# Higher-order illative combinatory logic

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#### Abstract

We show a model construction for a system of higher-order illative combinatory logic  $\mathcal{I}_{\omega}$ , thus establishing its strong consistency. We also use a variant of this construction to provide a complete embedding of first-order intuitionistic predicate logic with second-order propositional quantifiers into the system  $\mathcal{I}_0$  of Barendregt, Bunder and Dekkers, which gives a partial answer to a question posed by these authors.

This paper is a revised version of [Cza13] which appeared in the Journal of Symbolic Logic, vol. 78, issue 3, pp. 837-872. An error in Section 5 and some minor mistakes in Section 4 are corrected. Also, the construction in Section 4 is slightly simplified. © 2013 by the Association for Symbolic Logic.

## 1 Introduction

Illative systems of combinatory logic or lambda-calculus consist of type-free combinatory logic or lambda-calculus extended with additional constants intended to represent logical notions. In fact, early systems of combinatory logic and lambda calculus (by Schönfinkel, Curry and Church) were meant as very simple foundations for logic and mathematics. However, the Kleene-Rosser and Curry paradoxes led to this work being abandoned by most logicians.

It has proven surprisingly difficult to formulate and show consistent illative systems strong enough to interpret traditional logic. This was accomplished in [BBD93], [DBB98a] and [DBB98b], where several systems were shown complete for the universal-implicational fragment of first-order intuitionistic predicate logic.

The difficulty in proving consistency of illative systems in essence stems from the fact that, lacking a type regime, arbitrary recursive definitions involving logical operators may be formulated, including negative ones. In early systems containing an unrestricted implication introduction rule this was the reason for the Curry's paradox [BBD93, CFC58, §8A], where an arbitrary term X is derived using a term Y satisfying  $Y =_{\beta} Y \supset X$ . For an overview of and introduction to illative combinatory logic see [BBD93], [Sel09] or [CFC58].

Systems of illative combinatory logic are very close to Pure Type Systems. The rules of illative systems, however, have fewer restrictions, judgements have the form  $\Gamma \vdash t$  where t is an arbitrary term instead of  $\Gamma \vdash N : C$ . This connection has been explored in [BD05] where some illative-like systems were proven equivalent to more liberal variants of PTSs from [BD01]. Those illative systems, however, differ somewhat from what is in the literature.

In [Cza11] an algebraic treatment of a combination of classical first-order logic with type-free combinatory logic was given. On the face of it, the system of [Cza11] seems to be not quite like traditional illative combinatory logic, but the methods used in the present paper are a (substantial) extension of those from [Cza11].

In this work we construct a model for a system of classical higher-order illative combinatory logic  $\mathcal{I}_{\omega}^{c}$ , thus establishing a strong consistency result. We also use a variant of this construction to improve slightly on the results of [BBD93]. We show a complete embedding of the system PRED2<sub>0</sub> of first-order intuitionistic many-sorted predicate logic with second-order propositional quantifiers into the system  $\mathcal{I}_{0}$  which is an extension of  $\mathcal{I}\Xi$  from [BBD93].

To be more precise, we define a translation  $\lceil - \rceil$  from the language of PRED2<sub>0</sub> to the language of  $\mathcal{I}_0$ , and a mapping  $\Gamma$  from sets of formulas of PRED2<sub>0</sub> to sets of terms of  $\mathcal{I}_0$ . The embedding is proven to satisfy the following for any formula  $\varphi$  of PRED2<sub>0</sub> and any set of formulas  $\Delta$  of PRED2<sub>0</sub>:

$$\Delta \vdash_{\mathrm{PRED2_0}} \varphi \text{ iff } [\Delta], \Gamma(\Delta, \varphi) \vdash_{\mathcal{I}_0} [\varphi]$$

where  $\Delta, \varphi$  stands for  $\Delta \cup \{\varphi\}$ . The implication from left to right is termed soundness of the embedding, from right to left – completeness.

Our methods are quite different from those of [BBD93], where an entirely syntactic approach is adopted. We define a Kripke semantics for illative systems and prove it sound and complete<sup>1</sup>. Given a Kripke model  $\mathcal{N}$  for PRED2<sub>0</sub> we show how to construct an illative Kripke model  $\mathcal{M}$  for  $\mathcal{I}_0$  such that exactly the translations of statements true in a state of  $\mathcal{N}$  are true in the corresponding state of  $\mathcal{M}$ . This immediately implies completeness of the embedding.

The model constructions for  $\mathcal{I}_0$  and  $\mathcal{I}_\omega^c$  are similar, but the latter is much more intricate. The basic idea is to define for each ordinal  $\alpha$  a relation  $\leadsto_\alpha$  between terms and so called "canonical terms". To every canonical term we associate a unique type. In a sense, the set of all canonical terms of a given type fully describes this type. Intuitively,  $t \leadsto_\alpha \rho$  holds if  $\rho$  is a "canonical" representant of t in the type of  $\rho$ . This relation encompasses a definition of truth when  $\rho \in \{\top, \bot\}$ . Essentially,  $\leadsto_\alpha$  is defined by transfinite induction in a monotonous way. We show that there must exist some ordinal  $\zeta$  such that  $\leadsto_\alpha = \leadsto_\zeta$  for  $\alpha > \zeta$ . We use the relation  $\leadsto_\zeta$  to define our model. Then it remains to prove that what we obtain really is the kind of model we expect, which is the hard part.

### 2 Preliminaries

In this section we define the system PRED2<sub>0</sub> of first-order many-sorted intuitionistic predicate logic with second-order propositional quantifiers, together with its (simplified) Kripke semantics. We also briefly recapitulate the definition of full models for a system of classical higher-order logic PRED $\omega^c$ .

**Definition 2.1.** The system PRED $\omega$  of higher-order intutionistic logic is defined as follows.

• The *types* are given by

$$\mathcal{T} ::= o \mid \mathcal{B} \mid \mathcal{T} \rightarrow \mathcal{T}$$

where  $\mathcal{B}$  is a specific finite set of base types. The type o is the type of propositions.

• The set of terms of PRED $\omega$  of type  $\tau$ , denoted  $T_{\tau}$ , is defined by the following grammar, where for each type  $\tau$  the set  $V_{\tau}$  is a countable set of variables and  $\Sigma_{\tau}$  is a countable set of constants.

$$T_{\tau} ::= V_{\tau} \mid \Sigma_{\tau} \mid T_{\sigma \to \tau} \cdot T_{\sigma} \text{ for } \sigma \in \mathcal{T} \mid \lambda V_{\tau_{1}}.T_{\tau_{2}} \text{ if } \tau = \tau_{1} \to \tau_{2}$$

$$T_{o} ::= V_{o} \mid \Sigma_{o} \mid T_{\tau \to o} \cdot T_{\tau} \text{ for } \tau \in \mathcal{T} \mid T_{o} \supset T_{o} \mid \forall V_{\tau}.T_{o} \text{ for } \tau \in \mathcal{T}$$

Terms of type o are called formulas.

- We identify α-equivalent formulas, i.e., formulas differing only in the names of bound variables are considered identical.
- Every variable x has an associated unique type, i.e., there is exactly one  $\tau$  such that  $x \in V_{\tau}$ . We sometimes use the notation  $x_{\tau}$  for a variable such that  $x_{\tau} \in V_{\tau}$ .
- The system PRED $\omega$  is given by the following rules and an axiom, where  $\Delta$  is a finite set of formulas,  $\varphi, \psi$  are formulas. The notation  $\Delta, \varphi$  is a shorthand for  $\Delta \cup \{\varphi\}$ .

Axiom

 $\Delta, \varphi \vdash \varphi$ 

Rules

<sup>&</sup>lt;sup>1</sup>In fact, for completeness of the embedding the easier soundness of the semantics would suffice, i.e., the completeness of the semantics is not necessary for the main results of this paper.

$$\supset_{i}: \frac{\Delta, \varphi \vdash \psi}{\Delta \vdash \varphi \supset \psi} \qquad \supset_{e}: \frac{\Delta \vdash \varphi \supset \psi}{\Delta \vdash \psi} \qquad \Delta \vdash \varphi$$

$$\forall_{i}: \frac{\Delta \vdash \varphi}{\Delta \vdash \forall x_{\tau}.\varphi} \ x_{\tau} \notin FV(\Delta) \qquad \forall_{e}: \frac{\Delta \vdash \forall x_{\tau}.\varphi}{\Delta \vdash \varphi[x_{\tau}/t]} \ t \in T_{\tau}$$

$$\operatorname{conv}: \frac{\Delta \vdash \varphi}{\Delta \vdash \psi} \qquad \varphi =_{\beta\eta} \psi$$

The classical variant  $PRED\omega^c$  is defined by adding to  $PRED\omega$  the law of double negation as an axiom

$$\Delta \vdash ((\varphi \supset \bot) \supset \bot) \supset \varphi$$

where  $\perp \equiv \forall x_o.x_o \text{ and } x_o \in V_o.$ 

The system PRED2<sub>0</sub> is the fragment of second-order many-sorted predicate calculus restricted to formulas in which second-order quantifiers are only propositional. It is obtained from PRED $\omega$  by dropping the rule conv, restricting the types to

$$\mathcal{T} ::= o \mid \mathcal{B} \mid \mathcal{B} \to \mathcal{T}$$

and changing the definition of terms to

$$T_{\tau} ::= V_{\tau} \mid \Sigma_{\tau} \mid T_{\sigma \to \tau} \cdot T_{\sigma} \text{ for all } \tau \in \mathcal{T}, \ \sigma \in \mathcal{B}$$

$$T_{o} ::= V_{o} \mid \Sigma_{o} \mid T_{\sigma \to o} \cdot T_{\sigma} \mid T_{o} \supset T_{o} \mid \forall V_{\tau}.T_{o} \text{ for } \tau \in \mathcal{B} \cup \{o\}, \ \sigma \in \mathcal{B}$$

For an arbitrary set  $\Delta$  we write  $\Delta \vdash_S \varphi$  if  $\varphi$  is derivable from a subset of  $\Delta$  in system S. We drop the subscript when obvious or irrelevant. Note that we trivially have weakening with this definition, i.e., if  $\Delta \vdash \varphi$  then  $\Delta' \vdash \varphi$  for any  $\Delta' \supseteq \Delta$ .

In the rest of this section we assume a fixed set of base types and fixed sets of constants  $\Sigma_{\tau}$  for each type  $\tau \in \mathcal{T}$ . We assume  $\mathcal{T}$ ,  $T_{\tau}$ , etc. to refer either to PRED $\omega$  or PRED $\omega$ 0, depending on the context.

The systems contain only  $\supset$  and  $\forall$  as logical operators. However, it is well-known that all other connectives may be defined from these with the help of the second-order propositional universal quantifier.

We denote by t[x/t'] a term obtained from t by simultaneously substituting all free occurrences of x with t'.

**Definition 2.2.** A full model for PRED $\omega^c$  is a pair

$$\mathcal{M} = \langle \{ \mathcal{D}_{\tau} \mid \tau \in \mathcal{T} \}, I \rangle$$

where each  $\mathcal{D}_{\tau}$  is a nonempty set for  $\tau \in \mathcal{B}$ ,  $\mathcal{D}_{o} = \{\top, \bot\}$ , each  $\mathcal{D}_{\tau_{1} \to \tau_{2}}$  is the set of all functions from  $\mathcal{D}_{\tau_{1}}$  to  $\mathcal{D}_{\tau_{2}}$ , and I is a function mapping constants of type  $\tau$  to  $\mathcal{D}_{\tau}$ . The interpretation function  $\llbracket \rrbracket$  and the satisfaction relation  $\models$  are defined in the standard way. It is well-known and easy to show that  $\Delta \vdash_{\mathsf{PRED}\omega^{c}} \varphi$  implies  $\Delta \models \varphi$ .

The rest of this section is devoted to introducing a simplified variant of Kripke semantics for  $PRED2_0$  and proving it sound and complete. The development is mostly but not completely standard.

**Definition 2.3.** A Kripke pre-model of PRED2<sub>0</sub> is a tuple

$$\mathcal{M} = \langle \mathcal{S}, \leq, \{\mathcal{D}_{\tau} \mid \tau \in \mathcal{T}\}, \cdot, I, \varsigma \rangle$$

where S is a set of states,  $\leq$  is a partial order on S, the set  $\mathcal{D}_{\tau}$  is the domain for type  $\tau$ , the function  $\cdot$  is a binary application operation, I is an interpretation of constants, and  $\varsigma$  is a function assigning upward-closed (w.r.t.  $\leq$ ) subsets of S to elements of  $\mathcal{D}_o$ . A set  $X \subseteq S$  is upward-closed w.r.t.  $\leq$  when for all  $s_1, s_2 \in S$ , if  $s_1 \in X$  and  $s_1 \leq s_2$ , then  $s_2 \in X$  as well. We sometimes write  $\varsigma_{\mathcal{M}}$ ,  $S_{\mathcal{M}}$ , etc., to stress that they are components of  $\mathcal{M}$ . Furthermore, the following conditions are imposed on a Kripke pre-model:

- $\mathcal{D}_{\tau}$  is nonempty for any  $\tau$ .
- for any  $d_1 \in \mathcal{D}_{\tau_1 \to \tau_2}$  and  $d_2 \in \mathcal{D}_{\tau_1}$  we have  $d_1 \cdot d_2 \in \mathcal{D}_{\tau_2}$ ,
- $I(c) \in \mathcal{D}_{\tau}$  for any  $c \in \Sigma_{\tau}$ .

A valuation is a function that, for all types  $\tau$ , maps  $V_{\tau}$  into  $D_{\tau}$ . When we want to stress that a valuation is associated with a structure  $\mathcal{M}$ , we call it an  $\mathcal{M}$ -valuation. If u is a valuation,  $d \in D_{\tau}$  and  $x_{\tau}$  is a variable of type  $\tau$ , then by  $u[x_{\tau}/d]$  we denote a valuation u' such that u'(y) = u(y) for  $y \neq x_{\tau}$  and  $u(x_{\tau}) = d$ . For a given structure  $\mathcal{M}$  and an  $\mathcal{M}$ -valuation u, an interpretation  $\mathbf{u}$  (sometimes abbreviated by  $\mathbf{u}$ ) is a function mapping terms of type  $\tau$  to  $\mathcal{D}_{\tau}$ , and satisfying the following:

- $[x]^u = u(x)$  for a variable x,
- $[\![c]\!]^u = I(c)$  for  $c \in \Sigma_\tau$ ,
- $[t_1t_2]^u = [t_1]^u \cdot [t_2]^u$ .

For a formula  $\varphi$ , a state s and a valuation u we write  $s, u \Vdash_{\mathcal{M}} \varphi$  if  $s \in \varsigma(\llbracket \varphi \rrbracket_{\mathcal{M}}^u)$ . Given a set of formulas  $\Delta$ , we use the notation  $s, u \Vdash_{\mathcal{M}} \Delta$  if  $s, u \Vdash_{\mathcal{M}} \varphi$  for all  $\varphi \in \Delta$ . We drop the subscript  $\mathcal{M}$  when obvious or irrelevant.

A Kripke model is a Kripke pre-model  $\mathcal{M}$  satisfying the following for any state s and any valuation u:

- $s, u \Vdash \varphi \supset \psi$  iff for all  $s' \geq s$  such that  $s', u \Vdash \varphi$  we have  $s', u \Vdash \psi$ ,
- $s, u \Vdash \forall x_{\tau}.\varphi$  for  $x_{\tau} \in V_{\tau}$  iff for all  $s' \geq s$  and all  $d \in \mathcal{D}_{\tau}$  we have  $s', u[x_{\tau}/d] \Vdash \varphi$ ,
- $s, u \nvDash \forall p.p \text{ for } p \in V_o$ .

We write  $\Delta \Vdash \varphi$  if for every Kripke model  $\mathcal{M}$ , every state s of  $\mathcal{M}$ , and every valuation u, the condition  $s, u \Vdash_{\mathcal{M}} \Delta$  implies  $s, u \Vdash_{\mathcal{M}} \varphi$ .

**Remark 2.4.** What we call Kripke semantics is in fact a somewhat simplified version of the usual notion. It is not much more than a reformulation of the inference rules. There are no conditions for connectives other than  $\forall$  and  $\supset$ , so for instance with our definition  $s, u \Vdash \varphi \lor \psi$  need not imply  $s, u \Vdash \varphi$  or  $s, u \Vdash \psi$ , where  $\varphi \lor \psi$  is defined in the standard way as  $\forall x_o.(\varphi \supset x_o) \supset (\psi \supset x_o) \supset x_o$ . We also assume constant domains.<sup>2</sup> The resulting notion of a model is quite syntactic, which allows us to simplify the usual completeness proof considerably.

Another peculiarity is the presence of the function  $\varsigma$ . It may seem superfluous, but it is necessary in the Kripke semantics for illative systems in Section 3 where we do not know *a priori* which terms represent propositions. For the sake of uniformity we already introduce it here.

**Lemma 2.5.** If  $\mathcal{M}$  is a Kripke model,  $x \in V_{\tau}$ ,  $t_0 \in T_{\tau}$ ,  $t \in T_{\tau'}$  and  $\tau' \neq o$ , then:

$$[t[x/t_0]]_{\mathcal{M}}^u = [t]_{\mathcal{M}}^{u'}$$

where we use the notation u' for  $u[x/[t_0]]^u$ .

*Proof.* Straightforward induction on the size of t.

**Lemma 2.6.** If  $\mathcal{M}$  is a Kripke model,  $x \in V_{\tau}$ ,  $t \in T_{\tau}$ ,  $\varphi \in T_{\rho}$  and  $u' = u[x/[t]^u]$ , then for all states s:

$$s, u \Vdash \varphi[x/t]$$
 iff  $s, u' \Vdash \varphi$ 

*Proof.* We proceed by induction on the size of  $\varphi$ . If  $\varphi$  is a constant, a variable, or  $\varphi = t_1 t_2$ , then the claim follows from Lemma 2.5.

Assume  $\varphi = \varphi_1 \supset \varphi_2$ . Suppose  $s, u \Vdash \varphi_1[x/t] \supset \varphi_2[x/t]$  and let  $s' \geq s$  be such that  $s', u' \Vdash \varphi_1$ . By the IH we have  $s', u \Vdash \varphi_1[x/t]$ , hence  $s', u \Vdash \varphi_2[x/t]$ . Applying the IH again we obtain  $s', u' \Vdash \varphi_2$ . This implies that  $s, u' \Vdash \varphi$ . The other direction is analogous.

Assume  $\varphi = \forall y.\varphi_0$ . Without loss of generality  $y \neq x$  and  $y \notin FV(t)$ . Suppose  $s, u \Vdash \forall y.\varphi_0[x/t]$ , and let  $s' \geq s$  and  $d \in \mathcal{D}_{\tau}$ . We have  $s', u[y/d] \Vdash \varphi_0[x/t]$ . By the IH we obtain  $s', u'[y/d] \Vdash \varphi_0$ . This implies  $s, u' \Vdash \forall y.\varphi_0$ . The other direction is analogous.

<sup>&</sup>lt;sup>2</sup>A reader concerned by this is invited to invent an infinite Kripke model (as defined in Definition 2.3) falsifying the Grzegorczyk's scheme  $\forall x(\psi \lor \varphi(x)) \supset \psi \lor \forall x\varphi(x)$ . This scheme is not intuitionistically valid, but holds in all models with constant domains, in the usual semantics.

#### **Theorem 2.7.** The conditions $\Delta \Vdash \varphi$ and $\Delta \vdash \varphi$ are equivalent.

Proof. By induction on the length of derivation we first show that  $\Delta \vdash \varphi$  implies  $\Delta \Vdash \varphi$ . Note that it suffices to show this for finite  $\Delta$ . The implication is obvious for the axiom. Assume  $\Delta \vdash \varphi$  was obtained by rule  $\forall_i$ . Then  $\varphi = \forall x.\psi$  for  $x \in V_\tau$ ,  $x \notin FV(\Delta)$ . Let  $\mathcal{M}, s, u$  be such that  $s, u \Vdash_{\mathcal{M}} \Delta$ . Hence for all  $s' \geq s$  we have  $s', u \Vdash_{\mathcal{M}} \Delta$ , and  $s', u[x/d] \Vdash_{\mathcal{M}} \Delta$  for any  $d \in \mathcal{D}_\tau$  because  $x \notin FV(\Delta)$ . So by the inductive hypothesis we obtain  $s', u[x/d] \Vdash_{\mathcal{M}} \psi$  for any  $d \in \mathcal{D}_\tau$ . By the definition of a Kripke model, this implies  $s, u \Vdash_{\mathcal{M}} \forall x.\psi$ . The remaining cases are equally straightforward. Lemma 2.6 is needed for the rule  $\forall_e$ .

To prove the other direction, we assume that  $\Delta_0 \nvdash \varphi_0$  and construct a Kripke model  $\mathcal{M}$  and a valuation u such that for some state s of  $\mathcal{M}$  we have  $s, u \Vdash_{\mathcal{M}} \Delta_0$ , but  $s, u \nvDash_{\mathcal{M}} \varphi_0$ .

First, without loss of generality, we assume that there are infinitely many variables not occuring in the formulas of  $\Delta_0$ . We can do this because extending the language with infinitely many new variables is conservative. The states of  $\mathcal{M}$  are consistent sets of formulas  $\Delta' \supseteq \Delta_0$ , i.e.,  $\Delta' \not\vdash \bot$ , which differ from  $\Delta_0$  by only finitely many formulas. The ordering is by inclusion. For any type  $\tau$  as  $\mathcal{D}_{\tau}$  we take the set of terms of type  $\tau$ . Let v be a valuation. Given a term t, we denote by  $t^v$  a term obtained from t by simultaneously substituting any variable  $x \in FV(t)$  by the term v(x). We obviously assume that no variables are captured in these substitutions, which is possible because we treat formulas up to  $\alpha$ -equivalence. We define the interpretation I by I(c) = c. We also set  $t_1 \cdot t_2 = t_1 t_2$ . Notice that now  $[t]^v = t^v$ . Further, we define the function  $\varsigma$  of  $\mathcal{M}$  as follows:  $\varsigma(\varphi) = \{\Delta \mid \Delta \vdash \varphi\}$  for a formula  $\varphi$ , where  $\Delta$  ranges over sets of formulas which are valid states. Note that  $\Delta, v \Vdash_{\mathcal{M}} \varphi$  is now equivalent to  $\Delta \vdash \varphi^v$ . Finally, we set u(x) = x.

Given a formula  $\phi$ , a state  $\Delta$ , and a valuation v, we show by induction on the size of  $\phi$  that  $\Delta, v \Vdash_{\mathcal{M}} \phi$  satisfies the conditions required for a Kripke model. If  $\phi = \varphi \supset \psi$ , then we need to check that  $\Delta \vdash \varphi^v \supset \psi^v$  iff for all  $\Delta' \supseteq \Delta$  such that  $\Delta'$  is a valid state and  $\Delta' \vdash \varphi^v$ , we have  $\Delta' \vdash \psi^v$ . Suppose the right side holds and take  $\Delta' = \Delta \cup \{\varphi^v\}$ . If  $\Delta'$  is a valid state then  $\Delta' \vdash \psi^v$ , hence by rule  $\supset_i$  we obtain  $\Delta \vdash \varphi^v \supset \psi^v$ . Because  $\Delta$  extends  $\Delta_0$  by finitely many formulas, so does  $\Delta'$ . Hence if  $\Delta'$  is not a valid state, then it is inconsistent. Then obviously  $\Delta' \vdash \psi^v$  anyway, so we again obtain the left side by applying rule  $\supset_i$ . The other direction follows by applying  $\supset_e$  and weakening finitely many times.

Similarly, if  $\psi = \forall x.\varphi$ , then without loss of generality we assume  $v(x) = x, x \in V_{\tau}$ , and check that  $\Delta \vdash \forall x.\varphi^v$  iff for all valid states  $\Delta' \supseteq \Delta$  and all  $t_1 \in \mathcal{D}_{\tau}$  we have  $\Delta' \vdash \varphi^{v'}$  where  $v' = v[x/t_1]$ . If the right side of the equivalence holds, then it holds in particular for  $t_1 = y$  such that  $y \notin FV(\Delta, \varphi^v)$ , and  $\Delta' = \Delta$ . Such y exists, because we have assumed an infinite number of variables not occurring in the formulas of  $\Delta_o$ , and  $\Delta$  extends  $\Delta_o$  by only finitely many formulas. By rule  $\forall_i$  we obtain  $\Delta \vdash \forall_y \varphi^v$ , which is  $\alpha$ -equivalent to the left side, and we treat  $\alpha$ -equivalent formulas as identical. Conversely, if  $\Delta \vdash \forall x.\varphi^v$ , then by rule  $\forall_e$  and weakening we obtain  $\Delta' \vdash \varphi^v[x/t_1]$ . This is equivalent to  $\Delta' \vdash \varphi^{v'}$  where  $v' = v[x/t_1]$ .

It is now a matter of routine to check that  $\mathcal{M}$  is a Kripke model. Obviously, in this model we have  $\Delta_0, u \nvDash \varphi_0$ , i.e.,  $\Delta_0 \notin \llbracket \varphi_0 \rrbracket^u = \varsigma(\varphi_0)$ , because  $\Delta_0 \nvDash \varphi_0$ . On the other hand,  $\Delta_0, u \vDash \psi$  for every  $\psi \in \Delta_0$ . This proves the theorem.

## 3 Illative systems

In this section we define the higher-order illative systems  $\mathcal{I}_{\omega}$ ,  $\mathcal{I}_{\omega}^{c}$  and the second-order illative system  $\mathcal{I}_{0}$ . We also define a semantics for these systems.

**Definition 3.1.** By  $\mathbb{T}(\Sigma)$  we denote the set of type-free lambda-terms over some specific set  $\Sigma$  of primitive constants, which is assumed to contain  $\Xi$ , L and  $A_{\tau}$  for each  $\tau \in \mathcal{B}$  where  $\mathcal{B}$  is some specific set of base types.

We use the following abbreviations. The term ⊃ is usually written in infix notation and is assumed

to be right-associative.

$$I = \lambda x.x$$

$$S = \lambda xyz.xz(yz)$$

$$K = \lambda xy.x$$

$$H = \lambda x.L(Kx)$$

$$\supset = \lambda xy.\Xi(Kx)(Ky)$$

$$F = \lambda xyf.\Xi(x)(xy)$$

The constant  $\Xi$  functions as a restricted quantification operator, i.e.,  $\Xi AB$  is intuitively interpreted as  $\forall x.Ax \supset Bx$ . The intended interpretation of LA is "A is a type", or "A may be a range of quantification". The term H stands for the "type" of propositions, and FAB denotes the "type" of functions from A to B. The constants  $A_{\tau}$  denote base types, i.e., different sorts of individuals. We use a notion of types informally in this section.

