theorem holds for smaller values of n. We also proceed by induction on  $d := \dim(A)$ . It's clear for d = 0; so we assume that d > 0 and the theorem holds for definable bounded subsets Bof  $K^n$  with  $\dim(B) < d$ .

Case I: dim(A) = n. In this case A is an open, bounded, definable subset of  $K^n$ , so by using the inductive hypothesis and Lemma 3.5, we can reduce to the case where there are pairwise disjoint, definable  $B_1, \ldots, B_k \subset \partial A$  such that  $k \leq 2n$ , dim $(\partial A - \bigcup_{j=1}^k B_j) < n-1$  and each  $B_j$  is an  $\epsilon_n$ -flat submanifold, where  $\epsilon_n$  is as in Lemma 3.2. By Lemma 3.2, there is a hyperplane L such that for each  $B_j$  and all  $x \in B_j$ , we have  $\delta(L^{\perp}, T_x B_j) > \epsilon_n$ . Take a cell decomposition  $\mathcal{B}$  of  $K^n$ , with respect to orthonormal coordinates in the L,  $L^{\perp}$  axis, partitioning each  $B_j$ . Let

$$\mathcal{S} := \{ C \in \mathcal{B} : \dim(C) = n - 1, C \subset \bigcup_{j=1}^{k} B_j \}$$

and note that  $\dim(\partial A \setminus \bigcup_{C \in \mathcal{S}} C) < n-1$ . Furthermore,

$$\mathsf{BAD} := \{ x \in A : \pi_L^{-1}(\pi_L(x)) \cap \partial A \not\subset \bigcup_{c \in \mathcal{S}} C \}$$

has dimension smaller than n. Let  $U_1, \ldots, U_l$  be the elements of  $\{\pi_L(C) : C \in S\}$ . Then the set

$$\{x \in A : x \notin \pi_L^{-1}(\bigcup_{i=1}^l U_i)\}$$

is contained in BAD, and therefore has dimension smaller than n.

By using the inductive hypothesis, we only need to find the required partition for each of the sets  $A \cap \pi_L^{-1}(U_i)$ , i = 1, ..., l. Fix  $i \in \{1, ..., l\}$  and let  $U := U_i, A' := A \cap \pi_L^{-1}(U)$ . Take  $C \in S$  with  $\pi_L(C) = U$ . Then  $C = \Gamma(\phi)$ for a definable  $C^1$ -map  $\phi : U \to L^{\perp}$  and for all  $x \in C$ ,

$$T_x C = \{(v, \mathbf{D}\phi(v)) : v \in T_{\pi_L(x)}U\}.$$

Let  $v \in T_{\pi_L(x)}U$  be a unit vector; since  $\delta(L^{\perp}, T_xC) > \epsilon_n$  and  $|(v, D\phi(v))| = \sqrt{1 + |D\phi(v)|^2}$ , it follows from equation (3) that

$$\frac{1}{2}\epsilon_n \le \frac{1}{\sqrt{1+|\mathbf{D}\phi(v)|^2}} |\pi_{L^{\perp}}((v,\mathbf{D}\phi(v))) - (v,\mathbf{D}\phi(v))| = \frac{1}{\sqrt{1+|\mathbf{D}\phi(v)|^2}} |v|.$$

Therefore,

$$|\mathbf{D}\phi(v)| \le \sqrt{\frac{4}{\epsilon_n^2} - 1}.$$

Let  $M_n \in \mathbb{Q}$  be bigger than  $\max\left\{M_{n-1}, \sqrt{\frac{4}{\epsilon_n^2} - 1}\right\}$ .

We have proved that for each  $C_j \in \mathcal{S}$  with  $\pi_L(C_j) = U$  there is a definable  $\mathcal{C}^1$ -map  $\phi_j : U \to K$ , such that  $|\mathrm{D}\phi_j| < M_n$  and  $C_j = \Gamma(\phi_j)$ .

By the inductive hypothesis, there is a partition  $\mathcal{P}$  of U such that each piece  $P \in \mathcal{P}$  is a  $M_{n-1}$ -cell after a change of coordinates of L. We have

$$A' = \coprod_{\substack{P \in \mathcal{P} \\ (\phi_r, \phi_s)_P \subset A'}} (\phi_r, \phi_s)_P,$$

and  $(\phi_r, \phi_s)_P$  is a  $M_n$ -cell after a coordinate change.

Case II: dim(A) < n. In this case, by Lemma 3.4, we can partition A into cells which are  $\epsilon_n$ -flat. Therefore we may assume that A is an  $\epsilon_n$ -flat submanifold, where  $\epsilon_n$  is as in Lemma 3.2. As in case I, there is a hyperplane L such that A is the graph of a function  $f: U \to K, U \subset L$  and  $|Df| < M_n$ . By the inductive hypothesis, we can partition U into  $M_{n-1}$ -cells. The graphs of f over the cells in this partition give the required partition of A.

