

AN ANALOGUE OF THE BAIRE CATEGORY THEOREM

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ABSTRACT. Every definably complete expansion of an ordered field satisfies an analogue of the Baire Category Theorem.

1. INTRODUCTION

Let \mathbb{K} be an expansion of an ordered field $(K, <, +, \cdot)$. We say \mathbb{K} is *definably complete* if every bounded subset of K definable in \mathbb{K} has a supremum in K . Such structures were first studied by Miller in [7]. The main result of this paper is the following definable analogue of the Baire Category Theorem.

Theorem A. Let \mathbb{K} be definably complete. Then there exists *no* set $Y \subseteq K_{>0} \times K$ definable in \mathbb{K} such that

- (i) Y_t is nowhere dense for $t \in K_{>0}$,
- (ii) $Y_s \subseteq Y_t$ for $s, t \in K_{>0}$ with $s < t$, and
- (iii) $\bigcup_{t \in K_{>0}} Y_t = K$,

where Y_t denotes the set $\{a \in K : (t, a) \in Y\}$.

Theorem A is a positive answer to a conjecture of Fornasiero and Servi raised in [2, 3]. By their work, Theorem A implies that definable versions of many standard facts from real analysis hold in \mathbb{K} . Among these are a definable analogue of the Kuratowski-Ulam Theorem, a restricted version of Sard's Theorem and several results in the model theoretic study of Pfaffian functions (see [3, 4, 5] for details).

A short remark about the proof of Theorem A is in order. A definably complete structure does not need to be complete in the topological sense. For this reason the strategy of the classical proof of the Baire Category Theorem to define a sequence of real numbers by recursion is not viable in our setting, as such a sequence might not converge. However, by [2] (see Fact 3 below) it is enough to consider a definable complete \mathbb{K} that defines a closed and discrete set which is mapped by a definable function onto a dense subset of K . In such a situation techniques are available that are based on the idea of definable approximation schemes first used by the author in [6]. These ideas allow us to replace the use of recursion in the classical proof by an explicit definition of an appropriate sequence.

Notation. In the rest of the paper \mathbb{K} will always be a fixed definably complete expansion of an ordered field K . When we say a set is definable in \mathbb{K} , we always mean definable with parameters from \mathbb{K} . We will use a, b, c for elements of K . The letters d, e will always denote elements of a discrete set D . Given a subset X of

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$K^n \times K^m$ and $a \in K^n$, we denote the set $\{b : (a, b) \in X\}$ by X_a . We write \overline{X} for the topological closure of X in the usual order topology.

2. FACTS ABOUT DEFINABLY COMPLETE FIELDS

In this section we recall several facts about definably complete expansions of ordered fields. For more details and background, see [7]. Let \mathbb{K} be a definably complete expansion of an ordered field.

Fact 1. Let $Y \subseteq K$ be a non-empty closed set definable in \mathbb{K} . Then Y contains a minimum and a maximum iff Y is bounded.

Fact 2 ([7, Lemma 1.9]). Let $Y \subseteq K^2$ be definable in \mathbb{K} such that Y_a is closed and bounded and $Y_a \supseteq Y_b \neq \emptyset$ for every $a, b \in K$ with $a < b$. Then $\bigcap_{a \in K} Y_a \neq \emptyset$.

We say that \mathbb{K} is *definably Baire* if it satisfies the conclusion of Theorem A. The proof of Theorem A uses the following result of Fornasiero as a starting point.

Fact 3 ([2, Corollary 6.6]). If \mathbb{K} is *not* definably Baire, then there exists an unbounded, closed and discrete set $D \subseteq K_{\geq 0}$ and a function $f : D \rightarrow K$ such that f is definable in \mathbb{K} and $f(D)$ is dense in K .

The strategy for the proof of Theorem A is to establish the following statement: A definably complete expansion of an order field that defines an unbounded, closed and discrete set which is mapped by a definable function onto a dense set, is definably Baire. Note that there are many instances where we already know Theorem A holds. Since \mathbb{R} is a Baire space, every expansion of the real field is definably Baire. Moreover, any o-minimal expansion of an ordered field is definably Baire. For more examples in this direction and related results for expansions of ordered groups, see Dolich, Miller and Steinhorn [1, 3.5].