For systems of illative combinatory logic, judgements have the form  $\Gamma \vdash t$  where  $\Gamma$  is a finite subset of  $\mathbb{T}(\Sigma)$  and  $t \in \mathbb{T}(\Sigma)$ . The notation  $\Gamma$ , t is an abbreviation for  $\Gamma \cup \{t\}$ .

The system  $\mathcal{I}_{\omega}$  is defined by the following axioms and rules.

#### Axioms

- (1)  $\Gamma, t \vdash t$
- (2)  $\Gamma \vdash LH$
- (3)  $\Gamma \vdash LA_{\tau}$  for  $\tau \in \mathcal{B}$

Rules

$$\begin{aligned} & \operatorname{Eq} : \frac{\Gamma \vdash t_1}{\Gamma \vdash t_2} & t_1 =_{\beta\eta} t_2 \\ & \Xi_e : \frac{\Gamma \vdash \Xi t_1 t_2}{\Gamma \vdash t_2 t_3} & H_i : \frac{\Gamma \vdash t}{\Gamma \vdash H t} \end{aligned}$$

$$& \Xi_i : \frac{\Gamma, t_1 x \vdash t_2 x}{\Gamma \vdash \Xi t_1 t_2} & x \notin FV(\Gamma, t_1, t_2)$$

$$& \Xi_H : \frac{\Gamma, t_1 x \vdash H(t_2 x)}{\Gamma \vdash H(\Xi t_1 t_2)} & x \notin FV(\Gamma, t_1, t_2)$$

$$& F_L : \frac{\Gamma, t_1 x \vdash L t_2}{\Gamma \vdash L(F t_1 t_2)} & x \notin FV(\Gamma, t_1, t_2)$$

The system  $\mathcal{I}_{\omega}^{c}$  is  $\mathcal{I}_{\omega}$  plus the axiom of double negation:

$$\Gamma \vdash \Xi H (\lambda x. ((x \supset \bot) \supset \bot) \supset x)$$

where  $\perp = \Xi HI$ .<sup>3</sup>

The system  $\mathcal{I}_0$  is  $\mathcal{I}_{\omega}$  minus the rule  $F_L$ . The rule  $F_L$  allows us to quantify over functions and predicates. Obviously, the system becomes more useful if for  $\tau \in \mathcal{B}$  we can add constants c representing some elements of type  $\tau$ , axioms  $A_{\tau}c$ , and some axioms of the form e.g.  $p(fc_1)(gc_2)$  where f, g are constants representing functions and p is a predicate constant (i.e. of type  $\tau_1 \to \tau_2 \to o$ ). That most such simple extensions are consistent with  $\mathcal{I}_{\omega}$  is a consequence of the model construction in Section 4.

<sup>&</sup>lt;sup>3</sup>Note that here the symbol  $\perp$  is an abbreviation for a term in the syntax of  $\mathcal{I}_{\omega}$ , which is distinct from previous uses of  $\perp$ .

For an arbitrary set  $\Gamma$ , we write  $\Gamma \vdash_{\mathcal{I}} t$  if there is a finite subset  $\Gamma' \subseteq \Gamma$  and a derivation of  $\Gamma' \vdash t$  in an illative system  $\mathcal{I}$ . The subscript is dropped when obvious from the context.

**Lemma 3.2.** The following rules are admissible in  $\mathcal{I}_{\omega}$  and  $\mathcal{I}_{0}$ .

$$P_e: \frac{\Gamma \vdash t_1 \supset t_2 \qquad \Gamma \vdash t_1}{\Gamma \vdash t_2} \qquad P_i: \frac{\Gamma, t_1 \vdash t_2 \qquad \Gamma \vdash Ht_1}{\Gamma \vdash t_1 \supset t_2}$$

$$P_H: \frac{\Gamma, t_1 \vdash Ht_2 \qquad \Gamma \vdash Ht_1}{\Gamma \vdash H(t_1 \supset t_2)} \qquad \qquad \text{Weak}: \frac{\Gamma \vdash t}{\Gamma, t' \vdash t}$$

Proof. Routine.

**Definition 3.3.** A combinatory algebra C is a tuple  $\langle C, \cdot, S, K \rangle$ , where  $\cdot$  is a binary operation in C and  $S, K \in C$ , such that for any  $X, Y, Z \in C$  we have:

- $S \cdot X \cdot Y \cdot Z = (X \cdot Z) \cdot (Y \cdot Z)$ ,
- $\bullet \ K \cdot X \cdot Y = X.$

To save on notation we often write  $X \in \mathcal{C}$  instead of  $X \in \mathcal{C}$ . We assume  $\cdot$  associates to the left, and sometimes omit it.

A combinatory algebra is *extensional* if for any  $M_1, M_2 \in \mathcal{C}$ , whenever for all  $X \in \mathcal{C}$  we have  $M_1X = M_2X$ , then we also have  $M_1 = M_2$ .

It is well-known that any combinatory algebra contains a fixed-point combinator and satisfies the principle of combinatory abstraction, so any equation of the form  $z \cdot x = \Phi(z, x)$ , where  $\Phi(z, x)$  is an expression involving the variables z, x and some elements of  $\mathcal{C}$ , has a solution for z satisfying this equation for arbitrary x.

**Definition 3.4.** An *illative Kripke pre-model* for an illative system  $\mathcal{I}$  ( $\mathcal{I} \in \{\mathcal{I}_{\omega}, \mathcal{I}_{\omega}^{c}, \mathcal{I}_{0}\}$ ) with primitive constants  $\Sigma$ , is a tuple  $\langle \mathcal{S}, \leq, \mathcal{C}, I, \varsigma \rangle$ , where  $\mathcal{S}$  is a set of states,  $\leq$  is a partial order on the states,  $\mathcal{C}$  is an extensional combinatory algebra,  $I: \Sigma \to \mathcal{C}$  is an interpretation of primitive constants, and  $\varsigma$  is a function assigning upward-closed (w.r.t.  $\leq$ ) subsets of  $\mathcal{S}$  to elements of  $\mathcal{C}$ . We sometimes write  $\sigma_{\mathcal{M}}, \mathcal{S}_{\mathcal{M}}$ , etc., to stress that they are components of  $\mathcal{M}$ .

Given an illative Kripke pre-model  $\mathcal{M}$ , the value  $[\![t]\!]_{\mathcal{M}}^u$  of term t under valuation u, which is a function from variables to  $\mathcal{C}$ , is defined inductively:

- $[x]^u = u(x)$  for a variable x,
- $[c]^u = I(c)$  for a constant c,
- $[t_1t_2]^u = [t_1]^u \cdot [t_2]^u$ ,
- $[\![\lambda x.t]\!]^u$  is the element  $d \in \mathcal{C}$  satisfying  $d \cdot d' = [\![t]\!]^{u'}$  for any  $d' \in \mathcal{C}$ , where u' = u[x/d'].

Note that the element in the last point is uniquely defined because of extensionality and combinatorial completeness of C.

To save on notation, we often confuse  $\Xi$ , L, etc. with  $\llbracket\Xi\rrbracket_{\mathcal{M}}^u$ ,  $\llbracket L\rrbracket_{\mathcal{M}}^u$ , etc. The intended meaning is always clear from the context. The subscript  $\mathcal{M}$  is also often dropped.

Intuitively, for  $X \in \mathcal{C}$  the set  $\varsigma(X)$  is the set of all states s such that the element X is true in s. The relation  $\leq$  on states is analogous to an accessibility relation in a Kripke frame.

An illative Kripke model for  $\mathcal{I}_{\omega}$  is an illative Kripke pre-model where  $\varsigma$  satisfies the following conditions for any  $X,Y\in\mathcal{C}$ :

- (1) if  $s \in \varsigma(LX)$  and for all  $s' \geq s$  and all  $Z \in \mathcal{C}$  such that  $s' \in \varsigma(XZ)$  we have  $s' \in \varsigma(YZ)$ , then  $s \in \varsigma(\Xi XY)$ ,
- (2) if  $s \in \varsigma(\Xi XY)$  then for all  $Z \in \mathcal{C}$  such that  $s \in \varsigma(XZ)$  we have  $s \in \varsigma(YZ)$ ,
- (3) if  $s \in \varsigma(LX)$  and for all  $s' \geq s$  and all  $Z \in \mathcal{C}$  such that  $s' \in \varsigma(XZ)$  we have  $s' \in \varsigma(H(YZ))$ , then  $s \in \varsigma(H(\Xi XY))$ ,

- (4) if  $s \in \varsigma(LX)$  and for all  $s' \geq s$  such that  $s' \in \varsigma(XZ)$  for some  $Z \in \mathcal{C}$ , we have  $s' \in \varsigma(LY)$ , then  $s \in \varsigma(L(FXY))$ ,
- (5) if  $s \in \varsigma(X)$  then  $s \in \varsigma(HX)$ ,
- (6)  $s \in \varsigma(LH)$ ,
- (7)  $s \in \varsigma(LA_{\tau})$  for  $\tau \in \mathcal{B}$ .

An illative Kripke model for  $\mathcal{I}_0$  is defined analogously, but omitting condition (4). A model is a classical illative model if it satisfies the law of double negation: if  $s \in \varsigma(HX)$  and  $s \in \varsigma((X \supset \bot) \supset \bot)$  then  $s \in \varsigma(X)$ , where  $\bot = \Xi HI$ . It is not difficult to see that every one-state illative Kripke model is a classical illative model. For a classical illative model with a single state s we define the set  $\mathscr T$  of true elements by  $\mathscr T = \{X \in \mathcal C \mid s \in \varsigma(X)\}$ . Note that  $\varsigma(X)$  may be empty.

For a term t and a valuation u, we write  $s, u \Vdash_{\mathcal{M}} t$  whenever  $s \in \varsigma(\llbracket t \rrbracket_{\mathcal{M}}^u)$ . For a set of terms  $\Gamma$ , we write  $\Gamma \Vdash_{\mathcal{I}} t$  if for all Kripke models  $\mathcal{M}$  of an illative system  $\mathcal{I}$ , all states s of  $\mathcal{M}$ , and all valuations u such that  $s, u \Vdash_{\mathcal{M}} t'$  for all  $t' \in \Gamma$ , we have  $s, u \Vdash_{\mathcal{M}} t$ . Note that  $s, u \Vdash_{\mathcal{M}} t$  implies  $s', u \Vdash_{\mathcal{M}} t$  for  $s' \geq s$ , because  $\varsigma(X)$  is always an upward-closed subset of  $\mathcal{S}$ , for any argument X.

Informally, one may think of illative Kripke models as combinatory algebras with an added structure of a Kripke frame.

**Fact 3.5.** In any illative Kripke model the following conditions are satisfied:

- (1) if  $s \in \varsigma(HX)$  and for all  $s' \geq s$  such that  $s' \in \varsigma(X)$  we have  $s' \in \varsigma(Y)$ , then  $s \in \varsigma(X \supset Y)$ ,
- (2) if  $s \in \varsigma(X \supset Y)$  then  $s \in \varsigma(X)$  implies  $s \in \varsigma(Y)$ ,
- (3) if  $s \in \varsigma(HX)$  and for all  $s' \geq s$  such that  $s' \in \varsigma(X)$  we have  $s' \in \varsigma(HY)$ , then  $s \in \varsigma(H(X \supset Y))$ .

**Theorem 3.6.** The conditions  $\Gamma \Vdash_{\mathcal{I}} t$  and  $\Gamma \vdash_{\mathcal{I}} t$  are equivalent, where  $\mathcal{I} = \mathcal{I}_{\omega}$  or  $\mathcal{I} = \mathcal{I}_{0}$ .

Proof. We first check that  $\Gamma \vdash_{\mathcal{I}} t$  implies  $\Gamma \Vdash_{\mathcal{I}} t$ , by a simple induction on the length of derivation. It suffices to prove this for finite  $\Gamma$ . The implication is immediate for the axioms. Now assume  $\Gamma \vdash t_2t$  was obtained by rule  $\Xi_e$ , and we have  $s, u \Vdash_{\mathcal{M}} \Gamma$ . Hence, by the inductive hypothesis  $s, u \Vdash_{\mathcal{M}} \Xi t_1 t_2$  and  $s, u \Vdash_{\mathcal{M}} t_1 t$ , which by condition (2) in Definition 3.4 implies  $s, u \Vdash_{\mathcal{M}} t_2 t$ . Assume  $\Gamma \vdash \Xi t_1 t_2$  was obtained by rule  $\Xi_i$ , and that  $s, u \Vdash_{\mathcal{M}} \Gamma$ . Let  $s' \geq s$  and  $Z \in \mathcal{C}$  be such that  $s' \in \varsigma(\llbracket t_1 \rrbracket_{\mathcal{M}}^u \cdot Z)$ . We therefore have  $s', u' \Vdash_{\mathcal{M}} \Gamma, t_1 x$ , where u' = u[x/Z] and  $x \notin FV(\Gamma, t_1, t_2)$ . So by the inductive hypothesis we obtain  $s', u' \Vdash_{\mathcal{M}} t_2 x$ . Because  $x \notin FV(t_2)$ , this is equivalent to  $s' \in \varsigma(\llbracket t_2 \rrbracket_{\mathcal{M}}^u \cdot Z)$ . The inductive hypothesis implies also that  $s \in \varsigma(L \cdot \llbracket t_1 \rrbracket_{\mathcal{M}}^u)$ . We therefore obtain by condition (1) in Definition 3.4 that  $s, u \Vdash_{\mathcal{M}} \Xi t_1 t_2$ . The other cases are equally straightforward and we leave them to the reader. In the case of rule Eq the extensionality of  $\mathcal{C}$  is needed.

To prove the other direction, we assume  $\Gamma_0 \nvDash_{\mathcal{I}} t_0$ , and construct an illative Kripke model  $\mathcal{M}$  and a valuation u such that for some state s of  $\mathcal{M}$  we have  $s, u \Vdash_{\mathcal{M}} \Gamma_0$ , but  $s, u \nvDash_{\mathcal{M}} t_0$ .

We construct the model as follows. First of all, we assume without loss of generality that there are infinitely many variables not occuring in  $\Gamma_0$ . As states we take all sets of terms  $\Gamma' \supseteq \Gamma_0$  which extend  $\Gamma_0$  by only finitely many formulas. The ordering is by inclusion. The combinatory algebra  $\mathcal{C}$  is the set of equivalence classes of  $\beta\eta$ -equality on  $\mathbb{T}(\Sigma)$ . We denote the equivalence class of a term t by  $[t]_{\beta\eta}$ . We define  $I(c) = [c]_{\beta\eta}$  for  $c \in \Sigma$ . The function  $\varsigma$  is defined by the condition:  $\Gamma \in \varsigma([t]_{\beta\eta})$  iff  $\Gamma \vdash_{\mathcal{I}} t$  and  $\Gamma$  is a valid state. This is well-defined because of  $\beta\eta$ -equality in rule Eq. The valuation u is defined by  $u(x) = [x]_{\beta\eta}$ . Note that  $[t]_{\mathcal{M}}^{\parallel u} = [t]_{\beta\eta}$ .

We now show that this is an illative Kripke model. We only need to check the conditions on  $\varsigma$ . It is obvious that  $\varsigma(X)$  is upward-closed for any  $X \in \mathcal{C}$  because of weakening. Assume that  $\Gamma \vdash Lt_1$ , and for all  $\Gamma' \supseteq \Gamma$  and all terms  $t_3$  such that  $\Gamma' \vdash t_1t_3$  we have  $\Gamma' \vdash t_2t_3$ . Then, in particular, this holds for  $\Gamma' = \Gamma \cup \{t_1x\}$  and  $t_3 = x$ , where x is a variable,  $x \notin FV(\Gamma, t_1, t_2)$ . Such a variable x exists because  $\Gamma$  differs from  $\Gamma_0$  by only finitely many formulas, and there are infinitely many variables not occuring in the formulas of  $\Gamma_0$ . Therefore, by rule  $\Xi_i$  we have  $\Gamma \vdash \Xi t_1t_2$ , hence  $\Gamma \in \varsigma([\Xi t_1t_2]_{\beta\eta})$ . This verifies condition (1). Conditions (2), (3), (4) and (5) are verified in a similar manner, using rules  $\Xi_e$ ,  $\Xi_H$ ,  $F_L$  and  $H_i$ , respectively. Condition (6) is immediate from the axiom  $\Gamma \vdash LH$ . Condition (7) follows from the axioms  $\Gamma \vdash LA_{\tau}$  for  $\tau \in \mathcal{B}$ .

It is obvious that  $\Gamma_0, u \nvDash_{\mathcal{M}} t_0$ , i.e.,  $\Gamma_0 \notin \varsigma([t_0]_{\beta\eta})$ , because  $\Gamma_0 \nvDash_{\mathcal{I}} t_0$ . Clearly, we also have  $\Gamma_0, u \Vdash_{\mathcal{M}} t$  for all  $t \in \Gamma_0$ . This proves the theorem.

Remark 3.7. Note one subtlety here. The above theorem does not imply that  $\mathcal{I}_0$  or  $\mathcal{I}_\omega$  is consistent. This is because we allow trivial Kripke models, i.e., ones such that  $\varsigma(X) = \mathcal{S}$  for any  $X \in \mathcal{C}$ , and it is not obvious that nontrivial ones exist. Indeed, if we dropped the restriction  $s \in \varsigma(LX)$  in condition (1) in Definition 3.4, then all illative Kripke models would be trivial. To see this, let  $X \in \mathcal{C}$  and  $s \in \mathcal{S}$  be arbitrary and consider the element  $\Upsilon \in \mathcal{C}$  defined by the equation  $\Upsilon = \Upsilon \supset X$ . Note that dropping  $s \in \varsigma(LX)$  in condition (1) in Definition 3.4 means dropping  $s \in \varsigma(HX)$  in condition (1) in Fact 3.5. For any  $s' \geq s$  we obviously have  $s' \in \varsigma(\Upsilon \supset X)$  whenever  $s' \in \varsigma(\Upsilon)$ . By condition (2) in Fact 3.5 we conclude that  $s' \in \varsigma(X)$  whenever  $s' \in \varsigma(\Upsilon)$ . Therefore, by condition (1) in Fact 3.5, we have  $s \in \varsigma(\Upsilon)$ . Hence,  $s \in \varsigma(\Upsilon \supset X)$  as well, so again  $s \in \varsigma(X)$ . Thus  $\varsigma(X) = \mathcal{S}$ . This argument is essentially Curry's paradox.

For convenience of reference we state the following simple fact about one-state classical illative models for  $\mathcal{I}_{\omega}^{c}$ , as we will be constructing such a model in the next section. Recall that for a classical illative model with a single state s, the set  $\mathscr{T}$  of true elements is defined by  $\mathscr{T} = \{X \in \mathcal{C} \mid s \in \varsigma(X)\}.$ 

**Fact 3.8.** For a one-state classical illative model for  $\mathcal{I}^c_{\omega}$  the conditions on  $\varsigma$  may be reformulated as follows:

- (1) if  $LX \in \mathcal{T}$  and for all  $Z \in \mathcal{C}$  such that  $XZ \in \mathcal{T}$  we have  $YZ \in \mathcal{T}$ , then  $\Xi XY \in \mathcal{T}$ ,
- (2) if  $\exists XY \in \mathcal{T}$  then for all  $Z \in \mathcal{C}$  such that  $XZ \in \mathcal{T}$  we have  $YZ \in \mathcal{T}$ ,
- (3) if  $LX \in \mathcal{T}$  and for all  $Z \in \mathcal{C}$  such that  $XZ \in \mathcal{T}$  we have  $H(YZ) \in \mathcal{T}$ , then  $H(\Xi XY) \in \mathcal{T}$ ,
- (4) if  $LX \in \mathcal{T}$ , and either  $LY \in \mathcal{T}$  or there is no  $Z \in \mathcal{C}$  such that  $XZ \in \mathcal{T}$ , then  $L(FXY) \in \mathcal{T}$ ,
- (5) if  $X \in \mathcal{T}$  then  $HX \in \mathcal{T}$ ,
- (6)  $LH \in \mathcal{T}$ ,
- (7)  $LA_{\tau} \in \mathcal{T} \text{ for } \tau \in \mathcal{B}.$

#### 4 The model construction

In this section we construct a model for  $\mathcal{I}_{\omega}^{c}$ . The construction is parametrized by a full model for classical higher-order logic.

#### 4.1 Definitions

In this subsection we give definitions necessary for the construction and fix some notational conventions.

**Definition 4.1.1.** We define the set of types  $\mathcal{T}^+$  by the following grammar:

$$\begin{array}{lll} \mathcal{T}^{+} & ::= & \mathcal{T}_{1} \mid \omega \mid \epsilon \\ \\ \mathcal{T}_{1} & ::= & \mathcal{T} \mid \mathcal{T}_{1} \rightarrow \mathcal{T}_{1} \mid \omega \rightarrow \mathcal{T}_{1} \\ \\ \mathcal{T} & ::= & o \mid \mathcal{B} \mid \mathcal{T} \rightarrow \mathcal{T} \end{array}$$

where  $\mathcal{B}$  is a specific finite set of base types. Intuitively, the type o is the type of propositions,  $\omega$  is the type of arbitrary objects,  $\varepsilon$  is the empty type.

For the sake of simplicity we use the following notational convention: we sometimes write  $\tau \to \varepsilon$  for  $\varepsilon$  when  $\tau \neq \varepsilon$ ,  $\varepsilon \to \tau$  for  $\omega$ , and  $\tau \to \omega$  for  $\omega$ . There is never any ambiguity because  $\tau \to \varepsilon$  etc. are not valid types according to the grammar for  $\mathcal{T}^+$ . This convention is only to shorten some statements later on. We also use the abbreviation  $\tau_1^n \to \tau_2$  for  $\tau_1 \to \dots \to \tau_1 \to \tau_2$  where  $\tau_1$  occurs n times (possibly n = 0).

From now on we fix a full model  $\mathcal{N} = \langle \{\mathcal{D}_{\tau} \mid \tau \in \mathcal{T}\}, I \rangle$  of classical higher-order logic and construct a one-state classical illative model  $\mathcal{M}$  for  $\mathcal{I}_{\omega}^{c}$ . We assume that  $\mathcal{T} \subset \mathcal{T}^{+}$  defined above corresponds exactly to the types of  $\mathcal{N}$ , and that the base types  $\mathcal{B}$  correspond exactly to the base types used in the definition of the syntax of  $\mathcal{I}_{\omega}^{c}$ .

We will define the universe of the model as the set of equivalence classes of a certain relation on the set of type-free lambda-terms over a set  $\Sigma^+$  of primitive constants, to be defined below. We assume these terms to be different objects than the terms of the syntax of  $\mathcal{I}^c_{\omega}$ . We also treat lambda-terms up to  $\alpha$ -equivalence, i.e., terms differing only in the names of bound variables are considered identical.

**Definition 4.1.2.** We define a set of primitive constants  $\Sigma^+$ , and a set of canonical terms as follows. First, for every type  $\tau \neq \omega$  we define by induction on the size of  $\tau$  a set of canonical terms of type  $\tau$ , denoted by  $\mathbb{T}_{\tau}$ . We also define a set of constants  $\Sigma_{\tau}$  for every type  $\tau \notin \{\omega, \varepsilon\} \cup \{\omega \to \tau' \mid \tau' \in \mathcal{T}^+\}$ , i.e., we leave  $\Sigma_{\tau}$  undefined if  $\tau$  is not of the form required. First, we set  $\mathbb{T}_{\varepsilon} = \emptyset$ . In the inductive step we consider possible forms of  $\tau$ . If  $\tau \in \mathcal{T}$  (i.e. it does not contain  $\omega$  or  $\varepsilon$ ) then we define  $\Sigma_{\tau}$  to contain a unique constant for every element  $d \in \mathcal{D}_{\tau}$ . We set  $\mathbb{T}_{\tau} = \Sigma_{\tau}$ . If  $\tau \notin \mathcal{T}$ ,  $\tau = \tau_1 \to \tau_2$  and  $\tau_1 \neq \omega$ , then denote by  $\Sigma_{\tau}$  a set of new constants for every (set-theoretical) function from  $\mathbb{T}_{\tau_1}$  to  $\mathbb{T}_{\tau_2}$ . Again we set  $\mathbb{T}_{\tau} = \Sigma_{\tau}$ . If  $\tau = \omega \to \tau_2$  then  $\mathbb{T}_{\tau}$  consists of all terms of the form  $\lambda x.\rho$  where  $\rho \in \mathbb{T}_{\tau_2}$ .

The symbol  $\Sigma^A$  stands for a set consisting of distinct new constants  $A_{\tau}$  for each base type  $\tau \in \mathcal{B}$ . Finally, we set  $\Sigma^+ = \{\Xi, L\} \cup \Sigma^A \cup \bigcup_{\tau} \Sigma_{\tau}$  where the index in the sum ranges over  $\tau \notin \{\omega, \varepsilon\} \cup \{\omega \to \tau' \mid \tau' \in \mathcal{T}^+\}$ . For the sake of uniformity, we use the notation  $\mathbb{T}_{\omega}$  for the set of all type-free lambda terms over  $\Sigma^+$ . Note that terms in  $\mathbb{T}_{\omega}$  are not necessarily canonical and all canonical terms are closed.

Note that for  $\tau \in \mathcal{T}$  the set  $\Sigma_{\tau}$  contains a unique constant for every element of  $\mathcal{D}_{\tau}$ . Hence for each  $\tau \in \mathcal{T}$  there is a natural bijection from  $\Sigma_{\tau}$  onto  $\mathcal{D}_{\tau}$ . We denote this bijection by  $\pi_{\tau}$ .