**Definition 3.9.** Let  $A \subseteq K^n$  and  $e \leq n$ . A is basic *e*-rectifiable with bound M if, after a permutation of coordinates, A is the graph of an M-function  $f: U \to K^{n-e}$ , where  $U \subset K^e$  is an open M-cell for some finite M.

**Lemma 3.10.** Let  $A \subset \mathring{K}^n$  be an *M*-cell of dimension *e*. Then, *A* is a basic *e*-rectifiable set, and the bound of *A* can be chosen depending only on *M* and *n*.

*Proof.* We proceed by induction on n. If n = 0 or n = 1 the result is trivial, so assume  $n \ge 2$ . By definition, there exists an M-cell  $B \subset \mathring{K}^{n-1}$  such that

- (1) either  $A = \Gamma(g)$  for some *M*-function  $g: B \to \mathring{K}$ , or
- (2)  $A = (g, h)_B$  for some *M*-functions  $g, h : B \to \mathring{K}$ , with g < h.

By inductive hypothesis, there exists an open L-cell  $C \subset K^d$  (for some d and some  $L \geq M$  depending only on M and on n), and an L-function  $f: C \to K^{n-1-d}$ , such that  $B = \Gamma(f)$ .

In case (1) d = e. Define  $l : C \to K^{n-e}$  by  $l(x) = \langle f(x), g(x, f(x)) \rangle$ . It is easy to see that l is an L'-function for some L' depending only on M and n, and that  $A = \Gamma(l)$ . In case (2), d = e - 1. Define  $\tilde{g} := g \circ f$ ,  $\tilde{h} := h \circ f$ , and  $\tilde{B} := (\tilde{g}, \tilde{h})_C$ . Given  $\langle \bar{x}, y \rangle \in \tilde{B}$ , define  $l(\bar{x}, y) := f(\bar{x})$ . We have that  $\tilde{B}$  is an open *e*-dimensional *L*-cell,  $l : \tilde{B} \to K^{n-e}$  is an *L*-function, and  $A = \Gamma(l)$ .

**Corollary 3.11.** Let  $A \subseteq K^n$  be definable of dimension at most e. Then there is a partition  $A = \bigcup_{i=0}^k A_i$  such that  $\dim(A_0) < e$  and  $A_i$  is a basic e-rectifiable set for i > 0. Moreover, the bounds of each  $A_i$  can be chosen to depend only on n (and not on A). We call  $(A_0, \ldots, A_k)$  a basic e-rectifiable partition of A.

*Proof.* Apply Theorem 3.8 and 3.10.

Notice that a similar result has also been proved in [PW06, Theorem 2.3] (where they also take arbitrarily small bounds): however, in [PW06] they don't require that the functions parametrizing the set A are injective (which is essential for our later uses).

#### 4 Whitney decomposition

The fact that the functions that define an M-cell are actually Lipschitz function follows from the following property of M-cells:

Every pair of points x, y in an *M*-cell  $C \subset K^n$  can be connected by a definable  $\mathcal{C}^1$  curve  $\gamma : [0,1] \to C$  with  $|\gamma'(t)| < N|x - y|$ , where *N* is a constant depending only on *M* and *n* which is finite if *M* is (Lemma 4.3 or [VR06] 3.10 & 3.11).

The same property implies that a N-function f on an M-cell is Lipschitz where the Lipschitz constant is finite if M and N are (Corollary 4.5). This last property will be needed for defining Hausdorff measure.

**Remark 4.1.** Let  $U \subset \mathring{K}^n$  be open and definable, and  $f: U \to \mathring{K}$  be an M-function (for some finite M). It is not true in general that f is L-Lipschitz for some finite L: this is the reason why we needed to prove Theorem 3.8.

**Definition 4.2.** Let  $A \subset K^n$ ,  $B \subset K^m$  be definable sets. Let  $\lambda \subset A \times ([0,1] \times B) \subset K^n \times K^{1+m}$  be a definable set such that for every  $x \in A$ , the fiber over x

$$\lambda_x := \{ y \in [0, 1] \times B : \langle x, y \rangle \in \lambda \}$$

is a curve  $\lambda_x : [0,1] \to B$ . We view  $\lambda$  as describing the family of curves  $\{\lambda_x\}_{x \in A}$ . Such a family is a definable family of curves (in *B*, parametrized by *A*).

An L-cell is an L-Lipschitz cell if the functions that define the L-cell are L-Lipschitz.