3. PROOF OF THEOREM A

Let \mathbb{K} be a definably complete expansion of an ordered field $(K, <, +, \cdot)$. Towards a contradiction, we assume that there is an increasing family $(Y_t)_{t \in K_{>0}}$ of definable nowhere dense sets such that $K = \bigcup_{t \in K_{>0}} Y_t$. Set $Y_0 := \emptyset$. Define $X_t := K \setminus Y_t$. Then X_t is dense in K . By replacing Y_t by its topological closure, we can assume that X_t is open.

By Fact 3 there is also an unbounded, closed and discrete set $D \subseteq K_{\geq 0}$ definable in \mathbb{K} and a map $f : D \rightarrow K$ definable in \mathbb{K} such that the image of D under f is dense in K . Since D is cofinal in $K_{>0}$,

$$K = \bigcup_{d \in D} Y_d.$$

Let $\beta : K \rightarrow D \cup \{0\}$ be the function that maps c to the largest $d \in D \cup \{0\}$ such that $c \in X_d$. Note that β is unbounded. Further let $\gamma : K \rightarrow [0, 1]$ map $c \in K$ to the supremum of the set of elements b in $(0, 1)$ such that $(c - 2b, c) \subseteq X_{\beta(c)}$. Since $X_{\beta(c)}$ is open, $\gamma(c) > 0$. We will write I_c for the open interval $(c - \gamma(c), c)$. We will use the following properties of β and γ .

Lemma 4. Let $c \in K$. Then

$$\emptyset \neq \overline{I_c} \subseteq X_{\beta(c)}.$$

Definition 5. Let $c \in K$. Define $S_c \subseteq D$ as the set

$$\left\{ d \in D : f(d) > c \wedge \forall e \in D \ e < d \rightarrow (f(e) < c \vee f(d) < f(e)) \right\}.$$

Moreover, let $S_c^\beta \subseteq D$ be

$$\left\{ d \in S_c : \forall e \in D \ (e < d \wedge e \in S_c) \rightarrow \beta(f(e)) < \beta(f(d)) \right\}.$$

The elements of the set S_c can be interpreted as the set of best approximations of c from the right. Compare this to the approximation arguments used in [6]. Note that S_c is always unbounded, because it does not contain a maximum. The set S_c^β is always non-empty for every $c \in K$, since it contains the minimum of S_c . But a priori there is no reason why S_c^β should be unbounded. In fact, it might even be finite for some $c \in K$. The advantage of S_c^β over S_c is that the composition $\beta \circ f$ is strictly increasing on S_c^β .

Definition 6. Let $\delta : K \rightarrow D$ be the function that maps c to the largest $d \in S_c^\beta$ such that for all $e_1, e_2 \in S_c^\beta$ with $e_1 < e_2 \leq d$

$$f(e_2) \in I_{f(e_1)}.$$

Define $J_c \subseteq K$ by

$$J_c := \bigcap_{e \in S_c^\beta \cap [0, \delta(c)]} I_{f(e)}.$$

Lemma 7. The function δ is well-defined.

Proof. Let $c \in K$. Towards a contradiction, suppose that $\delta(c)$ is not defined. Then S_c^β is unbounded by Fact 1 and for all $e_1, e_2 \in S_c^\beta$ with $e_1 < e_2$

$$f(e_2) \in I_{f(e_1)}.$$

Then for every $e \in S_c^\beta$ the set

$$\bigcap_{e_1 \in S_c^\beta, e_1 \leq e} \overline{I_{f(e_1)}} \subseteq X_{\beta(f(e))}$$

contains $f(e)$, and hence is non-empty and closed. By Fact 2 and Lemma 4

$$\emptyset \neq \bigcap_{e \in S_c^\beta} \overline{I_{f(e)}} \subseteq \bigcap_{e \in S_c^\beta} X_{\beta(f(e))}.$$

Since S_c^β is unbounded and $\beta \circ f$ is strictly increasing on S_c^β , the set $\{\beta(f(e)) : e \in S_c^\beta\}$ does not contain a maximum. Thus by Fact 1 it is unbounded. Hence it is cofinal in D and

$$\bigcap_{e \in S_c^\beta} X_{\beta(f(e))} = \bigcap_{d \in D} X_d.$$

This is a contradiction, since $\bigcap_{d \in D} X_d$ is empty. \square

Lemma 8. Let $c \in K$. Then J_c is a non-empty open interval.