We now define a mapping  $\mathcal{F}$  such that for  $\rho \in \mathbb{T}_{\tau_1 \to \tau_2}$  we have  $\mathcal{F}(\rho) : \mathbb{T}_{\tau_1} \to \mathbb{T}_{\tau_2}$ , where  $\tau_1 \to \tau_2 \in \mathcal{T}_1$ . If  $\tau_1 \to \tau_2 \in \mathcal{T}$  then  $\tau_1, \tau_2 \in \mathcal{T}$ ,  $\mathbb{T}_{\tau_1} = \Sigma_{\tau_1}$ , and both  $\pi_{\tau_1}$  and  $\pi_{\tau_2}$  are defined. In this case we set  $\mathcal{F}(c)(c_1) = \pi_{\tau_2}^{-1}(\pi_{\tau_1 \to \tau_2}(c)(\pi_{\tau_1}(c_1)))$  for  $c \in \Sigma_{\tau_1 \to \tau_2}$ ,  $c_1 \in \Sigma_{\tau_1}$ . If  $\tau_1 \to \tau_2 \notin \mathcal{T}$  and  $\tau_1 \neq \omega$  then also  $\mathbb{T}_{\tau_1 \to \tau_2} = \Sigma_{\tau_1 \to \tau_2}$  and by our construction to each  $c \in \Sigma_{\tau_1 \to \tau_2}$  corresponds a set-theoretical function  $f_c$  from  $\mathbb{T}_{\tau_1}$  to  $\mathbb{T}_{\tau_2}$ . In this case we set  $\mathcal{F}(c) = f_c$ . Finally, if  $\rho \in \mathbb{T}_{\omega \to \tau}$  then  $\rho = \lambda x.\rho'$  and by  $\mathcal{F}(\rho)$  we denote the constant function from  $\mathbb{T}_{\omega}$  to  $\mathbb{T}_{\tau}$  whose value is always  $\rho'$ . Note that because  $\mathcal{N}$  is assumed to be a full model, so by our construction if  $\tau_1 \to \tau_2 \in \mathcal{T}_1$  and  $\tau_1 \neq \omega$  then for every set-theoretical function f from  $\mathbb{T}_{\tau_1}$  to  $\mathbb{T}_{\tau_2}$  there exists a constant  $\rho_f \in \Sigma_{\tau_1 \to \tau_2}$  such that  $\mathcal{F}(\rho_f) = f$ .

By  $\top \in \Sigma_o$  we denote the constant corresponding to the element  $\top \in \mathcal{D}_o$ , by  $\bot \in \Sigma_o$  the one corresponding to  $\bot \in \mathcal{D}_o$ . Note that  $\Sigma_o = \{\top, \bot\}$ , because  $\mathcal{D}_o = \{\top, \bot\}$ .

Note that if  $\tau_1, \tau_2 \neq \omega$  and  $\tau_1 \neq \tau_2$  then  $\mathbb{T}_{\tau_1} \cap \mathbb{T}_{\tau_2} = \emptyset$ . Hence every canonical term  $\rho$  may be assigned a unique type  $\tau \neq \omega$  such that  $\rho \in \mathbb{T}_{\tau}$ . When talking about the *canonical type*, or simply *the type*, of a canonical term we mean the type thus defined.

An n-ary context C is a lambda-term over the set of constants  $\Sigma^+ \cup \{\Box_1, \ldots, \Box_n\}$ , where  $\Box_1, \ldots, \Box_n \notin \Sigma^+$ . The constants  $\Box_1, \ldots, \Box_n$  are the boxes of C. If C is an n-ary context then by  $C[t_1, \ldots, t_n]$  we denote the term C with all occurrences of  $\Box_i$  replaced with  $t_i$  for  $i = 1, \ldots, n$ . Unless otherwise stated, we assume that the free variables of  $t_1, \ldots, t_n$  do not become bound in  $C[t_1, \ldots, t_n]$ . By a context we usually mean a unary context, unless otherwise qualified. In this case we write  $\Box$  instead of  $\Box_1$ .

In what follows  $\alpha$ ,  $\beta$ , etc. stand for ordinals; t,  $t_1$ ,  $t_2$ , r,  $r_1$ ,  $r_2$ , q,  $q_1$ ,  $q_2$  etc. stand for type-free lambda-terms over  $\Sigma^+$  from which we build the model; c,  $c_1$ ,  $c_2$ , etc. stand for constants from  $\Sigma^+$ ;  $\tau$ ,  $\tau_1$ ,  $\tau_2$ , etc. stand for types;  $\rho$ ,  $\rho_1$ ,  $\rho_2$  stand for canonical terms (i.e. terms  $\rho \in \mathbb{T}_{\tau}$  for  $\tau \neq \omega$ ); and C, C',  $C_1$ ,  $C_2$ , etc. denote contexts; unless otherwise qualified.

The following simple fact states some easy properties of canonical terms. It will sometimes be used implicitly in what follows.

### **Fact 4.1.3.** If $\rho$ is a canonical term then:

- (1)  $\rho \equiv \lambda x_1 \dots x_n.c$  where  $n \geq 0$ ,  $c \in \Sigma_{\tau}$  for some  $\tau$  (so  $\tau \neq \omega \to \tau_1$ ), and  $\rho \in \mathbb{T}_{\omega^n \to \tau}$ ,
- (2) if  $\rho \equiv C[t]$  then either  $C \equiv \lambda x_1 \dots x_k \square$  and t is a canonical term, or  $C \equiv \rho$ .

<sup>&</sup>lt;sup>4</sup>Formally, terms are  $\alpha$ -equivalence classes of certain strings, i.e., by  $\lambda x.\rho$  we mean the  $\alpha$ -equivalence class of the string " $\lambda x.\rho$ ", so e.g.  $\lambda x.\rho$  and  $\lambda y.\rho$  are the same, which we denote by  $\lambda x.\rho \equiv \lambda y.\rho$ .

For each ordinal  $\alpha$  we inductively define reduction systems  $R_{\alpha}$  and  $\widehat{R_{\alpha}}$ , a relation  $\sim_{\alpha}$  between terms and types in  $\mathcal{T}^+$ , and a relation  $\succ_{\alpha}$  between terms and canonical terms. Formally, all these notions are defined by one induction in a mutually recursive way, but we split up the definitions for the sake of readability. These definitions are monotone with respect to  $\alpha$ , so the induction closes at some ordinal, i.e., the relations do not get larger after this ordinal.

First, let us fix some notations. We write  $R_{<\alpha}$  for  $\bigcup_{\beta<\alpha} R_{\beta}$ ,  $\succ_{<\alpha}$  for  $\bigcup_{\beta<\alpha} \succ_{\beta}$ ,  $\sim_{<\alpha}$  for  $\bigcup_{\beta<\alpha} \sim_{\beta}$ . We use the notation  $\equiv$  for identity of terms up to  $\alpha$ -equivalence. By  $\rightarrow_{\leq \alpha}$  we denote the reduction relation of  $R_{\alpha}$ , by  $\rightarrow_{\leq \alpha}^{\equiv}$  the reflexive closure of  $\rightarrow_{\leq \alpha}$ , by  $\rightarrow_{\leq \alpha}$  the transitive reflexive closure of  $\rightarrow_{\leq \alpha}$ , and by  $=_{\leq \alpha}$  the transitive reflexive symmetric closure. We write  $[t]_{\alpha}$  for the equivalence class of a term t w.r.t. the relation  $=_{\leq \alpha}$ . Analogously, we use the subscript  $<_{\alpha}$  for relations corresponding to  $R_{<\alpha}$ , and  $=_{\alpha}$  for relations corresponding to  $R_{<\alpha}$ . We drop the subscripts when they are obvious or irrelevant.

**Notation 4.1.4.** In what follows a term of the form Kt should be read as  $\lambda x.t$  where  $x \notin FV(t)$ , a term Ht as  $L\lambda x.t$  where  $x \notin FV(t)$ , and  $Ft_1t_2$  as  $\lambda f.\Xi t_1(\lambda x.t_2(fx))$ . We adopt this convention to shorten notations.

Before embarking on the task of rigorously constructing the model we explain the intuitive meaning of various notions formally introduced later. This is necessarily informal and at points rather vague.

Informally speaking, we identify types with sets of terms. A base type corresponds to the set of all constants of this type, the type o to the set of all propositions, the type o to the set of all terms, the type o to the empty set, and a function type o to the set of all terms o type o to the empty set, and a function type o to the set of all terms o type o that for all terms o type o to the transfinite inductive definition exactly which terms have base types, but not so for type o or function types. During the course of the induction new terms may obtain types. If o is a term, and o an ordinal, then by o in the induction, o has been shown to be "true". If o if o if o in the induction, the term o has been shown to have type o in the induction, the term o has been shown to have type o in the induction of the induction. Our induction stops when no new typings may be obtained and no new terms may become true or false, i.e., when we have all information we need to construct the model.

Note that canonical terms may obtain types different from their canonical types. For instance, a term of the form  $\lambda x.c$  where  $c \in \Sigma_{\tau}$  will ultimately obtain the type  $\omega$  and all of the types  $\tau' \to \tau$  for any type  $\tau'$ . As far as canonical terms are concerned, we mostly care about their canonical types, and it is known beforehand what types these are.

In  $R_{\alpha}$  we will have reduction rules of  $\beta$ - and  $\eta$ -reduction, and rules of the form  $c\rho \to \mathcal{F}(c)(\rho)$ , where  $c \in \Sigma_{\tau_1 \to \tau_2}$  and  $\rho \in \mathbb{T}_{\tau_1}$ . We will also add some other rules to make certain terms "indistinguishable", as explained in the paragraph below.

Intuitively,  $t \succ_{\alpha} \rho$  is intended to hold if  $\rho \in \mathbb{T}_{\tau}$  is a "canonical" term which is "equivalent" to t in type  $\tau$ , basing on the information we have at stage  $\alpha$ . Let us give some examples to elucidate what we mean by this. For instance, suppose we have two distinct (hence disjoint) base types  $\tau_1$  and  $\tau_2$ , and two functions  $\mathrm{Id}_{\tau_1 \to \tau_1} \in \mathcal{D}_{\tau_1 \to \tau_1}$  and  $\mathrm{Id}_{\tau_2 \to \tau_2} \in \mathcal{D}_{\tau_2 \to \tau_2}$  which are identities on  $\mathcal{D}_{\tau_1}$  and  $\mathcal{D}_{\tau_2}$  respectively. In  $\Sigma^+$  we will have two canonical constants  $\mathrm{id}_{\tau_1 \to \tau_1}$  and  $\mathrm{id}_{\tau_2 \to \tau_2}$  of type  $\tau_1 \to \tau_1$  and  $\tau_2 \to \tau_2$  respectively, associated with the functions  $\mathrm{Id}_{\tau_1 \to \tau_1}$  and  $\mathrm{Id}_{\tau_2 \to \tau_2}$ , i.e., such that  $\mathcal{F}(\mathrm{id}_{\tau_1 \to \tau_1}) = \pi_{\tau_1}^{-1} \circ \mathrm{Id}_{\tau_1 \to \tau_1} \circ \pi_{\tau_1}$  and  $\mathcal{F}(\mathrm{id}_{\tau_2 \to \tau_2}) = \pi_{\tau_2}^{-1} \circ \mathrm{Id}_{\tau_2 \to \tau_2} \circ \pi_{\tau_2}$ . The reduction rules associated with  $\mathrm{id}_{\tau_1 \to \tau_1}$  will be  $\mathrm{id}_{\tau_1 \to \tau_1} c \to c$  for every canonical constant c of type  $\tau_1$ , and analogously for  $\mathrm{id}_{\tau_2 \to \tau_2}$ . Note that  $\mathrm{id}_{\tau_1 \to \tau_1} c$  will not form a redex if c is a canonical constant of type different from  $\tau_1$ . Now we have both  $\lambda x.x \succ_1 \mathrm{id}_{\tau_1 \to \tau_1}$  and  $\lambda x.x \succ_1 \mathrm{id}_{\tau_2 \to \tau_2}$ , because  $\lambda x.x$  behaves exactly like  $\mathrm{id}_{\tau_1 \to \tau_1}$  when given arguments of type  $\tau_1$ , and exactly like  $\mathrm{id}_{\tau_2 \to \tau_2}$  when given arguments of type  $\tau_2$ . In fact, we will define the reduction systems  $R_{\alpha}$  so as to make  $\lambda x.x$  and  $\mathrm{id}_{\tau_1 \to \tau_1}$  indistinguishable, for sufficiently large  $\alpha$ , wherever a term of type  $\tau_1 \to \tau_1$  is "expected". For instance, for any reduction rule in  $R_{\alpha}$  of the form  $\rho$  id  $\tau_1 \to \tau_1 \to c$ , where  $\rho$  is a canonical term of type  $(\tau_1 \to \tau_1) \to \tau_1$  for some  $\tau$ , we will add to  $R_{\alpha+1}$  a reduction rule  $\rho$  ( $\lambda x.x$ )  $\to c$ .

In the case  $\rho \in \{\top, \bot\}$ , the relation  $t \succ_{\alpha} \rho$  encompasses a definition of truth. The condition  $t \succ_{\alpha} \top$  means that t is certainly true, basing on the information from the earlier stages  $\beta < \alpha$  of the inductive definition. So if  $t \succ_{\alpha} \top$  then t should behave like  $\top$  wherever a truth-value is expected. If  $t \succ_{\alpha} \bot$ , then t is certainly not true.

If  $t \neq \rho$  then we never have  $t \succ_{\alpha} \rho$  for a canonical term  $\rho$  of some base type  $\tau \in \mathcal{B}$ , because no term different from  $\rho$  behaves like  $\rho$  if the type of  $\rho$  is an atomic type different from  $\rho$ .

**Notation 4.1.5.** We use the notation  $t \leadsto_{\alpha} \rho$  when  $t \twoheadrightarrow_{\leq \alpha} t' \succ_{\alpha} \rho$ . We write  $\leadsto_{<\alpha}$  for  $\bigcup_{\beta < \alpha} \leadsto_{\beta} t'$ 

Informally,  $t \leadsto_{\alpha} \rho$  holds if we can reduce t, using the rules of  $R_{\alpha}$ , to a term equivalent to a canonical term  $\rho$  in the type of  $\rho$  basing on what we know at stage  $\alpha$  of the inductive definition. A careful reader will notice that what we ultimately really care about is the relation  $\leadsto_{\alpha}$ , not  $\succ_{\alpha}$ , because we want to identify  $R_{\alpha}$ -equivalent terms. The relation  $\succ_{\alpha}$  is needed chiefly to facilitate the proofs.

The condition  $t \sim_{\alpha} \tau$  is intended to hold if t "represents" the type  $\tau$  basing on what we know at stage  $\alpha$ , i.e., it is a "predicate" which is true when applied to terms of type  $\tau$ , and is never true when applied to terms which are not of type  $\tau$ . In other words,  $Lt \leadsto_{\alpha} \top$  and for all terms r known to be of type  $\tau$  we have  $tr \leadsto_{\alpha} \top$ , but we should not have  $tr \leadsto_{\alpha} \top$  for any r which is not of type  $\tau$ . So for instance for each type  $\tau \in \mathcal{B}$  we should have  $A_{\tau} \leadsto_{\alpha} \tau$  for sufficiently large  $\alpha$ . Because  $\varepsilon$  is the empty type, if  $t \leadsto_{\alpha} \varepsilon$  then we should never have  $tr \leadsto_{<\alpha} \top$  for any term r. Since  $\omega$  is the type of arbitrary objects we should have  $t \leadsto_{\alpha} \omega$  if for all terms r we have  $tr \leadsto_{<\alpha} \top$ .

Having explained the intuitive meaning of the relations, we may proceed to formal definitions. The definition below depends on the definition of  $\succ_{<\alpha}$ , and thus on  $\succ_{\beta}$  for  $\beta < \alpha$ .

**Definition 4.1.6.** A reduction system is a set of reduction rules over a specified set of terms, i.e., a set of pairs of terms. In all reduction systems we consider we assume the set of terms to be the type-free lambda-terms over  $\Sigma^+$ . Instead of writing  $\langle t_1, t_2 \rangle \in R$  we usually say that  $t_1 \to t_2$  is a reduction rule of R. Given a reduction system R we define its associated reduction relation  $\to_R$  by:  $t_1 \to_R t_2$  iff there exists a context C with exactly one box and terms  $r_1, r_2$  such that  $t_1 \equiv C[r_1], t_2 \equiv C[r_2]$  and  $r_1 \to r_2$  is a rule of R. In contrast to all subsequent uses of contexts, here we allow the free variables of  $r_1$  and  $r_2$  to become bound in  $C[r_1]$  or  $C[r_2]$ .

We define  $\widehat{R}_{\alpha}$  to contain the following reduction rules:

- for  $\alpha = 0$ : rules of  $\beta$  and  $\eta$ -reduction,
- for  $\alpha > 0$ : rules  $ct \to \rho_2$  for every  $c \in \Sigma_{\tau_1 \to \tau_2}$  (so  $\tau_1 \neq \omega$ ), every  $\rho_2 \in \mathbb{T}_{\tau_2}$  and every term t such that  $t \succ_{<\alpha} \rho_1$  and  $\mathcal{F}(c)(\rho_1) \equiv \rho_2$ .

We set 
$$R_{\alpha} = R_{<\alpha} \cup \widehat{R_{\alpha}}$$
.

**Definition 4.1.7.** The relation  $\sim_{\alpha}$  is defined by the following rules. Recall that  $\tau_1 \to \varepsilon = \varepsilon$  for  $\tau_1 \neq \varepsilon$ ,  $\varepsilon \to \tau_2 = \omega$ , and  $\tau_1 \to \omega = \omega$ .

$$(A): \frac{\tau \in \mathcal{B}}{A_{\tau} \sim_{\alpha} \tau} \qquad (H): \frac{1}{H \sim_{\alpha} \sigma}$$

$$(K\omega): \frac{t \sim_{<\alpha} \top}{Kt \sim_{\alpha} \omega} \qquad (K\varepsilon): \frac{t \sim_{<\alpha} \bot}{Kt \sim_{\alpha} \varepsilon}$$

$$(F): \frac{t_{1} \sim_{<\alpha} \tau_{1}}{Ft_{1}t_{2} \sim_{\alpha} \tau_{1} \rightarrow \tau_{2}}$$

$$(F'): \frac{t_{1} \sim_{<\alpha} \tau_{1}}{\lambda f. \Xi t_{1} (\lambda x. t_{2}[z/fx]) \sim_{\alpha} \tau_{1} \rightarrow \tau_{2}}$$

$$(F''): \frac{t_{1} \sim_{<\alpha} \tau_{1}}{\lambda f. \Xi t_{1}t_{2} \sim_{\alpha} \tau_{2}} \qquad f, x \notin FV(t_{1}, t_{2})}{\lambda f. \Xi t_{1}t_{2} \sim_{\alpha} \tau_{2} \in \{\omega, \varepsilon\}} \qquad f \notin FV(t_{1}, t_{2})$$

$$(F'''): \frac{t_{1} \sim_{<\alpha} \tau_{1}}{\lambda f. \Xi t_{1}t_{2} \sim_{\alpha} \omega} \qquad (F\omega'): \frac{t_{1} \sim_{<\alpha} \varepsilon}{\Xi t_{1} \sim_{\alpha} \omega}$$

The above definition depends on the definitions of  $R_{\beta}$ ,  $\sim_{\beta}$  and  $\succ_{\beta}$  for  $\beta < \alpha$ . The next definition of  $\succ_{\alpha}$  depends on the definitions of  $R_{\beta}$  and  $\sim_{\beta}$  for  $\beta \leq \alpha$ , and on  $\succ_{<\alpha}$ .

**Definition 4.1.8.** We define the relation  $t \succ_{\alpha} \rho$  for canonical terms  $\rho$  by the following conditions:

- $\rho \succ_{\alpha} \rho$  if the canonical type of  $\rho$  is o or a base type,
- $t \succ_{\alpha} \rho$  if the canonical type of  $\rho$  is  $\tau_1 \to \tau_2$  and t is a term such that for any  $t_1 \in \mathbb{T}_{\tau_1}$  we have  $tt_1 \leadsto_{<\alpha} \mathcal{F}(\rho)(t_1)$ . Note that we allow  $\tau_1 = \omega$  but not  $\tau_1 = \varepsilon$ .

In particular,  $\top \succ_{\alpha} \top$  and  $\bot \succ_{\alpha} \bot$  by the above definition. For  $\rho \in \{\top, \bot\}$  we give additional postulates. For  $\alpha \geq 0$  we postulate  $t \succ_{\alpha} \top$  for all terms t such that at least one of the following holds:

- $(A_{\tau}^{\top})$   $t \equiv A_{\tau}c$  where  $\tau \in \mathcal{B}$  and  $c \in \Sigma_{\tau}$ ,
- $(\Xi^{\top})$   $t \equiv \Xi t_1 t_2$  where  $t_1, t_2$  are terms such that there exists  $\tau$  s.t.  $t_1 \sim_{\alpha} \tau$  and for all  $t_3 \in \mathbb{T}_{\tau}$  we have  $t_2 t_3 \leadsto_{<\alpha} \top$ ,
- $(L^{\top})$   $t \equiv Lt_1$  and  $t_1 \sim_{\alpha} \tau$  for some type  $\tau$ .

Finally, when  $\alpha \geq 0$  we postulate  $t \succ_{\alpha} \bot$  for all terms t such that:

- $(\Xi^{\perp})$   $t \equiv \Xi t_1 t_2$  and there exists a type  $\tau$  such that:
  - $t_1 \sim_{\alpha} \tau$ , and
  - for every term  $t_3 \in \mathbb{T}_{\tau}$  we have  $t_2t_3 \rightsquigarrow_{\leq \alpha} \top$  or  $t_2t_3 \rightsquigarrow_{\leq \alpha} \bot$ ,
  - there exists a term  $t_3 \in \mathbb{T}_{\tau}$  with  $t_2 t_3 \leadsto_{\leq \alpha} \bot$ .

The intuitive interpretation of  $\Xi t_1 t_2$  is restricted quantification  $\forall x.t_1 x \supset t_2 x$ , but  $t_1$  is required to represent a type, if  $\Xi t_1 t_2$  is to have a logical value. In illative combinatory logic the notions of being (representing) a type and being eligible to stand as a quantifier range are equivalent. It turns out that the types of  $\mathcal{I}^c_\omega$  are just the types defined by  $\mathcal{T}^+$ . This explains putting  $t_1 \sim_\alpha \tau$  in some of the cases above.

During the course of the transfinite inductive definition some previously untyped terms t will obtain types, e.g. a statement of the form  $FA_{\tau_1}A_{\tau_2}t$  will become true at some stage  $\alpha$ . At that point we need to decide which term among the canonical terms of type  $\tau_1 \to \tau_2$  behaves exactly like t. The whole correctness proof rests on the fact that this decision is always possible. That we may choose such a canonical term implies that quantifying over only canonical terms of a certain type  $\tau$  is equivalent to quantifying over all terms of type  $\tau$ . This justifies restricting quantification to canonical terms in the above definition of  $t \succ_{\alpha} \top$ .

Let us now give some examples illustrating the above definitions.

**Example 4.1.9.** Suppose we have a base type  $\tau$  and  $\widehat{\mathrm{Id}} \in \mathcal{D}_{\tau \to \tau}$  is the identity function on  $\mathcal{D}_{\tau}$ . Let  $\mathrm{Id} = \pi_{\tau}^{-1} \circ \widehat{\mathrm{Id}} \circ \pi_{\tau}$ , i.e.,  $\mathrm{Id}(c) = c$  for any  $c \in \Sigma_{\tau}$ . There is a constant  $\mathrm{Id} \in \Sigma_{\tau \to \tau}$  such that  $\mathcal{F}(\mathrm{id}) = \mathrm{Id}$ . We show  $\lambda x.x \succ_1$  id. Let  $c \in \Sigma_{\tau} = \mathbb{T}_{\tau}$ . We have  $(\lambda x.x)c \to_{<1} c$ , because  $R_{<1} = \bigcup_{n<1} R_n = R_0$  and  $R_0$  contains the rules of  $\beta$ -reduction. We also have  $c \succ_{<1} c \equiv \mathcal{F}(\mathrm{id})(c)$  by the first part of Definition 4.1.8. Therefore  $(\lambda x.x)c \leadsto_{<1} c$ . Since  $c \in \Sigma_{\tau}$  was arbitrary, we obtain  $\lambda x.x \succ_1$  id by the second part of Definition 4.1.8.

Now we show that  $\lambda yx.x \succ_2 \lambda y.\mathrm{id}$ . We have  $\lambda y.\mathrm{id} \in \mathbb{T}_{\omega \to \tau \to \tau}$ . So let  $t \in \mathbb{T}_{\omega}$ . We have  $(\lambda yx.x)t \to_{<2} \lambda x.x$ . We already proved that  $\lambda x.x \succ_1 \mathrm{id}$ . Note that  $\mathcal{F}(\lambda y.\mathrm{id})$  is the constant function from  $\mathbb{T}_{\omega}$  to  $\mathbb{T}_{\tau \to \tau}$  whose value is always id. This implies that  $(\lambda yx.x)t \leadsto_{<2} \mathrm{id} \equiv \mathcal{F}(\lambda y.\mathrm{id})(t)$  for any  $t \in \mathbb{T}_{\omega}$ . Hence  $\lambda yx.x \succ_2 \lambda y.\mathrm{id}$ .

Let  $\rho \in \Sigma_{((\omega \to \tau) \to \tau) \to \tau}$  be such that  $\mathcal{F}(\rho)(f) \equiv \mathcal{F}(f)(\lambda y.\mathrm{id})$  for  $f \in \Sigma_{(\omega \to \tau) \to \tau}$ . As another example we will show that  $\lambda z.z(\lambda yx.x) \succ_4 \rho$ . So suppose  $f \in \Sigma_{(\omega \to \tau) \to \tau} = \mathbb{T}_{(\omega \to \tau) \to \tau}$ . We have  $(\lambda z.z(\lambda yx.x))f \to_{<4} f(\lambda yx.x)$ . We proved in the previous paragraph that  $\lambda yx.x \succ_2 \lambda y.\mathrm{id}$ . By the second part of Definition 4.1.6 we obtain  $f(\lambda yx.x) \to_{=3} \mathcal{F}(f)(\lambda y.\mathrm{id})$ . Hence  $(\lambda z.z(\lambda yx.x))f \twoheadrightarrow_{<4} \mathcal{F}(f)(\lambda y.\mathrm{id})$  for any  $f \in \Sigma_{(\omega \to \tau) \to \tau}$ . Obviously we have  $\mathcal{F}(f)(\lambda y.\mathrm{id}) \succ_0 \mathcal{F}(f)(\lambda y.\mathrm{id})$  by the first part of Definition 4.1.8, because the range of  $\mathcal{F}(f)$  is included in  $\Sigma_{\tau}$ . Recalling that  $\mathcal{F}(\rho)(f) \equiv \mathcal{F}(f)(\lambda y.\mathrm{id})$  for any  $f \in \Sigma_{(\omega \to \tau) \to \tau}$  we obtain  $\lambda z.z(\lambda yx.x) \succ_4 \rho$  by the second part of Definition 4.1.8.

**Lemma 4.1.10.** For  $\alpha \leq \beta$  we have the following inclusions:  $R_{\alpha} \subseteq R_{\beta}$ ,  $\sim_{\alpha} \subseteq \sim_{\beta}$ , and  $\succ_{\alpha} \subseteq \succ_{\beta}$ .

*Proof.* Follows easily from definitions.