**Lemma 4.3.** Fix  $L \in K_{>0}$  and  $n \in \mathbb{N}_{>0}$ . Then, there is a constant  $K(n, L) \in K_{>0}$  depending only on n and L, that is finite if L is, such that for every L-Lipschitz cell  $C \subset K^n$  there is a definable family of curves  $\gamma \subset C^2 \times ([0, 1] \times C)$ such that: For all  $x, y \in C$ ,  $\gamma_{x,y} : [0, 1] \to C$  is a  $C^1$ -curve with

- (i)  $\gamma_{xy}(0) = x, \gamma_{xy}(1) = y;$
- (ii)  $|\gamma'_{xy}(t)| \le K(n,L)|x-y|$ , for all  $t \in [0,1]$ .

Proof. By induction on n. For n = 1 the lemma is clear. Take  $n \ge 1$ , and assume that the lemma holds for n. Let  $C \subset K^{n+1}$  be an L-Lipschitz cell. Then  $C = \Gamma(f)$  or  $C = (g, h)_X$  for some L-Lipschitz cell  $X \subset K^{n-1}$  and definable,  $\mathcal{C}^1$ , L-Lipschitz functions f, g, h with g < h, and  $|Df|, |Dg|, |Dh| \le L$ . By induction, there are a constant k := K(n-1, L) and a definable family of  $\mathcal{C}^1$ -curves  $\beta$  in X with the required properties. Let  $\pi_n : K^{n+1} \to K^n$  be the projection onto the first n coordinates.

If  $C = \Gamma(f)$ , we lift  $\beta$  to C via f: fix  $x, y \in C$  and let  $\gamma_{x,y}(t) := (\alpha(t), f(\alpha(t)))$ , where for all  $t \in [0, 1]$   $\alpha(t) := \beta_{\pi_n(x), \pi_n(y)}(t)$ . Then we have  $|\gamma'_{xy}(t)| \leq (1+L)k|x-y|$ .

If  $C = (g, h)_X$ , we lift  $\beta$  as follows: Fix  $x, y \in C$  and let  $\alpha := \beta_{\pi_n(x), \pi_n(y)}$ . Let  $\pi : K^{n+1} \to K$  be the projection onto the last coordinate and take  $u, v \in (0, 1)$  with

$$\pi(x) = uh(\alpha(0)) + (1 - u)g(\alpha(0))$$
  
$$\pi(y) = vh(\alpha(1)) + (1 - v)g(\alpha(1)).$$

Let l(t) := tv + (1-t)u, for  $t \in [0, 1]$ . We define  $\gamma_{x,y}(t) := (\alpha(t), l(t)h(\alpha(t)) + (1-l(t))g(\alpha(t)))$ , and note that

$$|\gamma'_{xy}(t)| \le k|x-y| + |(v-u)(h(\alpha(t)) - g(\alpha(t)))| + 2Lk|x-y|,$$

since l(t), 1 - l(t) are between 0 and 1 and  $|Dh(\alpha'(t))|, |Dg(\alpha'(t))| \le L|\alpha'(t)|$ . Let f := h - g. We want to bound  $|(v - u)f(\alpha(t))|$ , which equals

$$|\pi y - \pi x - v(f(\alpha(1)) - f(\alpha(t))) + u(f(\alpha(0)) - f(\alpha(t))) + g(\alpha(0)) - g(\alpha(1))|.$$

But

$$|f(\alpha(1)) - f(\alpha(t))| \le L|\alpha(1) - \alpha(t)| = L|1 - t| \left| \frac{\alpha(1) - \alpha(t)}{1 - t} \right| \le L|\alpha'(t_0)|$$

for some  $t_0$  between t and 1. Similarly,  $|f(\alpha(0)) - f(\alpha(t))| \le L|\alpha'(t_1)|$ , for some  $t_1$  between t and 1. Since  $u, v \in [0, 1]$ , we get

$$|(v - u)f(\alpha(t))| \le |\pi y - \pi x| + 2Lk|x - y| + L|x - y|;$$

thus  $|\gamma'_{xy}(t)| \leq K(n,L)|x-y|$  for some constant K(n,L) depending only on n and L which is finite if L is. The collection of the curves  $\gamma_{xy}$  for  $x, y \in C$  constitutes the required family of curves.

**Theorem 4.4.** Let L > 0, and let  $C \subset K^n$  be an L-cell. Then C is a k(n, L)-Lipschitz cell, where k(n, L) depends only on n and L, and is finite if L is.

*Proof.* By induction on n; the theorem is clear for n = 1. Assume that n > 1and that the theorem holds for n - 1. Then  $C = \Gamma(f)$  or  $C = (g, h)_X$ , where  $X \subset K^{n-1}$  is a k(n - 1, L)-Lipschitz cell and f, g, h are  $C^1$ -functions on Xsuch that  $|Df|, |Dg|, |Dh| \leq L$ . We need to show that f, g, h are Lipschitz.