Proof. Let $d_1 \in D$ be the largest element of $S_c^\beta \cap [0, \delta(c)]$ such that

$$\bigcap_{e \in S_c^\beta \cap [0, d_1]} I_{f(e)}$$

is a non-empty open interval. Such an element exists, since S_c^β is non-empty and I_a is a non-empty open interval for every $a \in K$. Towards a contradiction, suppose

that $d_1 < \delta(c)$. Let $d_2 \in D$ be the smallest element in $S_c^\beta \cap [0, \delta(c)]$ larger than d_1 . Since $d_2 \leq \delta(c)$, $f(d_2) \in I_{f(e)}$ for all $e \in S_c^\beta$ with $e < d_2$. Hence

$$I_{f(d_2)} \cap \bigcap_{e \in S_c^\beta \cap [0, d_1]} I_{f(e)}$$

is a non-empty open interval. This is a contradiction to the maximality of d_1 . Hence $d_1 = \delta(c)$. \square

Note that for every $c \in K$

$$f(\delta(c)) \in \overline{J_c} \subseteq X_{\beta(f(\delta(c)))}.$$

In order to find a counter-example to the statement $\bigcap_{d \in D} X_d = \emptyset$, we will amalgamate sets of the form $S_c^\beta \cap [0, \delta(c)]$. For this purpose we introduce the following notion of an extension.

Definition 9. For $c_1, c_2 \in K$, we say that c_2 *extends* c_1 if $\delta(c_1) < \delta(c_2)$ and

$$S_{c_1}^\beta \cap [0, \delta(c_1)] = S_{c_2}^\beta \cap [0, \delta(c_1)].$$

In the following we will construct an unbounded definable subset E_0 of D such that for all $d, e \in E_0$ with $d < e$, $f(e)$ extends $f(d)$. Given such a set E_0 , we will be able to create a contradiction as in the proof of Lemma 7 (see proof of Theorem A below). With that goal in mind, we start by establishing several properties of extensions. First note that being an extension is transitive. If c_3 extends c_2 and c_2 extends c_1 , then c_3 extends c_1 .

Lemma 10. Let $c_1, c_2 \in K$ be such that c_2 extends c_1 . Then

- (i) $J_{c_2} \subseteq J_{c_1}$, and
- (ii) $\beta(f(\delta(c_2))) > \beta(f(\delta(c_1)))$.

Proof. (i) Since c_2 extends c_1 , $S_{c_1}^\beta \cap [0, \delta(c_1)] \subseteq S_{c_2}^\beta \cap [0, \delta(c_2)]$. Hence $J_{c_2} \subseteq J_{c_1}$.
(ii) Since c_2 extends c_1 , $\delta(c_1) \in S_{c_2}^\beta$. Since $\beta \circ f$ is strictly increasing on $S_{c_2}^\beta$, $\beta(f(\delta(c_2))) > \beta(f(\delta(c_1)))$. \square

Lemma 11. Let $c \in K$ and $d \in D$. If the set

$$L := \{f(e) : e \in D, e < d, f(e) < c\}$$

is non-empty, then it contains a maximum.

Proof. Suppose L is non-empty. Then the set

$$\left\{ e_1 \in D : f(e_1) < c \wedge \forall e_2 \in D (e_2 < e_1 \rightarrow (f(e_2) > c \vee f(e_2) < f(e_1))) \right\} \cap (0, d)$$

is bounded and non-empty. Thus it contains a maximum, say e_3 . By the definition, the image of e_3 under f is the maximum of L . \square

Proposition 12. Let $c \in K$. Then there exists $d \in D$ such that $f(d)$ extends c .

Proof. By Lemma 8, the set

$$A := K_{>c} \cap J_c$$

is a non-empty open interval. We will construct $d, d_1 \in D$ such that $f(d_1) \in A$, $f(d)$ extends c and d_1 is the smallest element in $S_{f(d)}^\beta$ larger than $\delta(c)$. Because $f(d_1) \in A$, d_1 witnesses that $\delta(f(d)) > \delta(c)$.