It follows from Lemma 4.1.10 by appealing to the well-known Knaster-Tarski fixpoint theorem that there exists an ordinal  $\zeta$  such that  $\succ_{\zeta} = \succ_{<\zeta}$  and  $R_{\zeta} = R_{<\zeta}$ . This simple fact may also be shown directly as follows. Suppose  $\zeta$  is an ordinal with cardinality greater than  $(\mathbb{T}_{\omega} \cup \{\Box\})^4$  and there is no  $\alpha < \zeta$  such that  $R_{\alpha} = R_{<\alpha}$  and  $\succ_{\alpha} = \succ_{<\alpha}$ . Then for each  $\alpha < \zeta$  either  $R_{\alpha} \setminus R_{<\alpha}$  or  $\succ_{\alpha} \setminus \succ_{<\alpha}$  is non-empty. Because  $R_{\alpha} \subseteq \mathbb{T}_{\omega} \times \mathbb{T}_{\omega}$  and  $\succ_{\alpha} \subseteq \mathbb{T}_{\omega} \times \mathbb{T}_{\omega}$ , we may thus define, using the axiom of choice, an injection f from  $\zeta$  to  $(\mathbb{T}_{\omega} \cup \{\Box\})^4$  (recall that in set theory an ordinal  $\zeta$  is the set of all ordinals less than  $\zeta$ ). If  $R_{\alpha} \setminus R_{<\alpha}$  is non-empty, then let  $f(\alpha) = \langle t_1, t_2, \Box, \Box \rangle$  where  $\langle t_1, t_2 \rangle \in R_{\alpha} \setminus R_{<\alpha}$  is chosen arbitrarily. Analogously, if  $\succ_{\alpha} \setminus \succ_{<\alpha}$  is non-empty, then let  $f(\alpha) = \langle \Box, \Box, t_1, t_2 \rangle$  where  $\langle t_1, t_2 \rangle \in \succ_{\alpha} \setminus \succ_{<\alpha}$  is chosen arbitrarily. Since  $R_{\alpha} \subseteq R_{<\beta}$  and  $\succ_{\alpha} \subseteq \succ_{<\beta}$  for  $\alpha < \beta$ , we have  $f(\alpha) \neq f(\beta)$ , so f really is an injection. But this implies that the cardinality of  $\zeta$  is not greater than the cardinality of  $(\mathbb{T}_{\omega} \cup \{\Box\})^4$ . Contradiction.

Let  $\zeta$  be an ordinal such that  $R_{\zeta} = R_{<\zeta}$  and  $\succ_{\zeta} = \succ_{<\zeta}$ . We may assume without loss of generality that also  $\sim_{\zeta} = \sim_{<\zeta}$ . In what follows we will use the notations  $R, \succ, \leadsto$ , etc. for  $R_{\zeta}, \succ_{\zeta}, \leadsto_{\zeta}$ , etc.

Finally, we are ready to define the model  $\mathcal{M}$  for  $\mathcal{I}_{\omega}^{c}$ .

**Definition 4.1.11.** The one-state classical illative Kripke model  $\mathcal{M}$  is defined as follows. We take the combinatory algebra  $\mathcal{C}$  of  $\mathcal{M}$  to be the set of equivalence classes of  $=_R$ . We define the interpretation I of  $\mathcal{M}$  by  $I(c) = [c]_R$ . We define the set  $\mathcal{T}$  of true elements of  $\mathcal{M}$  by  $\mathcal{T} = \{d \in \mathcal{C} \mid \exists t . d = [t]_R \land t \leadsto \top\}$ .

## 4.2 Correctness proof

In this subsection we prove that the preceding lengthy definition of  $\mathcal{M}$  is actually correct, i.e., that  $\mathcal{M}$  is a classical illative Kripke model for  $\mathcal{I}^c_{\omega}$ .

Below we will silently use the following simple lemma, without mentioning it explicitly every time.

**Lemma 4.2.1.** If  $\lambda \vec{x}. \exists t_1 t_2 \twoheadrightarrow \lambda \vec{x}.t$  then  $t \equiv \exists t'_1 t'_2$  where  $t_1 \twoheadrightarrow t'_1$  and  $t_2 \twoheadrightarrow t'_2$ . An analogous result holds when  $\lambda \vec{x}. A_{\tau} t_1 \twoheadrightarrow \lambda \vec{x}.t$  for  $\tau \in \mathcal{B}$ , and when  $\lambda \vec{x}. L t_1 \twoheadrightarrow \lambda \vec{x}.t$ . Here the reduction  $\twoheadrightarrow$  may stand for any of  $\twoheadrightarrow \leq \alpha$ ,  $\twoheadrightarrow < \alpha$ , etc.

*Proof.* This follows from the fact that there are no reduction rules which involve  $\Xi$ , L, or  $A_{\tau}$  for  $\tau \in \mathcal{B}$ , so the reductions may happen only inside  $t_1$  and  $t_2$ .

Note that together with our convention stated in Notation 4.1.4 regarding the meaning of  $Ht_1$ , Lemma 4.2.1 implies that if  $Ht_1 \rightarrow t$  then  $t \equiv Ht'_1$  where  $t_1 \rightarrow t'_1$ .

The proof of the following lemma illustrates a pattern common to many of the proofs below. We give this single proof in full, but when later an argument follows this same pattern we treat only some of the cases to spare the reader excessive tedious details.

**Lemma 4.2.2.** If  $x_1, \ldots, x_n$  are variables,  $n \ge 1$ , and C is a context, then the following conditions hold:

- (1) if  $t \rightarrow <_{\alpha} t'$  and  $t \equiv C[x_1 \dots x_n]$  then  $t' \equiv C'[x_1 \dots x_n]$  and  $C[t''] \rightarrow <_{\alpha} C'[t'']$  for any term t'',
- (2) if  $C[x_1 \dots x_n] \succ_{\alpha} \rho$  then  $C[t] \succ_{\alpha} \rho$  for any term t,
- (3) if  $C[x_1 \dots x_n] \sim_{\alpha} \tau$  then  $C[t] \sim_{\alpha} \tau$  for any term t.

*Proof.* Induction on  $\alpha$ .

First, we show (1) by induction on the length of the reduction  $C[x_1 \dots x_n] \to_{\leq \alpha} t'$ . The only interesting case is when  $cC[x_1 \dots x_n] \to_{\leq \alpha} \rho_2$  by virtue of  $C[x_1 \dots x_n] \succ_{<\alpha} \rho_1$ . But then by part (2) of the IH we have  $C[t''] \succ_{<\alpha} \rho_1$ , so  $cC[t''] \to_{\leq \alpha} \rho_2$ .

Next we shall verify (3). If  $C[x_1 \dots x_n] \sim_{\alpha} \tau$  is obtained by rule (A) or (H), then  $C \equiv A_{\tau}$  for  $\tau \in \mathcal{B}$  or  $C \equiv H$ , and the claim is obvious.

If  $C[x_1...x_n] \sim_{\alpha} \tau$  is obtained by rule  $(K\omega)$  or  $(K\varepsilon)$  then  $\tau \in \{\omega, \varepsilon\}$  and  $C[x_1...x_n] \leadsto_{<\alpha} c$  for  $c \in \{\top, \bot\}$ , i.e.,  $C[x_1...x_n] \twoheadrightarrow_{<\alpha} t' \succ_{<\alpha} c$ . By part (1) of the IH we obtain  $t' \equiv C'[x_1...x_n]$  where  $C[t] \twoheadrightarrow_{<\alpha} C'[t]$ . Then by part (2) of the IH we have  $C'[t] \succ_{<\alpha} c$ . Hence  $C'[t] \leadsto_{<\alpha} c$ , so  $C'[t] \sim_{\alpha} \tau$  by rule  $(K\omega)$  or  $(K\varepsilon)$ .

If  $C[x_1 \dots x_n] \sim_{\alpha} \tau$  is obtained by rule  $(F\omega)$  then  $\tau = \omega$ ,  $C \equiv \lambda f. \exists C_1 C_2$  and  $C_1[x_1 \dots x_n] \sim_{<\alpha} \varepsilon$ . By part (3) of the IH we obtain  $C_1[t] \sim_{<\alpha} \varepsilon$ , and thus  $C[t] \equiv \lambda f. \exists C_1[t] C_2[t] \sim_{\alpha} \omega$  by rule  $(F\omega)$ . If  $C[x_1 \ldots x_n] \sim_{\alpha} \tau$  is obtained by rule  $(F\omega')$  then  $\tau = \omega$ ,  $C \equiv \Xi C_1$  and  $C_1[x_1 \ldots x_n] \sim_{<\alpha} \varepsilon$ . By part (3) of the IH we obtain  $C_1[t] \sim_{<\alpha} \varepsilon$ , and thus  $C[t] \equiv \Xi C_1[t] \sim_{\alpha} \omega$  by rule  $(F\omega')$ .

If  $C[x_1...x_n] \sim_{\alpha} \tau$  is obtained by rule (F) then  $\tau = \tau_1 \to \tau_2$  and  $C \equiv \lambda f. \exists C_1(\lambda x. C_2(fx))$  where  $C_1[x_1...x_n] \sim_{<\alpha} \tau_1$  and  $C_2[x_1...x_n] \sim_{<\alpha} \tau_2$ . But then by part (3) of the IH we have  $C_1[t] \sim_{<\alpha} \tau_1$  and  $C_2[t] \sim_{<\alpha} \tau_2$ , which implies  $C[t] \equiv \lambda f. \exists C_1[t](\lambda x. C_2[t](fx)) \sim_{\alpha} \tau$ .

If  $C[x_1...x_n] \sim_{\alpha} \tau$  is obtained by rule (F') then  $\tau = \tau_1 \to \tau_2$  and  $C \equiv \lambda f. \exists C_1(\lambda x. C_2[z/fx])$  where  $C_1[x_1...x_n] \sim_{<\alpha} \tau_1$  and  $\lambda z. C_2[x_1...x_n] \sim_{<\alpha} \tau_2$ . But then by part (3) of the IH we have  $C_1[t] \sim_{<\alpha} \tau_1$  and  $\lambda z. C_2[t] \sim_{<\alpha} \tau_2$ , which implies  $C[t] \equiv \lambda f. \exists C_1[t](\lambda x. C_2[z/fx][t] \sim_{\alpha} \tau$  (recall that by our convention regarding contexts, the free variables of t are assumed not to become bound in C[t]).

Finally, if  $C[x_1 \ldots x_n] \sim_{\alpha} \tau$  is obtained by rule (F") then  $\tau = \tau_1 \to \tau_2$  and  $C \equiv \lambda f. \exists C_1 C_2$  where  $C_1[x_1 \ldots x_n] \sim_{<\alpha} \tau_1$  and  $C_2[x_1 \ldots x_n] \sim_{<\alpha} \tau_2 \in \{\omega, \varepsilon\}$ . But then by parts (1) and (3) of the IH we have  $C_1[t] \sim_{<\alpha} \tau_1$  and  $C_2[t] \sim_{<\alpha} \tau_2$ , which implies  $C[t] \equiv \lambda f. \exists C_1[t] C_2[t] \sim_{\alpha} \tau$ .

Now we check condition (2). Suppose  $C[x_1 \dots x_n] \succ_{\alpha} \rho$  for a canonical term  $\rho$ . If  $C[x_1 \dots x_n] \equiv \rho$  then the claim is obvious because canonical terms are closed, so  $C[t] \equiv C \equiv C[x_1 \dots x_n] \equiv \rho$ . If the canonical type of  $\rho$  is  $\tau_1 \to \tau_2$  then by definition for any  $t_1 \in \mathbb{T}_{\tau_1}$  we have  $C[x_1 \dots x_n]t_1 \leadsto_{<\alpha} \mathcal{F}(\rho)(t_1)$ . By parts (2) and (3) of the IH and by the definition of  $\leadsto_{<\alpha}$  we obtain  $C[t]t_1 \leadsto_{<\alpha} \mathcal{F}(\rho)(t_1)$ . Hence  $C[t] \succ_{\alpha} \rho$ .

Suppose  $\rho \equiv \top$ . If  $C[x_1 \dots x_n] \not\succ_0 \top$  then one of the conditions  $(A_{\tau}^{\top})$ ,  $(\Xi^{\top})$  or  $(L^{\top})$  in Definition 4.1.8 must hold. If  $(A_{\tau}^{\top})$  holds then the claim is obvious, because  $C[x_1 \dots x_n]$  is closed.

If  $(\Xi^{\top})$  holds then  $C \equiv \Xi C_1 C_2$  and there exists  $\tau$  such that  $C_1[x_1 \dots x_n] \sim_{\alpha} \tau$  and for all  $t' \in \mathbb{T}_{\tau}$  we have  $C_2[x_1 \dots x_n]t' \leadsto_{<\alpha} \top$ . By claim (3), which has already been verified in this inductive step, we obtain  $C_1[t] \sim_{\alpha} \tau$ . By parts (1) and (2) of the IH we conclude that for all  $t' \in \mathbb{T}_{\tau}$  we have  $C_2[t]t' \leadsto_{<\alpha} \top$ . Therefore  $C[t] = \Xi C_1[t]C_2[t] \succ_{\alpha} \top$ .

If condition  $(L^{\top})$  holds then  $C \equiv LC_1$  and  $C_1[x_1 \dots x_n] \sim_{\alpha} \tau$  for some type  $\tau$ . By calim (3), which has already been verified in this inductive step, we obtain  $C_1[t] \sim_{\alpha} \tau$ . Therefore  $C[t] \succ_{\alpha} \top$ .

It remains to verify the case  $C[x_1 \dots x_n] \succ_{\alpha} \bot$ . Assuming  $C[x_1 \dots x_n] \not\succ_0 \bot$ , the condition  $(\Xi^{\bot})$  must hold. Then the claim again follows by applying the already verified condition (3) and parts (1) and (2) of the inductive hypothesis.

Corollary 4.2.3. If  $t_1 \to_{=\alpha} t'_1$  and the free variables of  $t_2$  do not become bound in  $t_1[x/t_2]$ , then  $t_1[x/t_2] \to_{=\alpha} t'_1[x/t_2]$ .

*Proof.* If  $\alpha = 0$  then this is obvious. If  $\alpha > 0$  then assume without loss of generality that  $ct_1 \to_{=\alpha} \rho_2 \equiv t_1'$  by virtue of  $t_1 \succ_{<\alpha} \rho_1$ . But then by part (2) of Lemma 4.2.2 we have  $t_1[x/t_2] \succ_{<\alpha} \rho_1$ , so  $ct_1[x/t_2] \to_{=\alpha} \rho_2 \equiv t_1'[x/t_2]$ , since the canonical term  $\rho_2$  is closed.

**Lemma 4.2.4.** If  $Kt \sim_{\alpha} \tau$  then  $\tau = \omega$  or  $\tau = \varepsilon$ .

Proof. Induction on  $\alpha$ . The non-obvious case is when  $Kt \equiv \lambda f. \exists t_1(\lambda x. t_2[z/fx]) \sim_{\alpha} \tau_1 \to \tau_2$  is obtained by rule (F'), and  $t_1 \sim_{<\alpha} \tau_1$  for  $\tau_1 \neq \varepsilon$ , and  $\lambda z. t_2 \sim_{<\alpha} \tau_2$ . But then  $t \equiv \exists t_1(\lambda x. t_2[z/fx])$  and  $z \notin FV(t_2)$ . Since  $Kt_2 \sim_{<\alpha} \tau_2$  by the inductive hypothesis we conclude  $\tau_2 = \omega$  or  $\tau_2 = \varepsilon$ . In either case  $\tau = \omega$  or  $\tau = \varepsilon$ 

The next lemma and Lemma 4.2.12 are the two key technical lemmas justifying the correctness of our model construction.

**Lemma 4.2.5.** For all ordinals  $\alpha$ ,  $\beta$  the following conditions hold:

- (1)  $R_{\alpha}$  and  $R_{\beta}$  commute, i.e., if  $t \twoheadrightarrow_{\leq \alpha} t_1$  and  $t \twoheadrightarrow_{\leq \beta} t_2$  then  $t_1 \twoheadrightarrow_{\leq \beta} t'$  and  $t_2 \twoheadrightarrow_{\leq \alpha} t'$  for some term t',
- (2) if  $t_1 \succ_{\alpha} \rho$  and  $t_1 \twoheadrightarrow_{\leq \beta} t_2$  then  $t_2 \succ_{\alpha} \rho$ ,
- (3) if  $t \succ_{\alpha} \rho_1$ ,  $t \succ_{\beta} \rho_2$  and  $\rho_1, \rho_2 \in \mathbb{T}_{\tau}$  then  $\rho_1 \equiv \rho_2$ ,
- (4) if  $t_1 \sim_{\alpha} \tau$  and  $t_1 \rightarrow_{\leq \beta} t_2$  then  $t_2 \sim_{\alpha} \tau$ ,
- (5) if  $t \sim_{\alpha} \tau_1$  and  $t \sim_{\beta} \tau_2$  then  $\tau_1 = \tau_2$ ,
- (6) if  $t \sim_{\alpha} \omega$  then  $tr \leadsto_{\leq \alpha} \top$  for all r, and if  $t \sim_{\alpha} \varepsilon$  then  $tr \leadsto_{\leq \alpha} \bot$  for all r.

*Proof.* Induction on pairs  $\langle \alpha, \beta \rangle$  ordered lexicographically. Together with every condition we show its dual, i.e., the condition with  $\alpha$  and  $\beta$  exchanged. We give proofs only for the original conditions, but it can be easily seen that in every case the dual condition follows by exactly the same proof with  $\alpha$  and  $\beta$  exchanged. Note that for a proof of a condition to be a proof of its dual, it suffices that we never use the inductive hypothesis with  $\beta$  increased.

First note that conditions (1) and (2) imply that if  $t_1 \leadsto_{\alpha} \rho$  and  $t_1 \twoheadrightarrow_{\leq \beta} t_2$ , then  $t_2 \leadsto_{\alpha} \rho$ . Indeed, if  $t_1 \twoheadrightarrow_{\leq \alpha} t'_1 \succ_{\alpha} \rho$  and  $t_1 \twoheadrightarrow_{\leq \beta} t_2$ , then by (1) we have  $t_2 \twoheadrightarrow_{\leq \alpha} t'_2$  and  $t'_1 \twoheadrightarrow_{\leq \beta} t'_2$ . Hence by (2) it follows that  $t'_2 \succ_{\alpha} \rho$ , so  $t_2 \leadsto_{\alpha} \rho$ .

Instead of (1) we prove a stronger claim that  $\widehat{R}_{\alpha}$  and  $\widehat{R}_{\beta}$  commute. Condition (1) follows from this claim by a simple tiling argument, similar to the proof of the Hindley-Rosen lemma.

If  $\alpha=\beta=0$  then the claim is obvious, because  $\widehat{R_0}=R_0$  is the ordinary  $\lambda\beta\eta$ -calculus. We therefore check that  $R_0$  commutes with  $\widehat{R_\alpha}$  for  $\alpha>0$ . We show that if  $t\to_{=\alpha}t_1$  and  $t\to_{\beta\eta}t_2$  then there exists  $t_3$  such that  $t_1\to_{\overline{\beta\eta}}t_3$  and  $t_2\to_{=\alpha}t_3$ . The claim then follows by a simple diagram chase. First suppose  $t\equiv (\lambda x.r_1)r_2\to_{\beta}r_1[x/r_2]\equiv t_2$  and  $r_1\to_{=\alpha}r_1'$ . Then by Corollary 4.2.3 we have  $r_1[x/r_2]\to_{=\alpha}r_1'[x/r_2]$ . Also obviously  $(\lambda x.r_1')r_2\to_{\beta}r_1'[x/r_2]$ . If  $t\equiv (\lambda x.r_1)r_2\to_{\beta}r_1[x/r_2]\equiv t_2$  and  $r_2\to_{=\alpha}r_2'$  then the claim is obvious. Suppose  $t\equiv \lambda x.rx\to_{\eta}r$  where  $x\notin FV(r)$ . The only interesting case is when  $r\equiv c$  and  $cx\to_{=\alpha}\rho_2$  by virtue of  $x\succ_{<\alpha}\rho_1$ . But then by part (2) of Lemma 4.2.2 we have  $\rho'\succ_{<\alpha}\rho_1$  for  $\rho'\neq\rho_1$  with  $\rho'$  of the same canonical type as  $\rho_1$ . This is, however, impossible by part (3) of the IH. Without loss of generality, the only remaining case is  $t\equiv ct'\to_{=\alpha}t_1\equiv \mathcal{F}(c)(\rho)$ ,  $t'\succ_{<\alpha}\rho$ ,  $t_2\equiv ct'_2$ , and  $t'\to_{\beta\eta}t'_2$ . By part (2) of the IH we obtain  $t'_2\succ_{<\alpha}\rho$ . Therefore  $t_2\equiv ct'_2\to_{=\alpha}\mathcal{F}(c)(\rho)\equiv t_1$ .

We now check that  $R_{\alpha}$  commutes with  $R_{\beta}$  for  $\alpha, \beta > 0$ . It suffices to show that if  $t \to_{=\alpha} t_1$  and  $t \to_{=\beta} t_2$  then there exists  $t_3$  such that  $t_1 \to_{=\beta}^{\equiv} t_3$  and  $t_2 \to_{=\alpha}^{\equiv} t_3$ . If the redexes do not overlap then this is obvious. Suppose they overlap at the root, i.e.,  $t \equiv ct'$ ,  $ct' \to_{=\alpha} t_1 \equiv \mathcal{F}(c)(\rho_1)$  where  $t' \succ_{<\alpha} \rho_1$ , and  $ct' \to_{=\beta} t_2 \equiv \mathcal{F}(c)(\rho_2)$  where  $t' \succ_{<\beta} \rho_2$ . But then  $\rho_1$  and  $\rho_2$  are canonical terms of the same type, which is determined by the type of c. So by part (3) of the IH we obtain  $\rho_1 \equiv \rho_2$ . Hence  $t_1 \equiv t_2$ . If the overlap does not happen at the root, then without loss of generality  $t \equiv ct'$ ,  $ct' \to_{=\alpha} t_1 \equiv \mathcal{F}(c)(\rho)$  where  $t' \succ_{<\alpha} \rho$ ,  $t_2 \equiv ct'_2$ , and  $t' \to_{=\beta} t'_2$ . By part (2) of the IH we obtain  $t'_2 \succ_{<\alpha} \rho$ , so  $t_2 \equiv ct'_2 \to_{=\alpha} \mathcal{F}(c)(\rho) \equiv t_1$ .

Now we shall prove (4). If  $t_1 \equiv A_\tau \sim_\alpha \tau$  for  $\tau \in \mathcal{B}$  or  $t_1 \equiv H$ , then the claim is obvious. If  $t_1 \equiv Kt'_1 \sim_\alpha \omega$  and  $t'_1 \leadsto_{<\alpha} \top$ , then  $t_2 = Kt'_2$ ,  $t'_1 \twoheadrightarrow_{\leq\beta} t'_2$ , and by parts (1) and (2) of the IH we have  $t'_2 \leadsto_{<\alpha} \top$ . Hence  $t_2 \sim_\alpha \omega$ . If  $t_1 \equiv Kt'_1 \sim_\alpha \varepsilon$  and  $t'_1 \leadsto_{<\alpha} \bot$  the proof is analogous.

If  $t_1 \sim_{\alpha} \tau_1 \to \tau_2$  follows by (F) then  $t_1 \equiv \lambda f.\Xi t_1^1(\lambda x.t_1^2(fx))$  where  $t_1^1 \sim_{<\alpha} \tau_1$  and  $t_1^2 \sim_{<\alpha} \tau_2$ . Without loss of generality, we may assume  $t_1 \to_{\leq \beta} t_2$ , i.e., the reduction  $t_1 \to_{\leq \beta} t_2$  consists of a single step. Then  $t_2 \equiv \lambda f.\Xi t_2^1 s$  with  $t_1^1 \to_{\leq \beta}^{\pm} t_2^1$  and  $\lambda x.t_1^2(fx) \to_{\leq \beta}^{\pm} s$ . By the IH we have  $t_2^1 \sim_{<\alpha} \tau_1$ . If  $t_1^2 \equiv \lambda z.s_1$  and  $s \equiv \lambda x.s_2[z/fx]$  then  $t_2 \sim_{\alpha} \tau_1 \to \tau_2$  by (F'). It is impossible that  $t_1^2(fx)$  is a redex with  $t_1^2$  a constant. Indeed, then  $fx \succ_{<\beta} \rho$  for some canonical  $\rho$ . Using the definition of  $\succ$  and noting that a term of the form  $fxw_1 \dots w_k$  is not a  $\to_{\gamma}$ -redex for any  $\gamma$  because f is a variable, we may conclude that  $fxw_1 \dots w_n \succ_{\gamma} \rho'$  for some  $\gamma, w_1, \dots, w_n$  and some canonical  $\rho'$  of type o or base type. But this contradicts the definition of  $\succ_{\gamma}$ . Therefore, the only remaining possibility is  $s \equiv \lambda x.t_2^2(fx)$  with  $t_1^2 \to_{\leq \beta}^{\equiv} t_2^2$ . By the IH we obtain  $t_2^2 \sim_{<\alpha} \tau_2$ . Therefore  $t_2 \sim_{\alpha} \tau_1 \to \tau_2$  by (F).

If  $t_1 \sim_{\alpha} \tau_1 \to \tau_2$  follows by (F') then  $t_1 \equiv \lambda f. \exists t_1^1 (\lambda x. t_1^2 [z/fx])$  where  $t_1^1 \sim_{<\alpha} \tau_1$  and  $\lambda z. t_1^2 \sim_{<\alpha} \tau_2$ . Without loss of generality, we may assume  $t_1 \to_{\leq \beta} t_2$ , i.e., the reduction  $t_1 \to_{\leq \beta} t_2$  consists of a single step. By Lemma 4.2.1 we have  $t_2 \equiv \lambda f. \exists t_2^1 s_2$  where  $t_1^1 \to_{\leq \beta}^{\equiv} t_2^1$  and  $\lambda x. t_1^2 [z/fx] \to_{\leq \beta}^{\equiv} s_2$ . We show that  $s_2 \equiv \lambda x. t_2^2 [z/fx]$  with  $t_1^2 \to_{\leq \beta} t_2^2$ . Suppose the contraction in  $\lambda x. t_1^2 [z/fx]$  occurs at the root. Then this must be an  $\eta$ -contraction, and because  $x \notin FV(t_1^2)$  we have  $t_1^2 \equiv z$ . But then  $\lambda z. z \sim_{<\alpha} \tau_2$ . By inspecting the definition of  $\sim_{<\alpha}$  this is seen to be impossible. Hence the contraction does not occur at the root, and thus it follows from Lemma 4.2.1 and Lemma 4.2.2 that  $s_2 \equiv \lambda x. t_2^2 [z/fx]$  with  $t_1^2 \to_{\leq \beta} t_2^2$ . By the IH we obtain  $t_2^1 \sim_{<\alpha} \tau_1$  and  $\lambda z. t_2^2 \sim_{<\alpha} \tau_2$ . Thus  $t_2 \sim_{\alpha} \tau_1 \to \tau_2$ .