Since X is a k-Lipschitz cell, k := k(n-1, L), it follows from Lemma 4.3 that there is a constant K(n-1, k) such that whenever  $x, y \in X$ , there is a definable,  $\mathcal{C}^1$ -curve  $\gamma$  joining x and y with  $|\gamma'(t)| \leq K(n-1, k)|x-y|$  for all  $t \in [0, 1]$ . Let  $g := f \circ \gamma$ , and let  $t_0 \in (0, 1)$  be such that

$$|f(x) - f(y)| = |g'(t_0)| = |\mathrm{D}f(\gamma'(t_0))| \le L|\gamma'(t_0)| \le LK(n-1,k)|x-y|.$$

Thus f is LK(n-1,k)-Lipschitz. We set k(n,L) := LK(n-1,k).

**Corollary 4.5.** Let C be an M-cell and f be a definable M-function. Then f is Lipschitz, and with finite Lipschitz constant if M is finite.

*Proof.* By Theorem 4.4, C has a definable family of curves as in Lemma 4.3. The result therefore follows from the mean value theorem.

**Definition 4.6.** A definable set  $A \subset K^n$  satisfies the Whitney arc property if there is a constant  $K \in \mathring{K}_{>0}$  such that for all  $x, y \in A$  there is a definable curve  $\gamma : [0,1] \to A$  with  $\gamma(0) = x$ ,  $\gamma(1) = y$  and  $\text{length}(\gamma) := \int_0^1 |\gamma'| \le K|x-y|$ .

**Lemma 4.7.** Let  $C \subset \mathring{K}^n$  be an *M*-cell,  $M \in \mathring{K}$ . Then, *C* satisfies the Whitney arc property.

*Proof.* It follows from Theorem 4.4 and Lemma 4.3.

**Theorem 4.8.** Let  $A \subset \mathring{K}^n$  be definable. Then, A can be partitioned into finitely many definable sets, each of them satisfying the Whitney arc property.

*Proof.* This follows from Lemma 4.7, Theorem 3.8 and the fact that the Whitney arc property is invariant under an orthonormal change of coordinates.  $\hfill \Box$ 

## 5 Hausdorff measure

For an introduction to geometric measure theory, and in particular to the Hausdorff measure, see [Morgan88].

**Definition 5.1.** Let  $U \subseteq K^n$  be open and let  $f : U \to \mathring{K}^m$  be a definable function. If  $a \in U$ ,  $e \leq n$  and M is the set of the  $e \times e$  minors of Df(a) we define

$$J_e f(a) = \begin{cases} +\infty & \text{if } f \text{ is not differentiable at } a \text{ or } \operatorname{rank}(\mathrm{D}f(a)) > e, \\ \sqrt{\sum_{m \in M} m^2} & \text{otherwise;} \end{cases}$$

(cf. [Morgan88, §3.6]).

Notice that if e = n = m, then  $J_n f = |\det(\mathbf{D}f)|$ .

**Definition 5.2.** Let  $U \subseteq \mathring{K}^e$  be an open *M*-cell for some  $M \in \mathbb{N}$ , and let  $f: U \to \mathring{K}^m$  be a definable function with finite derivative. Let  $F: U \to \mathring{K}^{m+e}$  be  $F(x) := \langle x, f(x) \rangle$  and  $C := \Gamma(f) = F(U)$  (notice that *C* has bounded diameter). We define

$$\mathcal{H}^e(C) := \int_U J_e F \, \mathrm{d}\mathcal{L}^e.$$

**Lemma 5.3.** If  $C \subseteq \mathring{K}^n$  is basic e-rectifiable, then  $\mathcal{H}^e(C) = \mathcal{H}^e_{\mathbb{R}}(\mathrm{st}(C))$ , where  $\mathcal{H}^e_{\mathbb{R}}$  is the e-dimensional Hausdorff measure on  $\mathbb{R}^n$ .

*Proof.* Let  $A \subset \mathring{K}^e$  and  $f : A \to \mathring{K}^{n-e}$  be as in Definition 3.9, and  $F : A \to \mathring{K}^n$  as in Definition 5.2. Let  $B := \operatorname{st}(A)$ . Then, using the real Area formula [Morgan88],

$$\int_{A} J_e F \, \mathrm{d}\mathcal{L}^e = \int_{B} J_e(\overline{F}) \, \mathrm{d}\mathcal{L}^e_{\mathbb{R}} = \mathcal{H}^e_{\mathbb{R}}(\overline{F}(B)) = \mathcal{H}^e_{\mathbb{R}}(\mathrm{st}(C)).$$

**Definition 5.4.** Let  $A \subseteq \mathring{K}^n$  be definable of dimension at most e, and  $(A_0, \ldots, A_k)$  be a basic *e*-rectifiable partition of A. Define

$$\mathcal{H}^e(A) := \sum_{i=1}^k \mathcal{H}^e(A_i),$$

where  $\mathcal{H}^{e}(A_{i})$  is defined using 5.2.