Since $f(D)$ is dense in K , we can define $d_1 \in D$ as the smallest element in D such that $f(d_1) \in A$ and $\beta(f(d_1)) > \beta(f(\delta(c)))$. Since $c < f(d_1) < f(\delta(c))$ and $\delta(c) \in S_c$, we have $d_1 > \delta(c)$. We now choose d . By Lemma 11, the set

$$\{f(e) : e \in D, e < d_1, f(e) < f(d_1)\}$$

has a maximum, say $f(d_2)$ for some $d_2 \in D$. By density of $f(D)$, we can choose $d \in D$ such that $f(d) \in A \cap (f(d_2), f(d_1))$.

Since $c < f(d) < f(\delta(c))$,

$$S_c^\beta \cap [0, \delta(c)] = S_{f(d)}^\beta \cap [0, \delta(c)].$$

It is only left to establish that $\delta(f(d)) > \delta(c)$. Since $f(d_1) \in A$, it is enough to show that d_1 is the smallest element in $S_{f(d)}^\beta$ larger than $\delta(c)$. By the choice of d , we have that for all $e \in D$ with $e < d_1$

$$(3.1) \quad c < f(d) < f(d_1) < f(e) \text{ or } f(e) < f(d).$$

Hence $d_1 \in S_{f(d)}$. Let $e \in D$ be such that $\delta(c) < e < d_1$. We will show that $e \notin S_{f(d)}^\beta$. This then directly implies that $d_1 \in S_{f(d)}^\beta$ and that d_1 is the smallest such element larger than $\delta(c)$. By minimality of d_1 either $f(e) \notin A$ or $\beta(f(e)) \leq \beta(f(\delta(c)))$. In both cases we have to check that $e \notin S_{f(d)}^\beta$. If $\beta(f(e)) \leq \beta(f(\delta(c)))$, then $e \notin S_{f(d)}^\beta$ because $\delta(c) \in S_{f(d)}^\beta$ and $\beta \circ f$ is strictly increasing on $S_{f(d)}^\beta$. Now consider the case that $f(e) \notin A$. Since A is an interval and $f(\delta(c)) \in \overline{A}$, either $f(e) < f(d_1)$ or $f(e) \geq f(\delta(c))$. If $f(e) \geq f(\delta(c))$, then $e \notin S_{f(d)}^\beta$ because $\delta(c) \in S_{f(d)}^\beta$. If $f(e) < f(d_1)$, then $f(e) < f(d)$ by (3.1). Hence $e \notin S_{f(d)}^\beta$. Hence d_1 is the smallest element in $S_{f(d)}^\beta$ larger than $\delta(c)$. \square

Definition 13. Define E as the set of $e \in D$ such that there is no $d \in D$ with $d < e$ and

$$S_{f(e)}^\beta \cap [0, \delta(f(e))] = S_{f(d)}^\beta \cap [0, \delta(f(e))].$$

The set E is defined in a way to make sure that if $e \in E$, $d \in D$ and $f(d)$ extends $f(e)$, then $e < d$.

Lemma 14. Let $c \in K$. The set $\{e \in E : f(e) \text{ extends } c\}$ is unbounded.

Proof. Let $d_1 \in D$ be the smallest element in D such that $f(d_1)$ extends c . It is easy to see that $d_1 \in E$. Towards a contradiction, suppose there exists $e \in E$ such that e is the largest element in E such that $f(e)$ extends c . By Proposition 12, let $d \in D$ be the smallest element of D such that $f(d)$ extends $f(e)$. Because $e \in E$, $d > e$. Since $f(e)$ extends c , so does $f(d)$. Moreover, since $e \in E$ and d is the smallest element in D such that $f(d)$ extends $f(e)$, d is in E as well. This contradicts the maximality of e . Hence the set $\{e \in E : f(e) \text{ extends } c\}$ is unbounded. \square

Lemma 15. Let $d_1, d_2, d_3 \in D$ be such that $d_2 \in E$ and $d_1 < d_2 < d_3$. If $f(d_3)$ extends $f(d_1)$ and $f(d_3)$ extends $f(d_2)$, then $f(d_2)$ extends $f(d_1)$.