IH we obtain  $t_2^1 \sim_{<\alpha} \tau_1$  and  $\lambda z. t_2^2 \sim_{<\alpha} \tau_2$ . Thus  $t_2 \sim_{\alpha} \tau_1 \to \tau_2$ .

If  $t_1 \sim_{\alpha} \tau_1 \to \tau_2$  follows by (F") then  $t_1 \equiv \lambda f. \exists t_1^1 t_1^2$  with  $t_1^1 \sim_{<\alpha} \tau_1$  and  $t_1^2 \sim_{<\alpha} \tau_2$ , with  $f \notin FV(t_1^2)$ . Since  $t_1^2 \not\equiv f$  we have  $t_2 \equiv \lambda f. \exists t_2^1 t_2^2$  with  $t_1^1 \twoheadrightarrow_{\leq\beta} t_2^1$  and  $t_1^2 \twoheadrightarrow_{\leq\beta} t_2^2$ . By the IH we obtain  $t_2^1 \sim_{<\alpha} \tau_1$  and  $t_2^2 \sim_{<\alpha} \tau_2$ . Thus  $t_2 \sim_{\alpha} \tau_1 \to \tau_2$ .

If  $t_1 \sim_{\alpha} \omega$  follows by  $(F\omega)$  then  $t_1 \equiv \lambda f. \Xi t_1^1 t_1^2$  and  $t_1^1 \sim_{<\alpha} \varepsilon$ . Without loss of generality we assume  $t_1 \to_{\leq\beta} t_2$ . There are two possibilities.

- $t_2 \equiv \lambda f. \exists t_2^1 t_2^2$  with  $t_1^1 \twoheadrightarrow_{\leq \beta} t_2^1$  and  $t_1^2 \twoheadrightarrow_{\leq \beta} t_2^2$ . Then  $t_2^1 \sim_{<\alpha} \varepsilon$  by the IH, so  $t_2 \sim_{\alpha} \omega$  by  $(F\omega)$ .
- $t_2 \equiv \Xi t_1^1$ . Then  $t_2 \sim_{\alpha} \omega$  follows by  $(F\omega')$ .

If  $t_1 \sim_{\alpha} \omega$  follows by  $(F\omega')$  then  $t_1 \equiv \Xi t_1'$ ,  $t_1' \sim_{<\alpha} \varepsilon$  and  $t_2 \equiv \Xi t_2'$  with  $t_1' \twoheadrightarrow_{\leq \beta} t_2'$ . Then  $t_2' \sim_{<\alpha} \varepsilon$  by the IH. Thus  $t_2 \sim_{\alpha} \omega$ .

We show (2). If  $t_1 \equiv \rho$  then  $t_1$  is in  $R_{\beta}$ -normal form, so there is nothing to prove. If  $t_1 \not\equiv \rho$ ,  $t_1 \succ_{\alpha} \rho$  and  $t_1 \twoheadrightarrow_{\leq \beta} t_2$ , where  $\rho \in \mathbb{T}_{\tau_1 \to \tau_2}$ , then by definition for all  $\rho_1 \in \mathbb{T}_{\tau_1}$  we have  $t_1 \rho_1 \leadsto_{<\alpha} \rho_2$ , where  $\rho_2 \equiv \mathcal{F}(\rho)(\rho_1)$ . But then by parts (1) and (2) of the inductive hypothesis  $t_2 \rho_1 \leadsto_{<\alpha} \rho_2$ , so  $t_2 \succ_{\alpha} \rho$ . Therefore suppose  $t_1 \succ_{\alpha} \top$ . When  $t_1 \succ_{\alpha} \bot$  the argument is similar. If  $\alpha = 0$  then the claim is obvious, because the right sides of the identities in the postulates for  $t_1 \succ_0 \top$  are normal forms. If  $\alpha > 0$  then assume  $t_1 \twoheadrightarrow_{\leq \beta} t_2$ ,  $t_1 \equiv \Xi t_1^1 t_1^2$  and condition  $(\Xi^\top)$  in the definition of  $t_1 \succ_{\alpha} \top$  is satisfied, i.e., there exists  $\tau$  s.t.  $t_1^1 \leadsto_{\alpha} \tau$  and for all  $t_3 \in \mathbb{T}_{\tau}$  we have  $t_1^2 t_3 \leadsto_{<\alpha} \tau$ . When any of the other conditions in the definition of  $t_1 \succ_{\alpha} \tau$  is satisfied instead of  $(\Xi^\top)$ , then the proof is analogous. By Lemma 4.2.1 we have  $t_2 \equiv \Xi t_2^1 t_2^2$  where  $t_1^1 \twoheadrightarrow_{\leq \beta} t_2^1$  and  $t_1^2 \twoheadrightarrow_{\leq \beta} t_2^2$ . By (4), which has already been verified in this inductive step, we obtain  $t_2^1 \leadsto_{\alpha} \tau$ . It therefore suffices to check that for all  $t_3 \in \mathbb{T}_{\tau}$  we have  $t_2^2 t_3 \leadsto_{<\alpha} \tau$ . But for  $t_3 \in \mathbb{T}_{\tau}$  obviously  $t_1^2 t_3 \leadsto_{<\alpha} \tau$ , so  $t_2^2 t_3 \leadsto_{<\alpha} \tau$  by parts (1) and (2) of the IH.

We show (6). Suppose  $t \sim_{\alpha} \omega$ . When  $t \sim_{\alpha} \varepsilon$  the argument is similar. If  $t \sim_{\alpha} \omega$  is obtained by rule (K $\omega$ ) then the claim is obvious. If  $t \sim_{\alpha} \omega$  is obtained by (F) then  $t \equiv \lambda f. \exists t_1(\lambda x. t_2(fx))$  with  $f, x \notin FV(t_1, t_2), t_1 \sim_{<\alpha} \tau_1$  and  $t_2 \sim_{<\alpha} \tau_2$ . Because  $\tau = \omega$  we must have  $\tau_1 = \varepsilon$  or  $\tau_2 = \omega$ . Since  $tr \to_{\beta} \exists t_1(\lambda x. t_2(rx))$  it suffices to show  $\exists t_1(\lambda x. t_2(rx)) \succ_{<\alpha} \top$ . If  $\tau_1 = \varepsilon$  then this follows from ( $\Xi^{\top}$ ) because  $\mathbb{T}_{\varepsilon} = \emptyset$ . So assume  $\tau_2 = \omega$ . Let  $\gamma < \alpha$  be such that  $t_1 \sim_{\gamma} \tau_1$  and  $t_2 \sim_{\gamma} \omega$ . Let  $t_3 \in \mathbb{T}_{\tau_1}$ . By part (6) of the IH we have  $(\lambda x. t_2(rx))t_3 \to_{\beta} t_2(rt_3) \leadsto_{<\gamma} \top$ . Hence  $\exists t_1 t_2 \succ_{<\alpha} \top$  by ( $\Xi^{\top}$ ). If  $t \sim_{\alpha} \omega$  is obtained by (F") then the argument is analogous to the case for (F). If the derivation of  $t \sim_{\alpha} \omega$  is obtained by (F $\omega$ ) then  $t \equiv \lambda f. \exists t_1 t_2$  with  $t_1 \sim_{<\alpha} \varepsilon$ . Then  $tr \to_{\beta} \exists t_1(t_2[f/r]) \succ_{<\alpha} \top$  by definition. When  $t \sim_{\alpha} \omega$  is obtained by (F $\omega$ ) the argument is analogous to the case for (F $\omega$ ). The only other possibility is that  $t \sim_{\alpha} \omega$  is obtained by rule (F'). Then  $t \equiv \lambda f. \exists t_1(\lambda x. t_2[z/fx]), t_1 \sim_{<\alpha} \tau_1$  and  $\lambda z. t_2 \sim_{<\alpha} \tau_2$ . It suffices to verify that  $\exists t_1(\lambda x. (\lambda z. t_2)(rx)) \succ_{<\alpha} \top$ , because for  $t' \equiv \exists t_1(\lambda x. t_2[z/rx])$  we have  $\exists t_1(\lambda x. (\lambda z. t_2)(rx)) \to_{\beta} t'$  and  $tr \to_{\beta} t'$ , so then  $t' \succ_{<\alpha} \top$  by part (2) of the IH, which implies  $tr \leadsto_{<\alpha} \top$ . But the argument to show  $\exists t_1(\lambda x. (\lambda z. t_2)(rx)) \succ_{<\alpha} \top$  is analogous to the case for (F).

We show (5). Suppose  $t \sim_{\alpha} \tau_1$  and  $t \sim_{\beta} \tau_2$ . If  $t \equiv A_{\tau}$  for  $\tau \in \mathcal{B}$  or  $t \equiv H$  then the claim is obvious. So suppose  $t \not\equiv A_{\tau}$  for  $\tau \in \mathcal{B}$  and  $t \not\equiv H$ . First assume that both  $t \sim_{\alpha} \tau_1$  and  $t \sim_{\beta} \tau_2$  are obtained by rule (F'). Hence  $\tau_1 = \tau_1^1 \to \tau_1^2$ ,  $\tau_2 = \tau_2^1 \to \tau_2^2$ , and  $t \equiv \lambda f. \exists t_1 (\lambda x. t_2 [z/fx])$  where  $t_1 \sim_{<\alpha} \tau_1^1$ ,  $\lambda z. t_2 \sim_{<\alpha} \tau_1^2$ ,  $t_1 \sim_{<\beta} \tau_2^1$  and  $\lambda z. t_2 \sim_{<\beta} \tau_2^2$ . By the IH we obtain  $\tau_1^1 = \tau_2^1$  and  $\tau_1^2 = \tau_2^2$ . Hence  $\tau_1 = \tau_2$ . If one of  $t \sim_{\alpha} \tau_1$  or  $t \sim_{\beta} \tau_2$  is obtained by (F $\omega$ ) and the other by (F), (F') or (F''), or one by (F) and the other by (F'), or both are obtained by (F), etc., then the argument is similar. If one is obtained by (K $\omega$ ) and the other by (K $\varepsilon$ ), then the claim follows from parts (2) and (3) of the IH. The only other possibility is, without loss of generality, when  $t \sim_{\alpha} \tau_1$  is obtained by (K $\omega$ ) or (K $\varepsilon$ ) and  $t \sim_{\beta} \tau_2$  by (F), (F'), (F'') or (F $\omega$ ). Then  $t \equiv Kt'$ . So by Lemma 4.2.4 we have  $\tau_1, \tau_2 \in \{\omega, \varepsilon\}$ . For instance, suppose  $\tau_1 = \omega$  and  $\tau_2 = \varepsilon$ . By (6) and its dual, which we have already verified in this inductive step, for all  $t_3$  we have  $t_3 \sim_{<\alpha} \top$  and  $t_3 \sim_{<\beta} \bot$ . By parts (1) and (2) of the IH this implies the existence of  $t_4$  such that  $t_4 \succ_{<\alpha} \top$  and  $t_4 \succ_{<\beta} \bot$ , which contradicts part (3) of the IH.

It remains to verify (3). If  $\tau \in \mathcal{B}$  then this is obvious. Suppose  $\tau = \tau_1 \to \tau_2 \in \mathcal{T}_1$ . Note that for all  $t_1 \in \mathbb{T}_{\tau_1}$  we have  $\mathcal{F}(\rho_1)(t_1) \equiv \mathcal{F}(\rho_2)(t_1)$ . This follows from the definition of  $\succ_{\alpha}$  for  $\tau = \tau_1 \to \tau_2 \in \mathcal{T}_1$ , from parts (1), (2) and (3) of the IH, and from the fact that canonical terms are in normal form. Now, if  $\tau_1 = \omega$  then  $\rho_1 \equiv \lambda x. \rho_1'$  and  $\rho_2 \equiv \lambda x. \rho_2'$ . Thus for any  $t_1$  we have  $\rho_1' \equiv \mathcal{F}(\rho_1)(t_1) \equiv \mathcal{F}(\rho_2)(t_1) \equiv \rho_2'$ , so  $\rho_1 \equiv \rho_2$ . If  $\tau_1 \neq \omega$  then the claim is immediate, because  $\mathbb{T}_{\tau_1 \to \tau_2} = \Sigma_{\tau_1 \to \tau_2}$  for  $\tau_1 \neq \omega$  was defined to contain exactly one constant for every function from  $\mathbb{T}_{\tau_1}$  to  $\mathbb{T}_{\tau_2}$ .

The last remaining case is  $\tau = o$ . Thus, suppose  $t \succ_{\alpha} \top$  and  $t \succ_{\beta} \bot$ . It is easily seen that this is possible only when the conditions  $(\Xi^{\top})$  and  $(\Xi^{\bot})$  are satisfied. So we have  $t \equiv \Xi t_1 t_2$  and there exists  $\tau_1$  such that  $t_1 \sim_{\alpha} \tau_1$  and for all  $t' \in \mathbb{T}_{\tau_1}$  we have  $t_2 t' \leadsto_{<\alpha} \top$ . There also exists  $\tau_2$  and  $t_3 \in \mathbb{T}_{\tau_2}$  such that  $t_1 \sim_{\beta} \tau_2$  and  $t_2 t_3 \leadsto_{<\beta} \bot$ . But by (5) we have  $\tau_1 = \tau_2$ . Hence  $t_2 t_3 \leadsto_{<\alpha} \top$  and  $t_2 t_3 \leadsto_{<\beta} \bot$ , which contradicts the inductive hypothesis.

**Corollary 4.2.6.** If  $t = \leq_{\alpha} t'$  then  $t \leadsto_{\alpha} \top$  is equivalent to  $t' \leadsto_{\alpha} \top$ .

*Proof.* Follows from conditions (1) and (2) in Lemma 4.2.5.

**Corollary 4.2.7.** If  $t \leadsto_{\alpha} \rho_1$  and  $t \leadsto_{\alpha} \rho_2$  where  $\rho_1$ ,  $\rho_2$  are canonical terms with the same canonical type, then  $\rho_1 \equiv \rho_2$ .

*Proof.* Follows from conditions (1)-(3) in Lemma 4.2.5.

**Lemma 4.2.8.** Let  $t_1$  and  $t_2$  be terms. If for all terms  $t_0$  we have  $t_1t_0 = \leq_{\alpha} t_2t_0$  then  $t_1 = \leq_{\alpha} t_2$ . In particular, the combinatory algebra of  $\mathcal{M}$ , as defined in Definition 4.1.11, is extensional.

*Proof.* If  $t_1t_0 = \le \alpha t_2t_0$  for all terms  $t_0$  then in particular  $t_1x = \le \alpha t_2x$  where x is variable which does not occur in  $t_1$  and  $t_2$ . Hence  $t_1 \not\leftarrow \lambda x.t_1x = \le \alpha \lambda x.t_2x \rightarrow_{\eta} t_2$ . Therefore  $t_1 = \le \alpha t_2$ .

The rank of a type  $\tau$ , denoted rank( $\tau$ ), is defined as follows. If  $\tau \in \mathcal{B} \cup \{o, \omega, \varepsilon\}$  then rank( $\tau$ ) = 1. Otherwise  $\tau = \tau_1 \to \tau_2 \in \mathcal{T}_1$  and we set rank( $\tau$ ) = max{rank( $\tau_1$ ) + 1, rank( $\tau_2$ )}. By the rank of a canonical term we mean the rank of its canonical type.

We write  $t \gg_{\alpha} t'$  if there exists an n-ary context C, terms  $t_1, \ldots, t_n$ , and canonical terms  $\rho_1, \ldots, \rho_n$ , such that  $t_i \succ_{\alpha} \rho_i$  for  $i = 1, \ldots, n$ ,  $t \equiv C[t_1, \ldots, t_n]$  and  $t' \equiv C[\rho_1, \ldots, \rho_n]$ . If the maximal rank of  $\rho_1, \ldots, \rho_n$  is at most k then we write  $t \gg_{\alpha}^{< k} t'$ , and if it is less than k we write  $t \gg_{\alpha}^{< k} t'$ .

Recall that whenever we write  $C[t_1, \ldots, t_n]$  we assume that the free variables of  $t_1, \ldots, t_n$  do not become bound in  $C[t_1, \ldots, t_n]$ .

**Lemma 4.2.9.** If  $t \succ_{\alpha} \rho$  and  $x_1, \ldots, x_n \notin FV(t)$  then  $\lambda x_1 \ldots x_k . t \succ_{\alpha+k} \lambda x_1 \ldots x_k . \rho$ .

*Proof.* Easy induction on k.

**Lemma 4.2.10.** If  $t \gg^n Fr'_1r'_2$  then  $t \equiv Fr_1r_2$  with  $r_1 \gg^n r'_1$  and  $r_2 \gg^n r'_2$ .

*Proof.* This follows from  $Fr'_1r'_2 \equiv \lambda f.\Xi r'_1(\lambda x.r'_2(fx))$  and from the fact that canonical terms are closed and do not contain  $\Xi$ .

**Lemma 4.2.11.** If  $t \gg^n \lambda f. \exists r'_1(\lambda x. r'_2[z/fx])$  with  $x, f \notin FV(r'_2)$  then one of the following holds:

- $t \equiv \lambda f \cdot \Xi r_1(\lambda x \cdot r_2[z/fx]), x, f \notin FV(r_2), r_1 \gg^n r'_1 \text{ and } r_2 \gg^n r'_2, \text{ or } r'_2 = r'_1 \text{ and } r_2 \approx^n r'_2 \text{ or } r'_2 = r'_2 \text{ or } r'_2 =$
- $t \equiv \lambda f. \exists r_1 r_2, z \notin FV(r'_2), r_1 \gg^n r'_1 \text{ and } r_2 \gg^n \lambda z. r'_2.$

Proof. Let  $q \equiv \lambda x.r_2'[z/fx]$ . Since  $t \gg^n \lambda f.\Xi r_1'q$  there exist contexts  $C_1, C_2$ , terms  $t_1, \ldots, t_k$ , and canonical terms  $\rho_1, \ldots, \rho_k$ , such that  $t_i \succ_{\alpha} \rho_i$  for  $i = 1, \ldots, k$ ,  $t \equiv \lambda f.\Xi C_1[t_1, \ldots, t_k]C_2[t_1, \ldots, t_k]$ ,  $r_1' \equiv C_1[\rho_1, \ldots, \rho_k]$  and  $q \equiv C_2[\rho_1, \ldots, \rho_k]$ . We take  $r_1 \equiv C_1[t_1, \ldots, t_k]$ . If  $C_2 \equiv \lambda x.(C_2')[z/(fx)]$ , then  $C_2'[\rho_1, \ldots, \rho_k] \equiv r_2'$  and we take  $r_2 \equiv C_2'[t_1, \ldots, t_k]$ . Otherwise  $z \notin FV(r_2')$  and  $\lambda x.r_2' \equiv \rho_i$  for some  $1 \leq i \leq k$ . Then  $C_2[t_1, \ldots, t_k] \equiv t_i \succ \rho_i$  and the second point in the statement of the lemma holds.  $\square$ 

Recall that we use the notations  $R, \succ, \leadsto, \gg$ , etc. without subscripts to denote  $R_{\zeta}, \succ_{\zeta}, \leadsto_{\zeta}, \gg_{\zeta}$ , etc., where  $\zeta$  is the ordinal introduced just before Definition 4.1.11. For this ordinal we have  $\succ_{\zeta} = \succ_{<\zeta}$ ,  $R_{\zeta} = R_{<\zeta}$ , etc.

**Lemma 4.2.12.** If  $t_1$ ,  $t_2$ ,  $t_3$  are terms,  $\rho$  is a canonical term, and  $\tau$  is a type, then for every ordinal  $\alpha$  and every natural number n the following conditions hold:

- (1) if  $t_1 \gg^n t_2 \succ_{\alpha} \rho$  then  $t_1 \succ \rho$ ,
- (2) if  $t_1 \gg^n t_2 \sim_{\alpha} \tau$  then  $t_1 \sim \tau$ ,
- (3) if  $t_1 \gg^n t_2 \twoheadrightarrow_{<\alpha} t_2'$  then  $t_1 \twoheadrightarrow_R t_1' \gg^n t_2'$ .

*Proof.* Induction on pairs  $\langle n, \alpha \rangle$  ordered lexicographically, i.e.,  $\langle n_1, \alpha_1 \rangle < \langle n_2, \alpha_2 \rangle$  iff  $n_1 < n_2$ , or  $n_1 = n_2$  and  $\alpha_1 < \alpha_2$ .

First we verify condition (2). Suppose  $t_1 \gg^n t_2 \sim_{\alpha} \tau$ . If  $t_2 \sim_{\alpha} \tau$  is obtained by rule (A) or (H) then  $t_2 \equiv A_{\tau}$  for  $\tau \in \mathcal{B}$  or  $t_2 \equiv H$ , so  $t_1 \equiv t_2$  and the claim is obvious.

If  $t_2 \sim_{\alpha} \tau$  is obtained by rule  $(K\omega)$  or  $(K\varepsilon)$  then  $t_2 \equiv Kt'_2$ ,  $\tau \in \{\omega, \varepsilon\}$  and  $t'_2 \sim_{<\alpha} c$  where  $c \in \{\top, \bot\}$ , i.e.,  $t'_2 \twoheadrightarrow_{<\alpha} t''_2 \succ_{<\alpha} c$  for some  $t''_2$ . Hence  $t_1 \equiv Kt'_1 \gg^n Kt'_2 \equiv t_2$ , and thus  $t'_1 \gg^n_{\zeta} t'_2 \twoheadrightarrow_{<\alpha} t''_2 \succ_{<\alpha} c$ . By part (3) the IH there exists  $t''_1$  such that  $t'_1 \twoheadrightarrow_R t''_1 \gg^n t''_2 \succ_{<\alpha} c$ . By part (1) of the IH we obtain  $t'_1 \leadsto c$ . Hence  $t_1 \sim \tau$ .

If  $t_2 \sim_{\alpha} \tau$  is obtained by rule  $(F\omega)$  then  $\tau = \omega$  and  $t_2 \equiv \lambda f. \exists r_1' r_2'$ . Then we must have  $t_1 \equiv \lambda f. \exists r_1 r_2 \gg^n \lambda f. \exists r_1' r_2'$  where  $r_1 \gg^n r_1' \sim_{<\alpha} \varepsilon$ . So by part (2) of the inductive hypothesis  $r_1 \sim \varepsilon$ . Therefore  $t_1 \sim \omega = \tau$  by rule  $(F\omega)$ . If  $t_2 \sim_{\alpha} \tau$  is obtained by rule  $(F\omega)$  then the argument is similar.

If  $t_2 \sim_{\alpha} \tau$  is obtained by rule (F) then  $t_2 \equiv Fr'_1r'_2$  and by Lemma 4.2.10 we obtain  $t_1 \equiv Fr_1r_2 \gg^n Fr'_1r'_2$  where  $r_1 \gg^n r'_1$  and  $r_2 \gg^n r'_2$ . We have  $\tau = \tau_1 \to \tau_2$ ,  $r_1 \gg^n r'_1 \sim_{<\alpha} \tau_1$  and  $r_2 \gg^n r'_2 \sim_{<\alpha} \tau_2$ . By part (2) of the IH we obtain  $r_1 \sim \tau_1$  and  $r_2 \sim \tau_2$ . Therefore  $t_1 \equiv Fr_1r_2 \sim \tau$  by rule (F).

If  $t_2 \sim_{\alpha} \tau$  is obtained by rule (F') then  $t_2 \equiv \lambda x. \exists r'_1(\lambda x. r'_2[z/fx]), \tau = \tau_1 \rightarrow \tau_2, r'_1 \sim_{<\alpha} \tau_1$  and  $\lambda z. r'_2 \sim_{<\alpha} \tau_2$ . By Lemma 4.2.11 there are two cases.

- $t_1 \equiv \lambda f. \exists r_1(\lambda x. r_2[z/fx]), \ x, f \notin FV(r_2), \ r_1 \gg^n r'_1 \text{ and } r_2 \gg^n r'_2.$  Then  $r_2 \gg^n r'_1 \sim_{<\alpha} \tau_1$  and  $\lambda z. r_2 \gg^n \lambda z. r'_2 \sim_{<\alpha} \tau_2.$  By part (2) of the IH we obtain  $r_1 \sim \tau_1$  and  $\lambda z. r_2 \sim \tau_2.$  Therefore  $t_1 \equiv Fr_1r_2 \sim \tau$  by rule (F').
- $t_1 \equiv \lambda f. \exists r_1 r_2, z \notin FV(r_2'), r_1 \gg^n r_1'$  and  $r_2 \gg^n \lambda z. r_2'$ . By part (2) of the IH we obtain  $r_1 \sim \tau_1$  and  $r_2 \sim \tau_2$ . Since  $Kr_2' \sim_{<\alpha} \tau_2$ , by Lemma 4.2.4 we have  $\tau_2 \in \{\omega, \varepsilon\}$ . Therefore  $t_1 \equiv Fr_1 r_2 \sim \tau$  by rule (F").

The remaining case is when  $t_2 \sim_{\alpha} \tau$  is obtained by rule (F"). Then  $t_2 \equiv \lambda f. \exists r_1' r_2', t_1 \equiv \lambda f. \exists r_1 r_2, r_1 \gg^n r_1' \sim_{<\alpha} \tau_1$  and  $r_2 \gg^n r_2' \sim_{<\alpha} \tau_2 \in \{\omega, \varepsilon\}$ . By part (2) of the IH we obtain  $r_2 \sim \tau_1$  and  $r_1 \sim \tau_2$ . Hence  $t_1 \sim \tau_1 \to \tau_2$ .

Now we verify condition (1). If  $t_2 \equiv \rho$  then  $t_1 \gg \rho$ . By (1) in Fact 4.1.3 we have  $\rho \equiv \lambda x_1 \dots x_n.c$ , so by definition of  $\gg$ , there exist a unary context C, a term t', and a canonical term  $\rho'$  such that  $t_1 \equiv C[t']$ ,  $\rho \equiv C[\rho']$  and  $t' \succ \rho'$ . If  $C \equiv \rho$  then the claim is obvious. Otherwise  $C \equiv \lambda x_1 \dots x_k.\Box$  where  $k \leq n$ ,  $\rho' \in \mathbb{T}_{\tau}$ , and  $\rho \in \mathbb{T}_{\omega^k \to \tau}$ , by (2) in Fact 4.1.3. By Lemma 4.2.9 we obtain  $t_1 \equiv C[t'] \equiv \lambda x_1 \dots x_k.t' \succ \lambda x_1 \dots x_k.\rho' \equiv C[\rho'] \equiv \rho$ .