**Lemma 5.5.** If A is as in the above definition, then  $\mathcal{H}^{e}(A)$  does not depend on the choice of the basic e-rectifiable partition  $(A_0, \ldots, A_k)$ . Proof. It suffices to prove the following: if C is a basic *e*-rectifiable set and  $(A_0, \ldots, A_k)$  is a basic *e*-rectifiable partition of C, then  $\mathcal{H}^e(C) = \sum_{i=1}^k \mathcal{H}^e(A_i)$ , where  $\mathcal{H}^e(C)$  and  $\mathcal{H}^e(A_i)$  are defined using 5.2. For every  $i = 1, \ldots, n$  let U and  $V_i$  be M-cells,  $f : U \to K^{n-e}$  and  $g_i : V_i \to K^{n-e}$  be definable functions with finite derivative,  $\sigma_i$  be a permutation of variables of  $K^n$ ,  $F : K^e \to K^n$  defined by F(x) := (x, f(x)), and  $G_i : K^e \to K^n$  defined by  $G(x) = \sigma_i(x, g_i(x))$  such that C = F(U) and  $A_i = G_i(V_i)$ . Define  $U_i := F^{-1}(A_i) \subseteq U$ , and  $H_i := G_i^{-1} \circ F : U_i \to V_i$ . Notice that each  $H_i$  is a bi-Lipschitz bijection, that U is the disjoint union of the  $U_i$ , and that  $\dim(U_0) < e$ . Hence,

$$\mathcal{H}^{e}(C) = \int_{U} J_{e}F \, \mathrm{d}\mathcal{L}^{e} = \sum_{i=1}^{n} \int_{U_{i}} J_{e}F \, \mathrm{d}\mathcal{L}^{e} = \sum_{i=1}^{n} \int_{U_{i}} J_{e}(G_{i} \circ H_{i}) \, \mathrm{d}\mathcal{L}^{e} =$$
$$= \sum_{i=1}^{n} \int_{U_{i}} (J_{e}(G_{i}) \circ H_{i}) \cdot |\det(\mathrm{D}H_{i})| \, \mathrm{d}\mathcal{L}^{e} = \sum_{i=1}^{n} \int_{V_{i}} J_{e}G_{i} \, \mathrm{d}\mathcal{L}^{e} = \sum_{i=1}^{n} \mathcal{H}^{e}(A_{i}),$$

where we used Lemma 2.9, the fact that each  $\sigma_i$  is a linear function with determinant  $\pm 1$ , and that  $J_e(G \circ H) = (J_e(G) \circ H) \cdot |\det(\mathrm{D}H)|$ .

**Lemma 5.6.**  $\mathcal{H}^e$  does not depend on n. That is, let  $m \ge n$ , and  $A \subset \check{K}^n$  definable, and  $\psi : K^n \to K^m$  be the embedding  $x \mapsto (x, 0)$ . Then,  $\mathcal{H}^e(A) = \mathcal{H}^e(\psi(A))$ .

*Proof.* Obvious from the definition and Lemma 5.5.

Notice that  $\mathcal{H}^0(C)$  is the cardinality of C.

It is clear that  $\mathcal{H}^e$  can be extended to the  $\sigma$ -ring generated by the definable subsets of  $K^n$  of finite diameter and dimension at most e; we will also denote the completion of this extension by  $\mathcal{H}^e$ .

**Lemma 5.7.**  $\mathcal{H}^e$  is a measure on the  $\sigma$ -ring generated by the definable subsets of  $K^n$  of bounded diameter and dimension at most e.

*Proof.* Since K is  $\aleph_1$ -saturated, it suffices to show that, for every A and B disjoint definable subsets of  $K^n$  of finite diameter and dimension at most e,  $\mathcal{H}^e(A \cup B) = \mathcal{H}^e(A) + \mathcal{H}^e(B)$ . But this follows immediately from Lemma 5.5.

**Example 5.8.** In Lemma 5.3, the assumption that C is basic e-rectifiable is necessary. For instance, take  $\epsilon > 0$  infinitesimal, and X be the following subset of  $K^2$ 

$$X := ([0,1] \times \{0\}) \cup \{ \langle x, y \rangle : 0 \le x \le 1 \& y = \epsilon x \}.$$

Then,  $\operatorname{st}(X) = [0, 1] \times \{0\}$ , and thus  $\mathcal{H}^1(X) = 2$ , while  $\mathcal{H}^1_{\mathbb{R}}(\operatorname{st}(X)) = 1$ . This is the source of complication in the theory, and one of the reasons why we had to wait until this section to introduce  $\mathcal{H}^e$ .