Proof. Towards a contradiction, suppose that $\delta(f(d_2)) \leq \delta(f(d_1))$. Since $f(d_3)$ extends both $f(d_1)$ and $f(d_2)$,

$$S_{f(d_2)}^\beta \cap [0, \delta(f(d_2))] = S_{f(d_1)}^\beta \cap [0, \delta(f(d_2))].$$

This contradicts $d_2 \in E$. Hence $\delta(f(d_2)) > \delta(f(d_1))$ and

$$S_{f(d_1)}^\beta \cap [0, \delta(f(d_1))] = S_{f(d_2)}^\beta \cap [0, \delta(f(d_1))].$$

□

Definition 16. Let d_0 be the smallest element in E . Define $E_0 \subseteq E$ as the set of elements d of E satisfying the following two properties:

- either $f(d)$ extends $f(d_0)$ or $d = d_0$,
- if there are $e_1, e_2 \in E$ such that $d_0 \leq e_1 < d$, $f(d)$ extends $f(e_1)$ and e_2 is the smallest element in E larger than e_1 such that $f(e_2)$ extends $f(e_1)$, then either $d = e_2$ or $f(d)$ extends $f(e_2)$.

The set E_0 is definable in \mathbb{K} , since both E and the property of being an extension are definable in \mathbb{K} .

Lemma 17. Let $d \in E_0$. If e is the smallest element in E larger than d such that $f(e)$ extends $f(d)$, then $e \in E_0$.

Proof. Since $f(e)$ extends $f(d)$, $f(e)$ extends $f(d_0)$. Let $e_1, e_2 \in E$ be such that $d_0 \leq e_1 < e$, $f(e)$ extends $f(e_1)$ and e_2 is the smallest element in E larger than e_1 such that $f(e_2)$ extends $f(e_1)$. If $e_1 = d$, we get $e_2 = e$ by minimality of e . If $e_1 < d$, then $f(d)$ extends $f(e_1)$ by Lemma 15. Since $d \in E_0$, either $d = e_2$ or $f(d)$ extends $f(e_2)$. Thus $f(e)$ extends $f(e_2)$. Hence we can reduce to the case that $e_1 > d$. Since $f(e)$ extends both $f(d)$ and $f(e_1)$, $f(e_1)$ extends $f(d)$ by Lemma 15. But then $e_1 = e$ by the minimality of e . Hence $e \in E_0$. □

Proposition 18. Let $d, e \in E_0$. If $d < e$, then $f(e)$ extends $f(d)$.

Proof. Consider the set

$$Z := \{d \in E_0 : \forall e_1, e_2 \in E_0 (e_1 \leq d \wedge e_1 < e_2) \rightarrow (f(e_2) \text{ extends } f(e_1))\}.$$

It is enough to show that Z is unbounded. Since $d_0 \in Z$ by definition of E_0 , Z is non-empty. For a contradiction, suppose $d_1 \in D$ is the largest element in Z . Let d_2 be the smallest element in E such that $f(d_2)$ extends $f(d_1)$. By Lemma 17, $d_2 \in E_0$. For every $e \in E_0$ with $e > d_1$, either $e = d_2$ or $f(e)$ extends $f(d_2)$ by definition of E_0 . Hence $d_2 \in Z$. This contradicts the maximality of d_1 . □

Proof of Theorem A. We will show that

$$\emptyset \neq \bigcap_{d \in E_0} \overline{J_{f(d)}} \subseteq \bigcap_{d \in D} X_d.$$

This contradicts the assumption that the family $(Y_d)_{d \in D}$ witnesses that \mathbb{K} is not definably Baire, and hence establishes Theorem A.

By Proposition 18 and Lemma 10, we have for all $d_1, d_2 \in E_0$ with $d_1 < d_2$

$$\overline{J_{f(d_2)}} \subseteq \overline{J_{f(d_1)}}.$$

By Fact 2,

$$\emptyset \neq \bigcap_{d \in E_0} \overline{J_{f(d)}} \subseteq \bigcap_{d \in E_0} X_{\beta(f(\delta(f(d))))}.$$

By Lemma 14 and 17, the set E_0 has no maximum and hence is unbounded by Fact 1. Hence $\{\beta(f(\delta(f(d)))) : d \in E_0\}$ is unbounded as well by Proposition 18 and Lemma 10. Thus the set $\bigcap_{d \in D} X_d$ is equal to $\bigcap_{d \in E_0} X_{\beta(f(\delta(f(d))))}$ and in particular non-empty. □

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