Next assume that  $\rho \in \mathbb{T}_{\tau}$  where  $\tau = \tau_1 \to \tau_2 \in \mathcal{T}_1$ . Thus for all  $t_3 \in \mathbb{T}_{\tau_1}$  there exists  $t_2'$  such that  $t_2t_3 \twoheadrightarrow_{<\alpha} t_2' \succ_{<\alpha} \mathcal{F}(\rho)(t_3)$ . Then obviously  $t_1t_3 \gg^n t_2t_3 \twoheadrightarrow_{<\alpha} t_2'$ , so by part (3) of the inductive hypothesis there exists  $t_1'$  such that  $t_1t_3 \twoheadrightarrow_R t_1' \gg^n t_2' \succ_{<\alpha} \mathcal{F}(\rho)(t_3)$ . Using part (1) of the IH we obtain  $t_1t_3 \twoheadrightarrow_R t_1' \succ \mathcal{F}(\rho)(t_3)$ . This implies  $t_1 \succ \rho$ .

The remaining case to check is  $\rho \in \mathbb{T}_o$ . Suppose  $\rho \equiv \top$ , so  $t_1 \gg^n t_2 \succ_{\alpha} \top$ . If  $\rho \equiv \bot$ , i.e.,  $t_1 \gg^n t_2 \succ_{\alpha} \bot$ , then proof is similar. We consider all possible forms of  $t_2$  according to the definition of  $t_2 \succ_{\alpha} \top$ . If  $t_2 \equiv A_{\tau}c$  for  $\tau \in \mathcal{B}$  then  $t_1 \equiv t_2$ , because if c is a canonical constant of a base type  $\tau$  then the condition  $t \succ c$  implies  $t \equiv c$ . If  $t_2 \equiv \top$  then  $t_1 \succ t_2 \equiv \top$  and the claim is obvious. Suppose condition  $(\Xi^\top)$  in the definition of  $t_2 \succ_{\alpha} \top$  is satisfied. Then  $t_1 \equiv \Xi r_1 r_2 \gg^n \Xi r_1' r_2' \equiv t_2$  where  $r_1 \gg^n r_1'$  and  $r_2 \gg^n r_2'$ . By definition of  $\succ_{\alpha}$  there exists  $\tau$  such that  $r_1' \sim_{\alpha} \tau$  and for all  $t_3 \in \mathbb{T}_{\tau}$  we have  $r_2' t_3 \leadsto_{<\alpha} \top$ , i.e.,  $r_2' t_3 \twoheadrightarrow_{<\alpha} t_3' \succ_{<\alpha} \top$ . Since  $r_1 \gg^n r_1' \sim_{\alpha} \tau$  we conclude that  $r_1 \sim \tau$  by condition (2) which we have already verified in this inductive step. Because for all  $t_3 \in \mathbb{T}_{\tau}$  we have  $r_2 t_3 \gg^n r_2' t_3 \twoheadrightarrow_{<\alpha} t_3' \succ_{<\alpha} \top$ , so by part (3) of the IH for all  $t_3 \in \mathbb{T}_{\tau}$  there exists  $t_3''$  such that  $r_2 t_3 \gg_n t_3'' \gg^n t_3' \succ_{<\alpha} \top$ . Hence  $r_2 t_3 \leadsto \top$  by applying part (1) of the IH. Therefore  $t_1 \succ \top$  by the definition of  $\succ$ . Finally, assume the condition  $(L^\top)$  in the definition of  $t_2 \succ_{\alpha} \top$  is satisfied. Then  $t_2 \equiv L t_2'$  with  $t_2' \sim_{\alpha} \tau$  for some type  $\tau$ . Since  $t_1 \gg^n t_2$  we must have  $t_1 \equiv L t_1'$  with  $t_1' \gg^n t_2' \sim_{\alpha} \tau$ . By condition (2), which we have already verified in this inductive step, we obtain  $t_1' \sim \tau$ . Therefore  $t_1 \equiv L t_1' \succ \top$ .

It remains to prove (3). It suffices to consider a single reduction step, i.e., to show that  $t_1 \gg^n t_2 \to_{\leq \alpha} t_2'$  implies  $t_1 \twoheadrightarrow_R t_1' \gg^n t_2'$ . We have  $t_1 \equiv C[r_1, \dots, r_k]$  and  $t_2 \equiv C[\rho_1, \dots, \rho_k]$  where  $r_i \succ \rho_i$  and rank $(\rho_i) \leq n$ , for  $i = 1, \dots, k$ . Denote by  $C_0[\rho_1, \dots, \rho_k]$  the contracted redex in  $t_2$ , where the boxes in  $C_0$  correspond to appropriate boxes in C. By  $C_e$  we denote the surrounding context satisfying  $C \equiv C_e[C_0, \square_1, \dots, \square_k]$ . It follows from the definition of  $R_\alpha$  that there are four possibilities:  $C_0 \equiv \lambda x. C_1 x$  where  $x \notin FV(C_1)$ ,  $C_0 \equiv (\lambda x. C_1)C_2$ ,  $C_0 \equiv c_0C_1$  for  $c_0 \in \Sigma_{\tau_1 \to \tau_2}$ , or  $C_0 \equiv \square_i C_1$  for some  $1 \leq i \leq k$ . In the first two cases we have  $t_2's \equiv C_e[C_0'[\rho_1, \dots, \rho_k], \rho_1, \dots, \rho_k]$  where  $C_0 \to_{\leq \alpha} C_0'$ , so we may just take  $t_1' \equiv C_e[C_0'[r_1, \dots, r_k], r_1, \dots, r_k]$ .

Otherwise the contraction in  $t_2$  produces some canonical term  $\rho$ , i.e.,  $C_0[\rho_1, \ldots, \rho_k] \to_{\leq \alpha} \rho$ . It suffices to prove:

(\*) there exists t such that  $C_0[r_1,\ldots,r_k] \twoheadrightarrow_R t \succ \rho$ , and if  $t \not\equiv \rho$  then  $\operatorname{rank}(\rho) \leq n$ .

Indeed, if  $(\star)$  holds then simply take  $t'_1 \equiv C_e[t, r_1, \ldots, r_k]$ . We have  $t_1 \equiv C_e[C_0[r_1, \ldots, r_k], r_1, \ldots, r_k] \twoheadrightarrow_R C_e[t, r_1, \ldots, r_k] \equiv t'_1$  and  $t'_2 \equiv C_e[\rho, r_1, \ldots, r_k]$ . Now it is easy to see that  $t'_1 \gg^n t'_2$ : if  $t \equiv \rho$  then we take  $C_e[\rho, \square_1, \ldots, \square_k]$  as the context required by the definition of  $\gg^n$ , otherwise we take  $C_e$  noting that  $t \succ \rho$  and rank $(\rho) \leq n$ .

If  $C_0 \equiv c_0 C_1$  then  $C_1[\rho_1, \ldots, \rho_k] \succ_{<\alpha} \rho'$  where  $\mathcal{F}(c)(\rho') \equiv \rho$ . We conclude  $C_1[r_1, \ldots, r_k] \succ \rho'$  by part (1) of the IH and the fact that  $C_1[r_1, \ldots, r_k] \gg^n C_1[\rho_1, \ldots, \rho_k]$ . Therefore  $C_0[r_1, \ldots, r_k] \equiv cC_1[r_1, \ldots, r_k] \rightarrow_R \rho$  and we are done.

Suppose  $C_0 \equiv \Box_i C_1$  where  $1 \leq i \leq k$ . First assume that  $\rho_i$  is a canonical constant of type  $\tau_1 \to \tau_2$ . As in the previous paragraph we have  $C_1[\rho_1, \ldots, \rho_k] \succ_{<\alpha} \rho'$  where  $\mathcal{F}(\rho_i)(\rho') \equiv \rho$ , so  $C_1[r_1, \ldots, r_k] \succ \rho'$  by part (1) of the IH. Obviously  $\operatorname{rank}(\rho) = \operatorname{rank}(\tau_2) \leq \operatorname{rank}(\tau_1 \to \tau_2) = \operatorname{rank}(\rho_i) \leq n$  and  $\operatorname{rank}(\rho') = \operatorname{rank}(\tau_1) < \operatorname{rank}(\tau_1) + 1 \leq \operatorname{rank}(\tau_1 \to \tau_2) = \operatorname{rank}(\rho_i) \leq n$ . Let  $r \equiv C_1[r_1, \ldots, r_k]$ . We have  $r \succ \rho'$  and  $\operatorname{rank}(\rho') < n$ , so  $r_i r \gg^{< n} r_i \rho'$  where the context required by the definition of  $\gg^{< n}$  is  $r_i \Box$ . Since  $r_i \succ \rho_i$  and the canonical type of  $\rho_i$  is a function type, we conclude by definition of  $\succ$  that  $r_i \rho' \leadsto \mathcal{F}(\rho_i)(\rho') \equiv \rho$ . Note that we may have  $r_i \equiv \rho_i$ , but then the condition  $r_i \rho' \leadsto \rho$  is satisfied anyway, by definition of  $\mathcal{F}$ . Therefore there exists t' such that  $r_i r \gg^{< n} r_i \rho' \twoheadrightarrow_R t' \succ \rho$ . By part (3) of the inductive hypothesis there exists t such that  $r_i r \gg_{\zeta}^{< n} t' \succ \rho$ . Applying part (1) of the IH we obtain  $t \succ \rho$ . Hence  $C_0[r_1, \ldots, r_k] \equiv r_i C_1[r_1, \ldots, r_k] \equiv r_i r \twoheadrightarrow_R t \succ \rho$  where  $\operatorname{rank}(\rho) \leq n$ , so  $(\star)$  holds.

Now suppose that  $\rho_i \equiv \lambda x_1 \dots x_m.c$  for m > 0. We have  $C_0[r_1, \dots, r_k] \equiv r_i C_1[r_1, \dots, r_k]$  with  $r_i \succ \rho_i$ . By the definition of  $\succ$  we conclude that there exists t such that  $r_i C_1[r_1, \dots, r_k] \twoheadrightarrow_R t \succ \lambda x_2 \dots x_m.c \equiv \rho$ . Obviously we also have  $\operatorname{rank}(\rho) \leq \operatorname{rank}(\rho_i) \leq n$ . Thus  $(\star)$  holds.

### Corollary 4.2.13. If $t \succ \rho_1$ and $C[\rho_1] \leadsto \rho_2$ , then $C[t] \leadsto \rho_2$ .

The above corollary states that our definition of  $\succ$  is correct. If  $t \succ \rho_1$  then t behaves exactly like  $\rho_1$  in every context C such that  $C[\rho_1]$  has an "interesting" interpretation.

The following final lemmas show that the conditions on  $\mathcal{T}$  required for a classical illative model are satisfied by  $\mathcal{M}$ .

#### **Lemma 4.2.14.** If $Ht \leadsto_{\alpha} \top$ then $t \leadsto_{<\alpha} \top$ or $t \leadsto_{<\alpha} \bot$ .

Proof. Keeping in mind the convention regarding the meaning of Ht, we note that if  $Ht \leadsto_{\alpha} \top$  then  $Ht \leadsto_{\leq \alpha} L(Kt') \succ_{\alpha} \top$  where  $t \leadsto_{\leq \alpha} t'$ . Thus it suffices to show that for any term t, if  $L(Kt) \succ_{\alpha} \top$  then  $t \leadsto_{\alpha} \top$  or  $t \leadsto_{\alpha+1} \bot$ . Assume  $L(Kt) \succ_{\alpha} \top$ . Then the condition  $(L^{\top})$  must hold, so  $Kt \leadsto_{\alpha} \tau$  for some type  $\tau$ . By Lemma 4.2.4 we have  $\tau = \omega$  or  $\tau = \varepsilon$ . Assume  $\tau = \omega$ . The other case is analogous. By (6) in Lemma 4.2.5 we have  $t \leadsto_{\leq \alpha} \top$ . Since  $t t \leadsto_{\beta} t$ , by Corollary 4.2.6 we have  $t \leadsto_{\leq \alpha} \top$ .

**Lemma 4.2.15.** If  $\rho \in \mathbb{T}_{\tau}$  and  $\tau \neq \omega$  then  $\rho \succ \rho$ .

*Proof.* Induction on the size of  $\tau$ .

#### **Lemma 4.2.16.** If $t \sim_{\alpha} \tau$ then for all $t_0 \in \mathbb{T}_{\tau}$ we have $tt_0 \leadsto \top$ .

Proof. Induction on  $\alpha$ . If  $t \sim_{\alpha} \tau$  is obtained by rule (A), (H), (K $\omega$ ) or (K $\varepsilon$ ), then the claim is obvious. If  $\tau = \omega$  then the claim follows from (6) in Lemma 4.2.5. If  $\tau = \varepsilon$  then the claim is also obvious. So we may assume  $\tau = \tau_1 \to \tau_2 \notin \{\omega, \varepsilon\}$ . Then the only remaining cases are when  $t \sim \tau$  is obtained by (F) or (F'). Then  $t =_{\beta} Ft_1t_2$ ,  $\tau = \tau_1 \to \tau_2$ ,  $t_1 \sim_{<\alpha} \tau_1$  and  $t_2 \sim_{<\alpha} \tau_2$ . Suppose  $t_0 \in \mathbb{T}_{\tau_1 \to \tau_2}$ . Then for all  $r_1 \in \mathbb{T}_{\tau_1}$  there exists  $r_2 \in \mathbb{T}_{\tau_2}$  such that  $t_0r_1 \twoheadrightarrow_R r_2$ , by Definition 4.1.8, because if  $\tau_1 \neq \omega$  then  $r_1 \succ r_1$  by Lemma 4.2.15. Also, we have  $Ft_1t_2t_0 =_{\leq 0} \Xi t_1 \lambda y. t_2(t_0y)$ . Hence  $(\lambda y. t_2(t_0y))r_1 \twoheadrightarrow_R t_2r_2$ . Because  $t_2 \sim_{<\alpha} \tau_2$ , we have  $t_2r_2 \leadsto \top$  by the IH, so  $(\lambda y. t_2(t_0y))r_1 \leadsto \top$ . Therefore  $\Xi t_1 \lambda y. t_2(t_0y) \succ \top$  by condition  $(\Xi_i^{\top})$ . Hence, by Corollary 4.2.6, we obtain  $Ft_1t_2t' \leadsto \top$ .

**Lemma 4.2.17.** If  $t_1 \sim_{\alpha} \tau$ ,  $\tau \neq \omega$ ,  $\tau \neq \varepsilon$  and  $t_1 t_2 \rightsquigarrow \top$ , then  $t_2 \rightsquigarrow \rho$  for some  $\rho \in \mathbb{T}_{\tau}$ .

*Proof.* Induction on  $\alpha$ . If  $t_1 \sim_{\alpha} \tau$  is obtained by rule (A) then  $t_1 \equiv A_{\tau}$  for  $\tau \in \mathcal{B}$ , and  $A_{\tau}t_2 \twoheadrightarrow_R t' \succ_{\top}$ . So  $t' \equiv A_{\tau}t'_2$  where  $t_2 \twoheadrightarrow_R t'_2$ . By Definition 4.1.8 we have  $t'_2 \equiv c$  for  $c \in \mathbb{T}_{\tau}$ . Hence  $t_2 \leadsto c$ . If  $t_1 \sim_{\alpha} \tau$  is obtained by rule (H) then  $t_1 \equiv H$  and  $t_2 \leadsto c \in \{\top, \bot\}$  by Lemma 4.2.14.

The only remaining case is when  $t_1 \sim_{\alpha} \tau = \tau_1 \to \tau_2$  is obtained by (F) or (F'). Then  $t_1 =_{\beta} Fr_1r_2 \sim_{\alpha} \tau = \tau_1 \to \tau_2$  where  $r_1 \sim_{<\alpha} \tau_1$ ,  $r_2 \sim_{<\alpha} \tau_2$ . We may assume  $\tau_1 \neq \varepsilon$ ,  $\tau_2 \neq \omega$  and  $\tau_2 \neq \varepsilon$ , since otherwise  $\tau = \omega$  or  $\tau = \varepsilon$ . By Corollary 4.2.6 we have  $\Xi r_1 \lambda y.r_2(t_2y) \leadsto \top$ , so  $\Xi r_1'r_2' \succ \top$  where  $r_1 \twoheadrightarrow_R r_1'$ ,  $\lambda y.r_2(t_2y) \twoheadrightarrow_R r_2'$ . By inspecting Definition 4.1.8 we see that the only possible way for  $\Xi r_1'r_2' \succ \top$  to hold is when condition ( $\Xi^{\top}$ ) is satisfied, i.e., there exists  $\tau'$  such that  $r_1' \sim \tau'$  and for all  $t_3 \in \mathbb{T}_{\tau'}$  we have  $r_2't_3 \leadsto \top$ . By (4) in Lemma 4.2.5 we have  $r_1' \sim_{<\alpha} \tau_1$ , so it follows from (5) in Lemma 4.2.5 that  $\tau' = \tau_1$ . Therefore for any  $t_3 \in \mathbb{T}_{\tau_1}$  we have  $r_2't_3 \leadsto \top$ . Since  $r_2(t_2t_3) =_{\leq 0} (\lambda y.r_2(t_2y))t_3 \twoheadrightarrow_R r_2't_3$ , we obtain by Corollary 4.2.6 that  $r_2(t_2t_3) \leadsto \top$  for any  $t_3 \in \mathbb{T}_{\tau_1}$ . Because  $r_2 \sim_{<\alpha} \tau_2$  where  $\tau_2 \neq \omega$  and  $\tau_2 \neq \varepsilon$ , we conclude by the inductive hypothesis that the following condition holds:

(\*) for all  $t_3 \in \mathbb{T}_{\tau_1}$  there exists  $\rho_2 \in \mathbb{T}_{\tau_2}$  such that  $t_2 t_3 \rightsquigarrow \rho_2$ .

Note that  $\rho_2$  depends on  $t_3$ .

If  $\tau_1 \neq \omega$  then  $\mathbb{T}_{\tau_1 \to \tau_2}$  contains a constant for every set-theoretical function from  $\mathbb{T}_{\tau_1}$  to  $\mathbb{T}_{\tau_2}$ . In particular it contains a constant c such that for every  $\rho_1 \in \mathbb{T}_{\tau_1}$  we have  $\mathcal{F}(c)(\rho_1) \equiv \rho_2$  where  $\rho_2 \in \mathbb{T}_{\tau_2}$  is a term depending on  $\rho_1$  such that  $t_2\rho_1 \rightsquigarrow \rho_2$ . Such a  $\rho_2$  exists by  $(\star)$ . Therefore by definition of  $\succ$  we have  $t_2 \succ c \in \mathbb{T}_{\tau}$ .

If  $\tau_1 = \omega$  then it suffices to show that there exists a single  $\rho' \in \mathbb{T}_{\tau_2}$  such that for all  $t_3$  we have  $t_2t_3 \rightsquigarrow \rho'$ . Indeed, if this holds then  $t_2 \succ K\rho' \in \mathbb{T}_{\omega \to \tau_2} = \mathbb{T}_{\tau}$ . Let x be a variable. Obviously  $x \in \mathbb{T}_{\omega}$ , so by  $(\star)$  there exists  $\rho' \in \mathbb{T}_{\tau_2}$  such that  $t_2x \rightsquigarrow \rho'$ , i.e.,  $t_2x \twoheadrightarrow_R t' \succ \rho'$  for some term t'. Taking  $C \equiv t_2 \square$ , we conclude by condition (1) in Lemma 4.2.2 that  $t' \equiv C'[x]$  where  $C[t_3] \twoheadrightarrow_R C'[t_3]$  for any term r. By condition (2) in Lemma 4.2.2 we have  $C'[t_3] > \rho'$  for any term  $t_3$ . Therefore for any  $t_3$  there exists  $t_3'$ such that  $t_2t_3 \rightarrow_R t_3' \succ \rho'$ , i.e.,  $t_2t_3 \sim \rho'$ . This  $\rho'$  depends only on x, but not on  $t_3$ , so our claim has been established.

#### Lemma 4.2.18. The following conditions are satisfied.

- If  $Lt_1 \hookrightarrow \top$  and for all  $t_3$  such that  $t_1t_3 \hookrightarrow \top$  we have  $t_2t_3 \hookrightarrow \top$ , then  $\Xi t_1t_2 \hookrightarrow \top$ .
- If  $Lt_1 \rightsquigarrow \top$  and for all  $t_3$  such that  $t_1t_3 \rightsquigarrow \top$  we have  $H(t_2t_3) \rightsquigarrow \top$ , then  $H(\Xi t_1t_2) \rightsquigarrow \top$ .
- If  $Lt_1 \leadsto \top$ , and either  $Lt_2 \leadsto \top$  or there is no  $t_3$  such that  $t_1t_3 \leadsto \top$ , then  $L(Ft_1t_2) \leadsto \top$ .

*Proof.* Suppose  $Lt_1 \rightsquigarrow \top$ . By definitions we have  $t_1 \twoheadrightarrow_R t'_1 \sim \tau$  for some type  $\tau$ .

Assume that for all  $t_3$  such that  $t_1t_3 \leadsto \top$  we have  $t_2t_3 \leadsto \top$ . Let  $t_0 \in \mathbb{T}_{\tau}$ . Then by Lemma 4.2.16 we obtain  $t'_1t_0 \leadsto \top$ . Because  $t_1t_0 = t'_1t_0$ , by Corollary 4.2.6 we conclude  $t_1t_0 \leadsto \top$ . Then by assumption  $t_2t_0 \leadsto \top$ . Therefore by  $(\Xi_i^\top)$  we obtain  $\Xi t_1't_2 \succ \top$ . Hence  $\Xi t_1t_2 \leadsto \top$ .

Assume that for all  $t_3$  such that  $t_1t_3 \rightsquigarrow \top$  we have  $H(t_2t_3) \rightsquigarrow \top$ , so  $t_2t_3 \rightsquigarrow \top$  or  $t_2t_3 \rightsquigarrow \bot$  by Lemma 4.2.14. If for all  $t_3$  such that  $t_1t_3 \to \top$  we have  $t_2t_3 \to \top$ , then  $\Xi t_1t_2 \to \top$  by the previous paragraph. Otherwise using Lemma 4.2.16, Corollary 4.2.6 and  $(\Xi^{\perp})$  we may conclude  $\Xi t_1 t_2 \rightsquigarrow \bot$  by an argument analogous to the previous paragraph. In any case  $H(\Xi t_1 t_2) \leadsto \top$  by  $(L^{\top})$ , and  $(K\omega)$  or  $(K\varepsilon)$ .

Assume  $Lt_2 \rightsquigarrow \top$ . Then  $t_2 \twoheadrightarrow_R t_2' \sim \tau'$ . Then  $Ft_1't_2' \sim \tau \rightarrow \tau'$ , so  $L(Ft_1t_2) \twoheadrightarrow_R L(Ft_1't_2') \succ \top$ . Finally, assume there is no  $t_3$  such that  $t_1t_3 \rightsquigarrow \top$ . Then there is no  $t_3$  such that  $t_1't_2 \rightsquigarrow \top$ . By Lemma 4.2.17 and (6) in Lemma 4.2.5 we must have  $\tau = \varepsilon$ . Then  $L(Ft_1t_2) \rightsquigarrow \top$  by  $(F\omega)$ ,  $(L^{\top})$  and Corollary 4.2.6. 

### **Lemma 4.2.19.** If $\exists t_1 t_2 \leadsto \top$ then for all terms $t_3$ such that $t_1 t_3 \leadsto \top$ we have $t_2 t_3 \leadsto \top$ .

*Proof.* If  $\Xi t_1 t_2 \rightsquigarrow \top$  then  $\Xi t_1 t_2 \twoheadrightarrow_R \Xi t_1' t_2' \succ \top$  where  $t_1 \twoheadrightarrow_R t_1'$  and  $t_2 \twoheadrightarrow_R t_2'$ . The only possibility for  $\Xi t_1' t_2' \succ \top$  to hold is that condition  $(\Xi^\top)$  holds for  $\Xi t_1' t_2'$ . Thus  $t_1' \sim \tau$  for some type  $\tau$ . Suppose  $t_1t_3 \rightsquigarrow \top$ . By Corollary 4.2.6 we have  $t_1't_3 \rightsquigarrow \top$ . Because  $t_2t_3 \twoheadrightarrow_R t_2't_3$ , it suffices to show that  $t_2't_3 \rightsquigarrow \top$ . If  $\tau = \omega$  then this is obvious by definition of  $(\Xi^{\top})$ . We cannot have  $\tau = \varepsilon$ , since if  $t'_1 \sim \varepsilon$  then by (6) in Lemma 4.2.5 and by Corollary 4.2.7 there is no t such that  $t_1't \sim T$ . If  $t_1' \sim \tau \neq \omega$  and  $\tau \neq \varepsilon$ , then we use Lemma 4.2.17 to conclude that there exist  $t_3'$  and  $\rho \in \mathbb{T}_{\tau}$  such that  $t_3 \to_R t_3' \succ \rho$ . Because  $(\Xi_i^\top)$ holds for  $\Xi t_1't_2'$ ,  $t_1' \sim \tau$  and  $\rho \in \mathbb{T}_{\tau}$ , we have  $t_2'\rho \sim \top$ . Since  $t_3' \succ \rho$ , taking  $C \equiv t_2'\square$  we conclude by Corollary 4.2.13 that  $t_2't_3' \rightsquigarrow \top$ , so  $t_2't_3 \rightsquigarrow \top$ .

**Theorem 4.2.20.** The systems  $\mathcal{I}^c_{\omega}$  and  $\mathcal{I}_{\omega}$  are strongly consistent, i.e.,  $\Xi HI$  is not derivable in them.

*Proof.* We verify that the structure  $\mathcal{M}$  constructed in Definition 4.1.11 is a one-state classical illative model for  $\mathcal{I}_{\omega}^{c}$ . It follows from Lemma 4.2.8 that the combinatory algebra of  $\mathcal{M}$  is extensional. Corollary 4.2.6 implies that  $[t]_R \in \mathscr{T}$  is equivalent to  $t \leadsto \top$ . We need to check the conditions stated in Fact 3.8. Conditions (1), (3) and (4) follow from Lemma 4.2.18. Condition (2) follows from Lemma 4.2.19. Conditions (5), (6) and (7) follow from definitions.