### 6 Cauchy-Crofton formula

Give  $e \leq n$ , define

$$\beta := \Gamma\left(\frac{e+1}{2}\right) \Gamma\left(\frac{n-e+1}{2}\right) \Gamma\left(\frac{n+1}{2}\right)^{-1} \pi^{-1/2}.$$

**Definition 6.1.** Let  $\mathcal{AG}_e(K^n)$  be the Grassmannian of affine *e*-dimensional subspaces of  $K^n$  and let  $\mathcal{AG}_e(\mathbb{R}^n)$  be the Grassmannian of affine *e*-dimensional subspaces of  $\mathbb{R}^n$ . Fix an embedding of  $\mathcal{AG}_e(\mathbb{R}^n)$  into some  $\mathbb{R}^m$ , such that  $\mathcal{AG}_e(\mathbb{R}^n)$  is a  $\emptyset$ -semialgebraic closed submanifold of  $\mathbb{R}^m$ , and the restriction to  $\mathcal{AG}_e(\mathbb{R}^n)$  of the dim $(\mathcal{AG}_e(\mathbb{R}^n))$ -dimensional Hausdorff measure coincides with the Haar measure on  $\mathcal{AG}_e(\mathbb{R}^n)$ .

**Definition 6.2.** Given  $A \subseteq K^n$  and  $E \in \mathcal{AG}_{n-e}(K^n)$ , let  $f_A(E) := \#(A \cap E)$ .

**Theorem 6.3** (Cauchy-Crofton Formula). Let  $A \subseteq \mathring{K}^n$  be definable of dimension e. Then,

$$\mathcal{H}^{e}(A) = \frac{1}{\beta} \int_{\mathcal{AG}_{n-e}(K^{n})} f_{A} \, \mathrm{d}\mathcal{L}^{\mathcal{AG}_{n-e}(K^{n})}.$$

We prove the theorem by reducing it to the known case of  $K = \mathbb{R}$ . This is done by showing that  $\#(A \cap E)$  equals  $\#(\operatorname{st} A \cap \operatorname{st} E)$  almost everywhere.

**Definition 6.4.** Let  $f: U \to \mathring{K}^m$  be definable, with  $U \subset \mathring{K}^n$  open. Let  $E \subset \mathbb{R}^n$  and  $\overline{f}$  be as in Lemma 2.10. We say that  $b \in \mathbb{R}^n$  is an S-regular point of  $\overline{f}$  if

- i)  $b \in \operatorname{st}(U) \setminus \overline{E};$
- ii) b is a regular point of  $\overline{f}$ .

Otherwise, we say that b is an S-singular point and  $\overline{f}(b)$  is an S-singular value of  $\overline{f}$ . If  $c \in \mathbb{R}^m$  is not an S-singular value, we say that c is an S-regular value of  $\overline{f}$ .

**Remark 6.5.** Let S be the set of S-regular points of  $\overline{f}$ . Then, S is open and definable in  $\mathbb{R}_K$ .

**Lemma 6.6** (Morse-Sard). Assume that  $m \ge n$ . Then, the set of S-singular values of  $\overline{f}$  is  $\mathcal{L}^m_{\mathbb{R}}$ -negligible,

Proof. By Lemma 2.10, E is negligible; since E is also  $\mathbb{R}_K$ -definable, it has empty interior and therefore dim(E) < n. Since  $m \ge n$ , it follows that  $\overline{f}(E)$ is negligible. The set of S-singular values of  $\overline{f}$  is the union of  $\overline{f}(E)$  and the set of singular values of  $\overline{f}$ ; it is therefore negligible.  $\Box$ 

**Lemma 6.7** (Implicit Function). Assume that m = n. Let  $b \in \mathbb{R}^n$ . If b is an S-regular point of  $\overline{f}$  then, for every  $y \in \operatorname{st}^{-1}(\overline{f}(b))$  there exists a unique  $x \in \operatorname{st}^{-1}(b)$  such that f(x) = y.

*Proof.* Choose  $x_0 \in \operatorname{st}^{-1}(b)$ . Let  $A := (Df(x_0))^{-1}$ . Since b is a regular point of  $\overline{f}$ , ||A|| is finite. Thus we can choose  $r, \rho \in \mathbb{Q}_{>0}$  such that  $B := \overline{B(b;\rho)}$  is contained in the set of S-regular points of  $\overline{f}$ , and

$$\|\overline{\mathrm{D}f}(b') - \overline{\mathrm{D}f}(b)\| < \frac{1}{2n\|A\|}, \text{ for every } b' \in B$$
$$r \le \frac{\rho}{2\|A\|}.$$

Moreover, we can pick  $\rho$  such that  $B' := \overline{B(x_0; \rho)} \subset U$ . Given  $y \in K^n$  such that  $|y - f(x_0)| < r$ , consider the mapping

$$T_y: B' \to K^n$$
  
$$T_y(x) := x + A \cdot (y - f(x)).$$

 $T_y$  is definable and Lipschitz, with Lipschitz constant 1/2. Therefore, for every  $y \in B(f(x_0); r)$  there exists a unique  $x \in B'$  such that  $T_y(x) = x$ . Thus, there is a unique  $x \in B$  with f(x) = y. It remains to show that, given  $y \in \operatorname{st}^{-1}(\overline{f}(b))$  and  $x \in B'$  such that f(x) = y, we have  $x \in \operatorname{st}^{-1}(b)$ . We can verify that

$$T_y: B \to B$$
  
$$\overline{T}_y(b') = b' + (D\overline{f}(b))^{-1} \cdot (\overline{f}(b) - \overline{f}(b'))$$

is also a contraction, and therefore it has a unique fixed point, namely b. Since  $\overline{T}_y(\operatorname{st}(x)) = \operatorname{st}(x)$ , we must have  $\operatorname{st}(x) = b$ .

**Remark 6.8.** Let  $U \subset \mathring{K}^m$ . If  $f: U \to \mathring{K}^n$  is definable and *M*-Lipschitz (for some finite *M*),  $n \geq m$  and *E* is  $\mathcal{L}^m_{\mathbb{R}}$ -negligible, then the set  $f(\mathrm{st}^{-1}(E))$  is  $\mathcal{L}^n$ -negligible.

*Proof.* We can cover E with a polyrectangle Y whose measure is an arbitrarily small rational number  $\lambda$  and such that Y covers st<sup>-1</sup>(E). Since f(Y) has measure at most  $CM^n\lambda$  (C depends only on m and n) the result follows.

**Lemma 6.9.** Let  $A \subseteq \mathring{K}^n$  be a basic e-rectifiable set of dimension e. Consider  $V := K^e$  as embedded in  $K^n$  via the map  $x \mapsto \langle x, 0 \rangle$ . Identify each  $p \in V$  with the (n - e)-dimensional affine space which is orthogonal to V and intersects V in p. Then, for almost every  $p \in V$ , we have  $\#(p \cap A) = \#(\operatorname{st}(p) \cap \operatorname{st}(A))$ .

Proof. Let  $\pi : K^n \to V$  be the orthogonal projection. Let  $U \subset \mathring{K}^e$  be an open *M*-cell and  $f : U \to K^{n-e}$  be a definable *M*-function (*M* finite) such that  $A = \Gamma(f)$ . Let  $F(x) := \langle x, f(x) \rangle$ . Let  $h := \pi \circ F : U \to V$ , and consider  $\overline{h} : C \to \operatorname{st}(V), C \subset \operatorname{st}(U)$  as in Lemma 2.10. For almost every  $p \in V$ ,  $\#(p \cap A) = \#(h^{-1}(p))$ , and  $\#(\operatorname{st} p \cap \operatorname{st} A) = \#(\overline{h}^{-1}(\operatorname{st} p))$  because  $F : U \to A$ and  $\overline{F} : C \to \operatorname{Im}(\overline{F})$  are bijections. Thus, it suffices to prove that, for almost every  $p \in V$ ,  $\#(h^{-1}(p)) = \#(\overline{h}^{-1}(\operatorname{st} p))$ . Let *E* be as in Lemma 2.10. By Remark 6.8,  $h(\operatorname{st}^{-1}(E))$  is  $\mathcal{L}^e$ -negligible. Let *S* be the set of *S*-singular values of  $\overline{h}$ , by Lemma 6.6, *S* is negligible.

Let  $p \in V \setminus (\operatorname{st}^{-1}(S) \cup h(\operatorname{st}^{-1}(E)))$ . Then for every x in  $h^{-1}(p)$ ,  $\operatorname{st}(x)$  is an S-regular point of  $\overline{h}$ , and therefore Lemma 6.7 implies  $\#(h^{-1}(p)) = \#(\overline{h}^{-1}(\operatorname{st} p))$ .

Notice that the above lemma does not hold if A is only definable, instead of basic e-rectifiable.