It is also easy to see that  $\not\vdash_{\mathcal{M}} \Xi HI$ . Indeed, otherwise we would have  $\Xi HI \leadsto \top$ , which is possible only when  $(\Xi^{\top})$  is satisfied for  $\Xi HI$ . Thus  $H \sim \tau$  for some type  $\tau$ , and for all  $t \in \mathbb{T}_{\tau}$  we have  $It \leadsto \top$ , so  $t \rightsquigarrow \top$  by Corollary 4.2.6. It is easily verified by inspecting the definitions that we must have  $\tau = o$ . But then  $\perp \rightsquigarrow \top$  which is impossible by Corollary 4.2.7.

Therefore, by the soundness part of Theorem 3.6, the term  $\Xi HI$  is not derivable in  $\mathcal{I}_{\omega}^{c}$ , and hence neither in  $\mathcal{I}_{\omega}$ , which is a subsystem of  $\mathcal{I}_{\omega}^{c}$ .

## 5 The embedding

In this section a syntactic translation from the terms of PRED2<sub>0</sub> into the terms of  $\mathcal{I}_0$  is defined and proven complete for  $\mathcal{I}_0$ . The translation is a slight extension of that from [BBD93]. The method of the completeness proof is by model construction analogous to that in the previous section. Relinquishing quantification over predicates and restricting arguments of functions to base types allows us to significantly simplify this construction and to extend it to more than one state.

We use the notation  $\mathcal{T}$  for the set of types of PRED2<sub>0</sub>. Recall that  $\mathcal{T}$  is defined by the grammar  $\mathcal{T} ::= o \mid \mathcal{B} \mid \mathcal{B} \to \mathcal{T}$ , where  $\mathcal{B}$  is a specific set of base types. We assume that  $\mathcal{B}$  corresponds exactly to the base types used in the definition of  $\mathcal{I}_0$ . We fix a signature for PRED2<sub>0</sub>, and by  $\Sigma_{\tau}$  denote the set of constants of type  $\tau$  in this signature. We always assume that all variables of PRED2<sub>0</sub> are present in the set of variables of  $\mathcal{I}_0$ .

Recall that by  $\mathbb{T}(\Sigma)$  we denote the set of type-free lambda terms over a set of primitive constants  $\Sigma$ , which is assumed to contain  $\Xi$ , L and  $A_{\tau}$  for each  $\tau \in \mathcal{B}$ . We also assume that  $\Sigma$  contains every constant  $c \in \Sigma_{\tau}$  for any  $\tau \in \mathcal{T}$ . For the sake of uniformity, we will sometimes use the notation  $A_o$  for H. For every composite type  $\tau = \tau_1 \to \tau_2 \in \mathcal{T}$  we inductively define  $A_{\tau} = FA_{\tau_1}A_{\tau_2}$ . We use the same notational conventions concerning Kt, Ht, etc. as in Section 4.

**Definition 5.1.** We define inductively a map [-] from the terms of PRED2<sub>0</sub> to  $\mathbb{T}(\Sigma)$  as follows:

- $\lceil x \rceil = x$  for a variable x,
- $\lceil c \rceil = c$  for a constant c,
- $\lceil t_1 t_2 \rceil = \lceil t_1 \rceil \lceil t_2 \rceil$ ,
- $\lceil \varphi \supset \psi \rceil = \lceil \varphi \rceil \supset \lceil \psi \rceil$ ,
- $[\forall x.\varphi] = \Xi A_{\tau} \lambda x. [\varphi] \text{ for } x \in V_{\tau}.$

We extend the map to finite sets of formulas by defining  $\lceil \Delta \rceil$  to be the image of  $\lceil - \rceil$  on  $\Delta$ . We also define a mapping  $\Gamma$  from sets of formulas to subsets of  $\mathbb{T}(\Sigma)$ , which is intended to provide a context for a set of formulas. For a finite set of formulas  $\Delta$  we define  $\Gamma(\Delta)$  to contain the following:

- $A_{\tau}x$  for all  $x \in FV(\Delta)$  s.t.  $x \in V_{\tau}$ , and all types  $\tau$ ,
- $A_{\tau}c$  for all  $c \in \Sigma_{\tau}$ , and all types  $\tau$ ,
- $LA_{\tau}$  for all  $\tau \in \mathcal{B}$ ,
- $A_{\tau}y$  for all  $\tau \in \mathcal{B}$  and some  $y \in V_{\tau}$  such that  $y \notin FV(\Delta)$ .

**Lemma 5.2.** For any  $\tau \in \mathcal{T}$  and any  $\Delta$  there exists a term t such that  $\Gamma(\Delta) \vdash_{\mathcal{I}_0} A_{\tau}t$ .

*Proof.* First note that by a straightforward induction on the size of  $\tau$  we obtain  $\Gamma(\Delta) \vdash LA_{\tau}$  for any type  $\tau$ .

We prove the lemma by induction on the size of  $\tau$ . If  $\tau \in \mathcal{B}$  then  $A_{\tau}y \in \Gamma(\Delta)$  for some variable y. If  $\tau = o$  then notice that e.g.  $\vdash H(LH)$ . If  $\tau = \tau_1 \to \tau_2$  then we need to prove that  $\Gamma(\Delta) \vdash FA_{\tau_1}A_{\tau_2}t$  for some term t. Because  $\Gamma(\Delta) \vdash LA_{\tau_1}$ , it suffices to show that  $\Gamma(\Delta)$ ,  $A_{\tau_1}x \vdash A_{\tau_2}(tx)$  for some term t and some  $x \notin FV(\Gamma(\Delta), t)$ . By the inductive hypothesis there exists a term  $t_2$  such that  $\Gamma(\Delta) \vdash A_{\tau_2}t_2$ . So just take  $x \notin FV(\Gamma(\Delta), t_2)$  and  $t \equiv Kt_2$ .

**Theorem 5.3.** The embedding is sound, i.e.,  $\Delta \vdash_{PRED2_0} \varphi$  implies  $[\Delta], \Gamma(\Delta, \varphi) \vdash_{\mathcal{I}_0} [\varphi]$ .

Proof. Induction on the length of derivation of  $\Delta \vdash_{\text{PRED2}_0} \varphi$ , using Lemma 3.2. The only interesting case is with modus-ponens, as from the inductive hypothesis we may only directly derive the judgement  $\lceil \Delta \rceil, \Gamma(\Delta, \psi), \Gamma(\varphi) \vdash_{\mathcal{I}_0} \lceil \psi \rceil$ . To get rid of  $\Gamma(\varphi)$  on the left, we note that if  $t \in \Gamma(\varphi) \setminus \Gamma(\Delta, \psi)$  then  $t \equiv A_\tau x$  for  $x \in FV(\varphi) \setminus FV(\Delta, \psi)$ . Now, by Lemma 5.2 there exists t' such that  $\Gamma(\Delta, \psi) \vdash_{\mathcal{I}_0} \Lambda_\tau t'$ . It is not difficult to show by induction on the length of derivation that  $\lceil \Delta \rceil, \Gamma(\Delta, \psi), \Gamma(\varphi) \lceil x/t' \rceil \vdash_{\mathcal{I}_0} \lceil \psi \rceil$ , i.e., that we may change  $A_\tau x$  on the left to  $A_\tau t'$ . To eliminate  $A_\tau t'$  altogether, it remains to notice that if  $\Gamma, t_1 \vdash_{\mathcal{I}_0} t_2$  and  $\Gamma \vdash_{\mathcal{I}_0} t_1$  then  $\Gamma \vdash_{\mathcal{I}_0} t_2$ .

If we had extended our semantics for PRED2<sub>0</sub> a bit by allowing non-constant domains, then we could also give a relatively simple semantic proof by transforming any illative Kripke model for  $\mathcal{I}_0$  to a Kripke model for PRED2<sub>0</sub>, and appealing to the completeness part of Theorem 3.6.

The rest of this section is devoted to proving that the embedding is also complete.

Let  $\mathcal{N}$  be a Kripke model for PRED2<sub>0</sub>. We will now construct an illative Kripke model  $\mathcal{M}$  such that  $\mathcal{M}$  will "mirror"  $\mathcal{N}$ , i.e., exactly the translations of true statements in a state of  $\mathcal{N}$  will be true in the corresponding state of  $\mathcal{M}$ . This construction is the crucial step in the completeness proof. It is similar to the construction given in Section 4. For the rest of this section we assume a fixed  $\mathcal{N}$ .

We define a set of primitive constants  $\Sigma^+$  and the sets  $\Sigma_{\tau}$  of canonical constants of type  $\tau$ , just like in Definition 4.1.2, but restricting ourselves only to the types in  $\mathcal{T}$  (i.e. the types of PRED2<sub>0</sub>). Note that there is a bijection  $\delta_{\tau}$  between  $\Sigma_{\tau}$  and  $\mathcal{D}_{\tau}^{\mathcal{N}}$ . We often drop the subscript in  $\delta_{\tau}$ . We also include in  $\Sigma^+$  an infinite set  $\Sigma^{\nu}$  of external constants. Note that  $\Sigma^+$  is disjoint from the signature  $\Sigma$  of  $\mathcal{M}$  which we defined earlier. The terms over  $\Sigma$  form the syntax. The terms over  $\Sigma^+$  are used to build the model. To every constant  $c \in \Sigma$  corresponds exactly one constant  $c^+ \in \Sigma^+$  such that  $[\![c]\!]_{\mathcal{N}} = \delta(c^+)$ . This correspondence, however, need not be injective, as there may be another constant  $c' \in \Sigma$ ,  $c' \neq c$ , such that  $[\![c']\!]_{\mathcal{N}} = \delta(c^+)$ .

Let  $\mathcal{S}$  be the set of states of  $\mathcal{N}$ . By  $\top \in \Sigma_o$  we denote the constant such that  $\varsigma_{\mathcal{N}}(\delta(\top)) = \mathcal{S}$ , and by  $\bot \in \Sigma_o$  the constant such that  $\varsigma_{\mathcal{N}}(\delta(\bot)) = \emptyset$ . In what follows  $\rho$ ,  $\rho'$ , etc., stand for  $\top$  or  $\bot$ . Note that  $\Sigma_o$  may contain other elements in addition to  $\top$  and  $\bot$ . In this section we use t,  $t_1$ ,  $t_2$ , etc., for closed terms, unless otherwise stated.

**Definition 5.4.** We construct a reduction system R as follows. The terms of R are the type-free lambda-terms over  $\Sigma^+$ . The reduction rules of R are as follows:

- rules of  $\beta$  and  $\eta$ -reduction,
- $cc_1 \to c_2$  for  $c \in \Sigma_{\tau_1 \to \tau_2}$ ,  $c_1 \in \Sigma_{\tau_1}$  and  $c_2 \in \Sigma_{\tau_2}$  such that  $\mathcal{F}(c)(c_1) = c_2$ .

It is easy to see that R has the Church-Rosser property.

**Definition 5.5.** For each ordinal  $\alpha$  and each state  $s \in \mathcal{S}$  we inductively define a relation  $\succ_{\alpha}^{s}$  between terms and  $\top$  or  $\bot$ . The notations  $\succ_{<\alpha}^{s}$ ,  $\leadsto_{<\alpha}^{s}$ , etc., have analogous meaning to those in Section 4.

We postulate  $t \succ_{\alpha}^{s} \top$  for  $\alpha \geq 0$  and all closed terms t such that:

- (1)  $t \equiv c$  for some  $c \in \Sigma_o$  such that  $s \in \varsigma_{\mathcal{N}}(\delta(c))$ , or
- (2)  $t \equiv LA_{\tau}$  for some  $\tau \in \mathcal{B}$ , or
- (3)  $t \equiv LH$ , or
- (4)  $t \equiv A_{\tau}c$  for  $\tau \in \mathcal{B}$  and  $c \in \Sigma_{\tau}$ , or
- (5)  $t \equiv Hc$  for  $c \in \Sigma_o$ .

When  $\alpha > 0$  we postulate  $t \succ_{\alpha}^{s} \top$  for all closed terms t such that one of the following holds:

- $(\Xi_{\top})$   $t \equiv \Xi A_{\tau} t_1$  where  $\tau \in \mathcal{B} \cup \{o\}$  and  $t_1$  is such that for all  $s' \geq s$  and all  $c \in \Sigma_{\tau}$  we have  $t_1 c \leadsto_{<\alpha}^{s'} \top$ ,
- $(P_{\top})$   $t \equiv \Xi(Kt_1)t_2$  where
  - $t_1 \sim_{\leq \alpha}^s \top$  or  $t_1 \sim_{\leq \alpha}^s \bot$ , and
  - for all  $s' \geq s$  such that  $t_1 \rightsquigarrow_{<\alpha}^{s'} \top$  we have  $t_2 \twoheadrightarrow_R Kt_2'$  with  $t_2' \succ_{<\alpha}^{s'} \top$ ,
- $(H_{\top})$   $t \equiv Ht_1$ , and  $t_1 \rightsquigarrow_{\leq \alpha}^s \top$  or  $t_1 \rightsquigarrow_{\leq \alpha}^s \bot$ .

Finally, we postulate  $t \succ_{\alpha}^{s} \bot$  for  $\alpha \ge 0$  and all closed terms t such that one of the following holds:

- $(c_{\perp})$   $t \equiv c \in \Sigma_o$  and  $s \notin \varsigma_{\mathcal{N}}(\delta(c))$ ,
- $(\Xi_{\perp})$   $t \equiv \Xi A_{\tau} t_1$  and  $\tau \in \mathcal{B} \cup \{o\}$ , and
  - for all  $c \in \Sigma_{\tau}$  and all  $s' \geq s$  we have  $t_1 c \leadsto_{<\alpha}^{s'} \top$  or  $t_1 c \leadsto_{<\alpha}^{s'} \bot$ ,
  - there exist a constant  $c \in \Sigma_{\tau}$  and a state  $s' \geq s$  such that  $t_1 c \rightsquigarrow_{<\alpha}^{s'} \bot$ ,
- $(\mathsf{P}_{\perp}) \ t \equiv \Xi(Kt_1)(Kt_2), \text{ and}$

- $t_1 \leadsto_{<\alpha}^s \top$  or  $t_1 \leadsto_{<\alpha}^s \bot$ , and
- for all  $s' \geq s$  such that  $t_1 \rightsquigarrow_{\leq \alpha}^{s'} \top$  we have  $t_2 \rightsquigarrow_{\leq \alpha}^{s'} \top$  or  $t_2 \rightsquigarrow_{\leq \alpha}^{s'} \bot$ .
- there exists  $s' \geq s$  such that  $t_1 \leadsto_{<\alpha}^{s'} \top$  and  $t_2 \leadsto_{<\alpha}^{s'} \bot$ .

In [Cza13] this definition is incorrect. In fact, Lemma 5.9 of [Cza13] is false, because of the presence of type  $\varepsilon$ . To correct this we need to separately consider the case when  $\Xi$  encodes implication, which is done here by means of the rules ( $P_{\top}$ ) and ( $P_{\perp}$ ). This change requires reworking the subsequent correctness proof.

With the corrected definition, it is not obvious that for  $\alpha \leq \beta$  we have  $\succ_{\alpha}^{s} \subseteq \succ_{\beta}^{s}$ . We will show this only in Lemma 5.12. However, for  $\alpha \leq \beta$  we obviously have  $\succ_{<\alpha}^{s} \subseteq \succ_{<\beta}^{s}$ , and consequently  $\leadsto_{<\alpha}^{s} \subseteq \leadsto_{<\beta}^{s}$ .

**Lemma 5.6.** If  $t_1 \succ_{\alpha}^s \rho$  and  $t_1 \twoheadrightarrow_R t_2$  then  $t_2 \succ_{\alpha}^s \rho$ .

*Proof.* This follows by an easy induction on  $\alpha$ , using the Church-Rosser property of R.

Corollary 5.7. If  $t =_R t'$  then  $t \leadsto_{\alpha}^s \rho$  is equivalent to  $t' \leadsto_{\alpha}^s \rho$ .

**Corollary 5.8.** If  $t \leadsto_{\alpha}^{s} \top$  and  $t \leadsto_{\alpha}^{s} \bot$  then there exists t' such that  $t' \succ_{\alpha}^{s} \top$  and  $t' \succ_{\alpha}^{s} \bot$ .

**Lemma 5.9.** For all ordinals  $\alpha$  and all  $s \in \mathcal{S}$  we have:

- (1) if  $t \succ_{\alpha}^{s} \top$  and  $s' \geq s$  then  $t \succ_{\alpha}^{s'} \top$ ,
- (2) if  $t \succ_{\alpha}^{s} \perp$  and  $s' \geq s$  then  $t \succ_{\alpha}^{s'} \top$  or  $t \succ_{\alpha}^{s'} \perp$ .

*Proof.* Induction on  $\alpha$ .

- (1) Follows directly from the inductive hypothesis.
- (2) The only non-obvious cases are with  $(\Xi_{\perp})$  and  $(P_{\perp})$ . Suppose  $t \equiv \Xi A_{\tau} t_1 \succ_{\alpha}^{s} \bot$  with:
  - for all  $c \in \Sigma_{\tau}$  and all  $s'' \geq s$  we have  $t_1 c \leadsto_{<\alpha}^{s''} \top$  or  $t_1 c \leadsto_{<\alpha}^{s''} \bot$ ,
  - there exist a constant  $c \in \Sigma_{\tau}$  and a state  $s'' \geq s$  such that  $t_1 c \leadsto_{<\alpha}^{s''} \bot$ ,

Let  $s' \geq s$ . The first condition obviously still holds with s' substituted for s. If the second condition does not hold, then by the first condition:

• for all  $c \in \Sigma_{\tau}$  and all  $s'' \geq s'$  we have  $t_1 c \rightsquigarrow_{\leq \alpha}^{s''} \top$ .

This implies  $t \succ_{\alpha}^{s'} \top$ . The argument for  $(P_{\perp})$  is analogous.

Corollary 5.10. If  $t \leadsto_{\alpha}^{s} \top$  then  $t \leadsto_{\alpha}^{s'} \top$  for  $s' \ge s$ .

Remark 5.11. The necessity of the above corollary is precisely the reason why it is not easy to extend this construction to the case of full higher-order intuitionistic logic, i.e., when we have functions and predicates of all types and more than one state. In that case we would need separate reduction systems  $R_{\alpha}^{s}$  for each s and  $\alpha$ , similarly to what is done in Section 4. But then it would not be the case that  $R_{\alpha}^{s} \subseteq R_{\alpha}^{s}$  for  $s' \geq s$ . Roughly speaking, this is because  $t \succ_{\alpha}^{s} \bot$  is interpreted as "t is not true in state s basing on what we know at stage  $\alpha$ ", and not as "t is false in state s". Thus we may have  $t \succ_{\alpha}^{s} \bot$  and  $t \succ_{\alpha}^{s'} \top$  for some  $s' \geq s$ . This by itself is not yet a fatal obstacle, because we really only care about  $t \leadsto_{\alpha}^{s} \top$  being monotonous w.r.t. state ordering. However, the condition  $t \succ_{\alpha}^{s} \bot$  would be used to define  $R_{\alpha}^{s}$ , which would make  $R_{\alpha}^{s}$  non-monotonous w.r.t s. Thus  $t \leadsto_{\alpha}^{s} \top$  would not be monotonous either, as it is equivalent to  $t \leadsto_{R_{\alpha}^{s}} t' \succ_{\alpha}^{s} \top$ . Hence the corollary would fail. This explains why we do not simply give a single construction generalizing both the present one and the one from Section 4.

**Lemma 5.12.** For all ordinals  $\alpha$  and all  $s \in \mathcal{S}$  we have:

- (1) if  $t \succ_{\leq \alpha}^{s} \rho$  then  $t \succ_{\alpha}^{s} \rho$ ,
- (2) if  $t \succ_{\alpha}^{s} \top$  then  $t \not\succeq_{\alpha}^{s} \bot$ .

*Proof.* Induction on  $\alpha$ . First note that the inductive hypothesis and Corollary 5.8 imply:

 $(\star)$  if  $t \rightsquigarrow_{\leq \alpha}^s \top$  then  $t \not\rightsquigarrow_{\leq \alpha} \bot$ .

Now, we check the conditions (1) and (2).

- (1) The problem is with the universal quantification in  $(P_{\perp})$  and  $(P_{\perp})$ . For instance, consider  $(P_{\perp})$ , i.e.,  $t \equiv \Xi(Kt_1)t_2 \succ_{\beta}^s \top$  for some  $\beta < \alpha$ , with:
  - $t_1 \leadsto_{<\beta}^s \top$  or  $t_1 \leadsto_{<\beta}^s \bot$ ,
  - for all  $s' \geq s$  such that  $t_1 \rightsquigarrow_{\leq \beta}^{s'} \top$  we have  $t_2 \twoheadrightarrow_R Kt_2'$  with  $t_2' \succ_{\leq \beta}^{s'} \top$ .

Of course, we have  $t_1 \rightsquigarrow_{<\alpha}^s \top$  or  $t_1 \rightsquigarrow_{<\alpha}^s \top$ . Suppose  $s' \geq s$  and  $t_1 \rightsquigarrow_{<\alpha}^{s'} \top$ . If  $t_1 \rightsquigarrow_{<\beta}^s \top$ , then  $t_1 \rightsquigarrow_{<\beta}^{s'} \top$  by Lemma 5.9. Thus  $t_2 \twoheadrightarrow_R Kt_2'$  with  $t_2' \succ_{<\beta}^{s'} \top$ , so also  $t_2' \succ_{<\alpha}^{s'} \top$ . If  $t_1 \rightsquigarrow_{<\beta}^s \bot$ , then  $t_1 \rightsquigarrow_{<\beta}^{s'} \top$  or  $t_1 \rightsquigarrow_{<\beta}^{s'} \top$  by Lemma 5.9. The case  $t_1 \rightsquigarrow_{<\beta}^{s'} \top$  has just been considered. So suppose  $t_1 \rightsquigarrow_{<\beta}^{s'} \bot$ . Then  $t_1 \rightsquigarrow_{<\alpha}^{s'} \bot$  which contradicts  $(\star)$ .

(2) The claim is immediate for  $\alpha = 0$ . Suppose  $t \succ_{\alpha}^{s} \top$  and  $t \succ_{\alpha}^{s} \bot$ . Then either  $t \equiv \Xi A_{\tau} t_{1}$  or  $t \equiv \Xi (Kt_{1})(Kt_{2})$ .

Assume  $t \equiv \Xi A_{\tau} t_1$ . Then, because  $t \succ_{\alpha}^{s} \bot$ , there exist  $c \in \Sigma_{\tau}$  and  $s' \geq s$  such that  $t_1 c \leadsto_{<\alpha}^{s'} \bot$ . On the other hand, because  $t \succ_{\alpha}^{s} \top$ , we have  $t_1 c \leadsto_{<\alpha}^{s'} \top$ . This contradicts  $(\star)$ .

If  $t \equiv \Xi(Kt_1)(Kt_2)$  then the argument is analogous.

It follows from Lemma 5.12, by a simple cardinality argument, that there exists an ordinal  $\zeta$  such that  $\succ_{\zeta}^{s} = \succ_{\zeta\zeta}^{s}$  for all  $s \in \mathcal{S}$ . We use the notations  $\succ^{s}$  and  $\leadsto^{s}$  without subscripts for  $\succ_{\zeta}^{s}$  and  $\leadsto_{\zeta}^{s}$ .

**Definition 5.13.** The structure  $\mathcal{M}$  is defined as follows. We define the extensional combinatory algebra  $\mathcal{C}$  of  $\mathcal{M}$  to be the set of equivalence classes of  $=_R$  on closed terms. We take the set  $\mathcal{S}$  of states of  $\mathcal{N}$  to be the set of states of  $\mathcal{M}$  as well. For  $c \in \Sigma$  we define the interpretation I of  $\mathcal{M}$  by  $I(c) = [c^+]_R$ , where  $c^+ \in \Sigma^+$  corresponds to the element  $[\![c]\!]_{\mathcal{N}}$ . The function  $\varsigma_{\mathcal{M}}$  is given by  $\varsigma_{\mathcal{M}}(d) = \{s \in \mathcal{S} \mid \exists t.d = [t]_R \land t \leadsto^s \top \}$ , where t is required to be closed.

**Lemma 5.14.** Let  $t_1$  and  $t_2$  be closed terms. If for all closed  $t_3$  we have  $t_1t_3 =_R t_2t_3$ , then  $t_1 =_R t_2$ .

Proof. If  $t_1t_3 =_R t_2t_3$  for all closed  $t_3$ , then in particular  $t_1\nu =_R t_2\nu$  for an external constant  $\nu$  not occurring in  $t_1$  and  $t_2$ . By the Church-Rosser property of R there exists t such that  $t_1\nu \twoheadrightarrow_R t$  and  $t_2\nu \twoheadrightarrow_R t$ . Because there are no rules in R involving  $\nu$ , and  $\nu$  cannot be produced by any of the reductions, it is easy to verify by induction on the number of reduction steps that  $t \equiv C'[\nu]$ ,  $t_1\nu \equiv C_1[\nu]$ ,  $t_2\nu \equiv C_2[\nu]$ ,  $C_1 \twoheadrightarrow_R C'$  and  $C_2 \twoheadrightarrow_R C'$ , where  $\nu$  does not occur in  $C_1$ ,  $C_2$  or C'. Hence  $t_1x \equiv C_1[x] =_R C_2[x] \equiv t_2x$  for a variable x, and thus  $\lambda x.t_1x =_R \lambda x.t_2x$ . Because R contains the rule of  $\eta$ -reduction, we conclude that  $t_1 =_R t_2$ .

**Lemma 5.15.** Let C be a context and let  $\rho \in \{\top, \bot\}$ . If  $C[\rho] \twoheadrightarrow_R t$  then there exists a context C' such that  $C \twoheadrightarrow_R C'$  and  $t = C'[\rho]$ .

*Proof.* Because there are no rules in R involving  $\rho$ , the claim is easy to verify by induction on the number of reduction steps.

The following lemma is a much simplified analogon of Lemma 4.2.12.

**Lemma 5.16.** If  $t \succ^s \rho_1$  and  $C[\rho_1] \leadsto_{\alpha}^s \rho_2$  then  $C[t] \leadsto^s \rho_2$ .

*Proof.* Induction on  $\alpha$ .

Suppose  $t \succ^s \rho_1$  and  $C[\rho_1] \leadsto_{\alpha}^s \rho_2$ . By Lemma 5.15 we have  $C \twoheadrightarrow_R C'$  where  $C'[\rho_1] \succ_{\alpha}^s \rho_2$ . It suffices to show that  $C'[t] \succ^s \rho_2$ .

First assume  $\alpha = 0$ . The claim is obvious if C' does not contain  $\square$ , so assume it does. Then by inspecting the definitions we see that there are the following two possibilities.