Proof of Theorem 6.3. By Corollary 3.11, w.l.o.g. A is basic e-rectifiable. Let  $B := \operatorname{st}(A)$ , and  $f_B(F) := \#(B \cap F)$ , for every  $F \in \mathcal{AG}_e(\mathbb{R}^n)$ . By Lemma 6.9,

$$\int_{\mathcal{AG}_{n-e}(K^n)} f_A \, \mathrm{d}\mathcal{L}^{\mathcal{AG}_{n-e}(K^n)} = \int_{\mathcal{AG}_{n-e}(\mathbb{R}^n)} f_B \, \mathrm{d}\mathcal{L}^{\mathcal{AG}_{n-e}(\mathbb{R}^n)}.$$

By the usual Cauchy-Crofton formula [Morgan88, 3.16], the right-hand side in the above identity is equal to  $\mathcal{H}^{e}_{\mathbb{R}}(B) = \mathcal{H}^{e}(A)$ , where we applied Lemma 5.3.

# 7 Further properties of Hausdorff measure and the Co-area formula

**Theorem 7.1.** Let  $e \leq n$  and  $C \subseteq K^n$  be bounded and definable of dimension at most e.

- 1.  $\mathcal{H}^e$  is invariant under isometries.
- 2. For every  $r \in \mathring{K}$ ,  $\mathcal{H}^e(rC) = \operatorname{st}(r)^e \mathcal{H}^e(C)$ .

- 3. If C is  $\emptyset$ -semialgebraic, then  $\mathcal{H}^e(C) = \mathcal{H}^e(C_{\mathbb{R}}) = \mathcal{H}^e(\operatorname{st}(C))$ .
- 4. if  $\dim(C) < e$ , then  $\mathcal{H}^e(C) = 0$ ; the converse is not true.
- 5.  $\mathcal{H}^e(C) < +\infty$ .
- 6. If  $(C(r))_{r \in K^d}$  is a definable family of bounded subsets of  $K^n$ , then there exists a natural number M such that  $\mathcal{H}^n(C(r)) < M$  for every  $r \in K^d$ .
- 7. If K' is either an elementary extension or an o-minimal expansion of K, then  $\mathcal{H}^e(C_{K'}) = \mathcal{H}^e(C)$ .
- 8. If n = e, then  $\mathcal{H}^e(C) = \mathcal{L}^n(C)$ .
- 9. If C is a subset of an e-dimensional affine space E, then  $\mathcal{H}^e(C) = \mathcal{L}^E(C)$ .

Proof.

- (1) Use the Cauchy-Crofton formula.
- (2), (4) and (7) Apply the definition of  $\mathcal{H}^e$  and Lemma 5.5.
  - (3) Apply Corollary 3.11 to  $C_{\mathbb{R}}$  and use Lemma 5.3.
  - (5) and (6) Apply the Cauchy-Crofton formula: see [Dries03].
    - (8) Apply Lemma 5.3.
    - (9) Since  $\mathcal{H}^e$  is invariant under isometries, w.l.o.g. E is the coordinate space  $K^e$ . By Lemma 5.6, the measure  $\mathcal{H}^e$  inside  $K^n$  is equal to the measure  $\mathcal{H}^e$  inside  $K^e$ , and the latter is equal to  $\mathcal{L}^e$ . The conclusion follows from Remark 2.1.

The following theorem is the adaption to o-minimal structures of the Coarea formula, a well-known generalization of Fubini's theorem. Let  $D := [0, 1] \subset K$ .

**Theorem 7.2** (Co-area Formula). Let  $A \subset D^m$  be definable, and  $f: D^m \to D^n$  be a definable Lipschitz function, with  $m \geq n$ . Then,  $J_n f$  is  $\mathcal{L}_K^m$ -integrable, and

$$\int_{A} J_n f \, \mathrm{d}\mathcal{L}^m = \int_{D^n} \mathcal{H}^{m-n}(A \cap f^{-1}(y)) \, \mathrm{d}\mathcal{L}^n(y).$$

Sketch of Proof. W.l.o.g., A is an open subset of  $D^m$ . By Lemma 6.6, w.l.o.g. all points of A are S-regular for  $\overline{f}$ . Apply the real co-area formula [Morgan88] to  $g := \overline{f}$  and  $B := \operatorname{st}(A)$ , and obtain

$$\int_{A} J_n f \, \mathrm{d}\mathcal{L}^m = \int_{B} J_n g \, \mathrm{d}\mathcal{L}^m_{\mathbb{R}} = \int_{D^n_{\mathbb{R}}} \mathcal{H}^{m-n}_{\mathbb{R}}(B \cap g^{-1}(z)) \, \mathrm{d}\mathcal{L}^n_{\mathbb{R}}(z).$$

By the Implicit Function Theorem and Lemma 5.3, for almost every  $y \in D^n_{\mathbb{R}}$ , we have

$$\mathcal{H}^{m-n}(A \cap f^{-1}(y)) = \mathcal{H}^{m-n}_{\mathbb{R}}(B \cap g^{-1}(\operatorname{st} y)).$$

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