- If  $C' \equiv \Box$  and  $\rho_1 \equiv \rho_2$  then the claim is obvious.
- If  $C' \equiv H \square$  and  $\rho_2 \equiv \top$ , then either  $t \succ \top$  or  $t \succ \bot$ . Thus  $Ht \succ \top$  by condition  $(H_\top)$ .

Now let  $\alpha > 0$ . If  $C' \equiv \Xi A_{\tau} C_1$  and  $\rho_2 = \top$  then for all  $c \in \Sigma_{\tau}$  and all  $s' \geq s$  we have  $C_1[\rho_1]c \leadsto_{<\alpha}^{s'} \top$ . We conclude by the inductive hypothesis that for all  $c \in \Sigma_{\tau}$  and all  $s' \geq s$  we have  $C_1[t]c \leadsto^{s'} \top$ . Hence  $C'[t] \succ^s \top$ .

If  $C' \equiv \Xi(KC_1)C_2$  and  $\rho_2 = \top$  then

- $C_1[\rho_1] \leadsto_{\leq \alpha}^s \top$  or  $C_1[\rho_1] \leadsto_{\leq \alpha}^s \bot$ , and
- for all  $s' \geq s$  such that  $C_1[\rho_1] \rightsquigarrow_{\leq \alpha}^{s'} \top$  we have  $C_2 \twoheadrightarrow_R KC_2'$  with  $C_2'[\rho_1] \succ_{\leq \alpha}^{s'} \top$ .

By Lemma 5.9 for all s' > s we have:

$$(\star)$$
  $C_1[\rho_1] \leadsto_{\leq \alpha}^{s'} \top \text{ or } C_1[\rho_1] \leadsto_{\leq \alpha}^{s'} \bot.$ 

By the inductive hypothesis  $C_1[t] \rightsquigarrow^s \top$  or  $C_1[t] \rightsquigarrow^s \bot$ . Let  $s' \geq s$  be such that  $C_1[t] \rightsquigarrow^{s'} \top$ . By  $(\star)$  we have  $C_1[\rho_1] \rightsquigarrow^{s'}_{<\alpha} \top$ , because if  $C_1[\rho_1] \rightsquigarrow^{s'}_{<\alpha} \bot$  then  $C_1[t] \rightsquigarrow^{s'} \bot$  by the inductive hypothesis, which contradicts  $C_1[t] \rightsquigarrow^{s'} \top$  by part 2 of Lemma 5.12. Hence,  $C_2 \twoheadrightarrow_R KC_2'$  with  $C_2'[\rho_1] \succ^{s'}_{<\alpha} \top$ , and by the inductive hypothesis  $C_2'[t] \rightsquigarrow^{s'}_{<\alpha} \top$ . This implies  $C_1'[t] \succ^s_{\alpha} \top$ .

In all other cases the proof is similar.

This finishes the more difficult part of the construction correctness proof. As in Section 4 it remains to prove several simple lemmas implying that  $\mathcal{M}$  satisfies the conditions imposed on an illative Kripke model for  $\mathcal{I}_0$ . For convenience we reformulate the definition of an illative Kripke model for  $\mathcal{I}_0$  in terms of the notions used to construct  $\mathcal{M}$ .

**Fact 5.17.** If the following conditions hold, then  $\mathcal{M}$  is an illative Kripke model for  $\mathcal{I}_0$ .

- (1) If  $t_1 =_R t_2$  then  $t_1 \rightsquigarrow^s \top$  is equivalent to  $t_2 \rightsquigarrow^s \top$ .
- (2) If  $t \rightsquigarrow^s \top$  then  $t \rightsquigarrow^{s'} \top$  for all  $s' \ge s$ .
- (3) If for all  $t_3$  we have  $t_1t_3 =_R t_2t_3$  then  $t_1 =_R t_2$ .
- (4) If  $Lt_1 \rightsquigarrow^s \top$  and for all  $s' \geq s$  and all  $t_3$  such that  $t_1t_3 \rightsquigarrow^{s'} \top$  we have  $t_2t_3 \rightsquigarrow^{s'} \top$ , then  $\Xi t_1t_2 \rightsquigarrow^s \top$ .
- (5) If  $\Xi t_1 t_2 \rightsquigarrow^s \top$  then for all  $t_3$  such that  $t_1 t_3 \rightsquigarrow^s \top$  we have  $t_2 t_3 \rightsquigarrow^s \top$ .
- (6) If  $Lt_1 \rightsquigarrow^s \top$  and for all  $s' \geq s$  and all  $t_3$  such that  $t_1t_3 \rightsquigarrow^{s'} \top$  we have  $H(t_2t_3) \rightsquigarrow^{s'} \top$ , then  $H(\Xi t_1t_2) \rightsquigarrow^s \top$ .
- (7) If  $t \rightsquigarrow^s \top$  then  $Ht \rightsquigarrow^s \top$ .
- (8)  $LH \sim^s \top$ ,
- (9)  $LA_{\tau} \leadsto^s \top \text{ for } \tau \in \mathcal{B}.$

*Proof.* Condition (1) ensures that  $s \in \varsigma_{\mathcal{M}}([t]_R)$  is equivalent to  $t \rightsquigarrow^s \top$ . Condition (2) implies that for any  $d \in \mathcal{M}$  the set  $\varsigma_{\mathcal{M}}(d)$  is upward-closed. Condition (3) implies that the combinatory algebra of  $\mathcal{M}$  is extensional. The remaining conditions are a reformulation of the conditions imposed on  $\varsigma$  in an illative Kripke model for  $\mathcal{I}_0$ .

**Lemma 5.18.**  $Ht \rightsquigarrow^s \top iff t \rightsquigarrow^s \top or t \rightsquigarrow^s \bot$ .

*Proof.* Follows directly from definitions.

**Lemma 5.19.** If  $Lt \sim^s \top$  then exactly one of the following holds:

- $t \twoheadrightarrow_R A_{\tau}$  for some  $\tau \in \mathcal{B}$ ,
- $t \rightarrow_R H$ ,

•  $t \rightarrow_R Kt'$  with  $t' \sim^s \top$  or  $t' \sim^s \bot$ .

*Proof.* Easy inspection of the rules in the definition of  $\succ_{\alpha}^{s}$ . That the conditions are exclusive is a consequence of the Church-Rosser property of R.

Lemma 5.20. The following conditions are satisfied.

- If  $Lt_1 \rightsquigarrow^s \top$  and for all  $s' \geq s$  and all  $t_3$  such that  $t_1t_3 \rightsquigarrow^{s'} \top$  we have  $t_2t_3 \rightsquigarrow^{s'} \top$ , then  $\Xi t_1t_2 \rightsquigarrow^s \top$ .
- If  $Lt_1 \rightsquigarrow^s \top$  and for all  $s' \geq s$  and all  $t_3$  such that  $t_1t_3 \rightsquigarrow^{s'} \top$  we have  $H(t_2t_3) \rightsquigarrow^{s'} \top$ , then  $H(\Xi t_1t_2) \rightsquigarrow^s \top$ .

*Proof.* Suppose  $Lt_1 \rightsquigarrow^s \top$  and for all  $s' \geq s$  and all  $t_3$  such that  $t_1t_3 \rightsquigarrow^{s'} \top$  we have  $t_2t_3 \rightsquigarrow^{s'} \top$ . We consider possible cases according to Lemma 5.19.

- $t_1 \twoheadrightarrow_R A_\tau$  for  $\tau \in \mathcal{B} \cup \{o\}$ . If  $c \in \Sigma_\tau$  and  $s' \geq s$  then  $A_\tau c \succ^{s'} \top$ , so also  $t_1 c \rightsquigarrow^{s'} \top$  by Corollary 5.7, and thus  $t_2 c \rightsquigarrow^{s'} \top$ . Hence  $\Xi A_\tau t_2 \succ^s \top$  by  $(\Xi_\top)$ . Therefore,  $\Xi t_1 t_2 \rightsquigarrow^s \top$ .
- $t_1 \to_R Kt_1'$  with  $t_1' \leadsto^s \top$  or  $t_1' \leadsto^s \bot$ . Let  $s' \geq s$  be such that  $t_1' \leadsto^{s'} \top$ . Then  $t_1t_3 \leadsto^{s'} \top$  for arbitrary closed  $t_3$ , so  $t_2t_3 \leadsto^{s'} \top$  for any closed  $t_3$ , in particular for  $t_3 \equiv \nu$  an external constant not occurring in  $t_2$ . We have  $t_2\nu \to_R t_2' \succ^{s'} \top$ . It is easy to see by inspecting the definitions that  $\nu$  cannot occur in  $t_2'$ . Thus we also have  $t_2x \to_R t_2'$ . Therefore  $t_2 \leftarrow_{\eta} \lambda x.t_2x \to_R Kt_2'$ . So if there exists  $s' \geq s$  such that  $t_1' \leadsto^{s'} \top$  then  $t_2 =_R Kt_2'$ , and for every such  $s' \geq s$  we have  $t_2' \succ^{s'} \top$ . Thus  $\Xi(Kt_1')(Kt_2') \succ^s \top$ , so  $\Xi t_1t_2 \leadsto^s \top$ , by Corollary 5.7. If there does not exist  $s' \geq s$  such that  $t_1' \leadsto^{s'} \top$ , then also  $\Xi t_1t_2 \leadsto^s \top$ .

The second claim is verified in a similar manner using Lemma 5.19, Lemma 5.18, Corollary 5.10 and Corollary 5.7.  $\Box$ 

**Lemma 5.21.** If  $\exists t_1 t_2 \rightsquigarrow^s \top$  then for all  $s' \geq s$  and all terms  $t_3$  such that  $t_1 t_3 \rightsquigarrow^{s'} \top$  we have  $t_2 t_3 \rightsquigarrow^{s'} \top$ .

*Proof.* Suppose  $\Xi t_1 t_2 \rightsquigarrow^s \top$ . Then  $\Xi t_1 t_2 \twoheadrightarrow_R \Xi t_1' t_2' \succ^s \top$  with  $t_i \twoheadrightarrow_R t_i'$ . There are three cases.

- $t_1' \equiv A_{\tau}$  where  $\tau \in \mathcal{B}$  and for all  $s' \geq a$  and all  $c \in \Sigma_{\tau}$  we have  $t_2' c \leadsto^{s'} \top$ . Assume  $s' \geq s$  and  $t_1 t_3 \leadsto^{s'} \top$ . Then also  $A_{\tau} t_3 \leadsto^{s'} \top$  by Corollary 5.7. This is only possible when  $t_3 \in \Sigma_{\tau}$ . This implies  $t_2' t_3 \leadsto^{s'} \top$ , so also  $t_2 t_3 \leadsto^{s'} \top$  because  $t_2 \twoheadrightarrow_R t_2'$ .
- $t_1' \equiv H$  and for all  $s' \geq s$  and all  $\rho \in \{\top, \bot\} \subseteq \Sigma_{\tau}$  we have  $t_2' \rho \sim^{s'} \top$ . Assume  $s' \geq s$  and  $t_1 t_3 \sim^{s'} \top$ . Then also  $H t_3 \sim^{s'} \top$  by Corollary 5.7. By Lemma 5.18 either  $t_3 \sim^{s'} \top$  or  $t_3 \sim^{s'} \bot$ . In any case, we may use Lemma 5.16 to conclude  $t_2 t_3 \sim^{s'} \top$ .
- $t'_1 \equiv Kt''_1$  and for all  $s' \geq s$  such that  $t''_1 \leadsto^{s'} \top$  we have  $t'_2 \twoheadrightarrow_R Kt''_2$  with  $t''_2 \succ^{s'} \top$ . Assume  $s' \geq s$  and  $t_1t_2 \leadsto^{s'} \top$ . Then  $t''_1 \leadsto^{s'} \top$  by Corollary 5.7. So also  $t_2t_3 \leadsto^{s'} \top$  by Corollary 5.7, because  $t_2 \twoheadrightarrow_R Kt''_2$  with  $t''_2 \succ^{s'} \top$ .

Corollary 5.22. The structure  $\mathcal{M}$  constructed in Definition 5.13 is an illative Kripke model for  $\mathcal{I}_0$ .

*Proof.* It suffices to check the conditions of Fact 5.17. Condition (1) follows from Corollary 5.7. Condition (2) is a consequence of Corollary 5.10. Condition (3) follows from Lemma 5.14. Conditions (4) and (6) follow from Lemma 5.20. Lemma 5.21 implies condition (5). Conditions (7), (8) and (9) are obvious from definitions.

**Lemma 5.23.** If  $\tau \in \mathcal{T}$  and  $c \in \Sigma_{\tau}$  then for all states s we have  $A_{\tau}c \rightsquigarrow^{s} \top$ .

*Proof.* Straightforward induction on the size of  $\tau$ .

It remains to prove that the values in  $\mathcal{N}$  of formulas of PRED2<sub>0</sub> are faithfully represented by the values of their translations in  $\mathcal{M}$ . From this completeness will directly follow.

**Definition 5.24.** Recall that for  $c \in \Sigma^+$ , we denote by  $\delta(c)$  the element of  $\mathcal{N}$  corresponding to c, if there is one. We say that an  $\mathcal{M}$ -valuation  $\widetilde{w}$  mirrors an  $\mathcal{N}$ -valuation w, if for every variable x there exists  $c \in \Sigma^+$  such that  $w(x) = \delta(c)$  and  $\widetilde{w}(x) = [c]_R$ . In other words,  $\widetilde{w}$  is the valuation assigning to each variable x the equivalence class of the constant corresponding to the element w(x). Note that given w the valuation  $\widetilde{w}$  is uniquely determined.

To avoid confusion, from now on we use  $q_1$ ,  $q_2$ , etc. for terms of PRED2<sub>0</sub>. By  $t_1$ ,  $t_2$ , etc. we denote closed terms from  $\mathbb{T}(\Sigma^+)$ . We use c,  $c_1$ ,  $c_2$ , etc. for constants from  $\Sigma^+$ .

**Lemma 5.25.** For any  $\mathcal{N}$ -valuation w and any term q of  $PRED2_0$  which is not a formula, we have  $\llbracket [q] \rrbracket_{\mathcal{M}}^{\widetilde{w}} = [c]_R$  for some  $c \in \Sigma^+$  such that  $\delta(c) = \llbracket q \rrbracket_{\mathcal{N}}^w$ .

Proof. Induction on the size of q. If q is a constant then  $\lceil q \rceil = q$  and  $\llbracket \lceil q \rceil \rrbracket_{\mathcal{M}}^{\widetilde{w}} = \llbracket q \rrbracket_{\mathcal{M}}^{\widetilde{w}} = I_{\mathcal{M}}(q) = [c]_R$  for some  $c \in \Sigma^+$  such that  $\delta(c) = \llbracket q \rrbracket_{\mathcal{N}}$ . If q = x is a variable of type  $\tau \in \mathcal{B}$  then  $\lceil q \rceil = x$ . So  $\llbracket \lceil q \rceil \rrbracket_{\mathcal{M}}^{\widetilde{w}} = \widetilde{w}(x) = [c]_R$  for  $c \in \Sigma^+$  such that  $w(x) = \delta(c)$ , by definition of  $\widetilde{w}$ .

Otherwise  $q \equiv q_1q_2$ . Neither  $q_1$  nor  $q_2$  is a formula, so by the inductive hypothesis  $\llbracket \lceil q_1 \rceil \rrbracket_{\mathcal{M}}^{\tilde{w}} = [c_1]_R$  and  $\llbracket \lceil q_2 \rceil \rrbracket_{\mathcal{M}}^{\tilde{w}} = [c_2]_R$  where  $\delta(c_1) = \llbracket q_1 \rrbracket_{\mathcal{M}}^{w}$  and  $\delta(c_2) = \llbracket q_2 \rrbracket_{\mathcal{M}}^{w}$ . We have  $\lceil q \rceil = \lceil q_1 \rceil \lceil q_2 \rceil$ , so  $\llbracket \lceil q \rceil \rrbracket_{\mathcal{M}}^{\tilde{w}} = \llbracket \lceil q_1 \rceil \rrbracket_{\mathcal{M}}^{\tilde{w}} : \mathcal{M} = [c_1]_R : \mathcal{M} = [c_1]_R : \text{Let } c \in \Sigma^+ \text{ be such that } \delta(c) = \delta(c_1) : \mathcal{N} \delta(c_2).$  In R there is a reduction rule  $c_1c_2 \to c$  because  $\mathcal{F}(c_1)(c_2) = c$ . Thus  $[c_1c_2]_R = [c]_R$ . We also have  $\delta(c) = \llbracket q_1 \rrbracket_{\mathcal{N}}^{w} : \mathcal{N} = \llbracket q_1q_2 \rrbracket_{\mathcal{N}}^{w} = \llbracket q_1q_2 \rrbracket_{\mathcal{N}}^{w} : \llbracket q_1q_2 \rrbracket_{\mathcal{N}}^{w} : \mathbb{Q}[q_1]_{\mathcal{N}}^{w} : \mathbb{Q}[q$ 

**Lemma 5.26.** For any formula  $\phi$  of PRED2<sub>0</sub>, any state s, and any N-valuation w we have:

$$s, w \Vdash_{\mathcal{N}} \phi \text{ iff } s, \widetilde{w} \Vdash_{\mathcal{M}} [\phi]$$

*Proof.* Induction on the size of  $\phi$ .

If  $\phi$  is a variable or a constant, then our claim follows easily from definitions. If  $\phi = q_1q_2$ , then neither  $q_1$  nor  $q_2$  is a formula, so by Lemma 5.25 we have  $[\![q_1]\!]_{\mathcal{M}}^{\widetilde{w}} = [c_1]_R$  and  $[\![q_2]\!]_{\mathcal{M}}^{\widetilde{w}} = [c_2]_R$  where  $c_1, c_2 \in \Sigma^+$  and  $\delta(c_1) = [\![q_1]\!]_{\mathcal{N}}^w$ ,  $\delta(c_2) = [\![q_2]\!]_{\mathcal{N}}^w$ . We have  $[\![c_1]\!]_R \cdot [\![c_2]\!]_R = [\![c_1]\!]_R \cdot [\![c_2]\!]_R$  for  $c \in \Sigma^+$  such that  $\delta(c) = \delta(c_1) \cdot_{\mathcal{N}} \delta(c_2) = [\![t_1t_2]\!]_{\mathcal{N}}^w$ . The claim now follows from the definition of  $\succ_0^s$ .

If  $\phi = \varphi \supset \psi$  then  $\lceil \phi \rceil = \lceil \varphi \rceil \supset \lceil \psi \rceil$ . Suppose  $s, \widetilde{w} \Vdash_{\mathcal{M}} \lceil \varphi \rceil \supset \lceil \psi \rceil$ . Let  $s' \geq s$  be such that  $s', w \Vdash_{\mathcal{N}} \varphi$ . By the inductive hypothesis  $s', \widetilde{w} \Vdash_{\mathcal{M}} \lceil \varphi \rceil$ . Note that we also have  $s', \widetilde{w} \Vdash_{\mathcal{M}} \lceil \varphi \rceil \supset \lceil \psi \rceil$ . By condition (2) in Fact 3.5 we obtain  $s', \widetilde{w} \Vdash_{\mathcal{M}} \lceil \psi \rceil$ , which implies  $s', w \Vdash_{\mathcal{N}} \psi$  by the IH. From Definition 2.3 it now follows that  $s, w \Vdash_{\mathcal{N}} \varphi \supset \psi$ . The other direction is analogous.

If  $\phi = \forall x. \varphi$  where  $x \in V_{\tau}, \ \tau \in \mathcal{B} \cup \{o\}$ , then  $\lceil \forall x. \varphi \rceil = \Xi A_{\tau} \lambda x. \lceil \varphi \rceil$ .

Suppose  $s, \widetilde{w} \Vdash_{\mathcal{M}} [\forall x.\varphi]$ , i.e.,  $s, \widetilde{w} \Vdash_{\mathcal{M}} \Xi A_{\tau} \lambda x. [\varphi]$ . Let  $s' \geq s$ ,  $d \in \mathcal{D}_{\tau}^{\mathcal{N}}$ , and u = w[x/d]. There exists  $c \in \Sigma^+$  such that  $\widetilde{u}(x) = [c]_R$  and  $\delta(c) = d$ . The constant c is a canonical constant of type  $\tau \in \mathcal{B} \cup \{o\}$ , so  $s', \widetilde{w} \Vdash_{\mathcal{M}} A_{\tau}c$ , by definition of  $\mathcal{M}$ . We also have  $s', \widetilde{w} \Vdash_{\mathcal{M}} \Xi A_{\tau} \lambda x. [\varphi]$ , so we conclude that  $s', \widetilde{w} \Vdash_{\mathcal{M}} (\lambda x. [\varphi])c$ . This implies  $s', \widetilde{u} \Vdash_{\mathcal{M}} [\varphi]$ , and hence  $s', w[x/d] \Vdash_{\mathcal{N}} \varphi$  by the IH. Therefore  $s, w \Vdash_{\mathcal{N}} \forall x.\varphi$ , by Definition 2.3.

For the other direction, we need to show that if  $s, w \Vdash_{\mathcal{M}} \forall x.\varphi$  then  $s, \widetilde{w} \Vdash_{\mathcal{M}} \Xi A_{\tau} \lambda x. \lceil \varphi \rceil$ , where  $\tau \in \mathcal{B} \cup \{o\}$ . If v is an  $\mathcal{M}$ -valuation and  $t \in \mathbb{T}(\Sigma^+)$ , then by  $t^v$  we denote the term t with every free variable x substituted for a representant of the equivalence class v(x). By induction on the size of t one may easily verify that  $\llbracket t \rrbracket_{\mathcal{M}}^v = \llbracket t^v \rrbracket_{\mathcal{M}}$ , but Lemma 5.14 is needed for the case of lambda-abstraction. Hence  $s, v \Vdash_{\mathcal{M}} t$  is equivalent to  $t^v \rightsquigarrow^s \top$ . Now the condition  $s, \widetilde{w} \Vdash_{\mathcal{M}} \Xi A_{\tau} \lambda x. \lceil \varphi \rceil$  may be reformulated as  $\Xi A_{\tau}(\lambda x. \lceil \varphi \rceil)^{\widetilde{w}} \rightsquigarrow^s \top$ . Therefore it suffices to prove, assuming  $s, w \Vdash_{\mathcal{M}} \forall x.\varphi$ , that for all canonical constants  $c \in \Sigma_{\tau}$  of type  $\tau \in \mathcal{B} \cup \{o\}$  and all  $s' \geq s$  we have  $(\lambda x. \lceil \varphi \rceil)^{\widetilde{w}} c \rightsquigarrow^{s'} \top$ . Let  $u = w[x/\delta(c)]$ . We have  $\widetilde{u} = \widetilde{w}[x/c]$ . Hence  $(\lambda x. \lceil \varphi \rceil)^{\widetilde{w}} c \rightsquigarrow^{s'} \top$  is equivalent to  $\lceil \varphi \rceil^{\widetilde{u}} \rightsquigarrow^{s'} \top$ , which is the same as  $s, \widetilde{u} \Vdash_{\mathcal{M}} \lceil \varphi \rceil$ . Because  $s, w \Vdash_{\mathcal{N}} \forall x.\varphi, s' \geq s$  and  $u = w[x/\delta(c)]$ , we conclude that  $s', u \Vdash_{\mathcal{N}} \varphi$ . By the inductive hypothesis we obtain  $s, \widetilde{u} \Vdash_{\mathcal{M}} \lceil \varphi \rceil$  which completes the proof.

**Theorem 5.27.** The embedding is complete, i.e.,  $[\Delta], \Gamma(\Delta, \varphi) \vdash_{\mathcal{I}_0} [\varphi]$  implies  $\Delta \vdash_{PRED2_0} \varphi$ .

Proof. Suppose  $\Delta \nvdash_{\mathrm{PRED2_0}} \varphi$ . Let  $\mathcal{N}$  be a Kripke model, v an  $\mathcal{N}$ -valuation and s a state of  $\mathcal{N}$  such that  $s, v \Vdash_{\mathcal{N}} \Delta$ , but  $s, v \nvDash_{\mathcal{N}} \varphi$ . We use the construction in Definition 5.13 to obtain an illative Kripke model  $\mathcal{M}$ . By Lemma 5.26 the condition  $s, v \Vdash_{\mathcal{N}} \psi$  is equivalent to  $s, \widetilde{v} \Vdash_{\mathcal{M}} \lceil \psi \rceil$ . Therefore  $s, \widetilde{v} \Vdash_{\mathcal{M}} \lceil \Delta \rceil$  but  $s, \widetilde{v} \nvDash_{\mathcal{M}}$ . Using Lemma 5.23, it is a matter of routine to verify that also  $s, \widetilde{v} \Vdash_{\mathcal{M}} \Gamma(\Delta, \varphi)$ . By the soundness part of Theorem 3.6 this implies  $\lceil \Delta \rceil, \Gamma(\Delta, \varphi) \nvdash_{\mathcal{I}_0} \lceil \varphi \rceil$ .

## 6 Remarks and open problems

**Remark 6.1.** In this paper we use lambda-calculus with  $\beta\eta$ -equality. Lambda-calculus with  $\beta$ -equality or combinatory logic with weak equality could be used instead. The proofs and definitions would only need minor adjustments.

Remark 6.2. It is clear that the methods presented here may be used to prove completeness of the embedding of propositional second-order logic into an extension of  $\mathcal{I}P$  from [BBD93]. This extension of  $\mathcal{I}P$  is essentially  $\mathcal{I}_0$  but with rules  $P_i$ ,  $P_e$ ,  $P_H$  from Lemma 3.2 instead of the more general rules for  $\Xi$ . Whether such an extension is complete for second-order propositional logic was posed as an open problem in [BBD93].

The open problem related to  $\mathcal{I}_0$  given in [BBD93] was whether full second-order predicate logic may be faithfully embedded into it. We do not know the answer to this question. One problem with extending our methods was already noted in Remark 5.11. It is not straightforward to extend our construction to obtain a model with quantification over predicates and more than one state. Another obstacle is that our construction of a model for  $\mathcal{I}_{\omega}^{c}$  crucially depends on the fact that the model of higher-order logic being transformed is a full model. Thus the construction cannot be used to show completeness of an embedding of higher-order logic into  $\mathcal{I}_{\omega}^{c}$ . Informally speaking, a full model is needed to ensure that no "essentially new" functions may be "created" at later stages  $\alpha$  of the inductive definition.

In [DBB98a] and [DBB98b] two indirect propositions-as-types translations of first-order propositional and predicate logic were shown complete for two illative systems  $\mathcal{I}F$  and  $\mathcal{I}G$ , which are stronger than  $\mathcal{I}P$  and  $\mathcal{I}\Xi$ , respectively. It is interesting whether our methods may be used to obtain these results, or improve on them.

Remark 6.3. In [Cza11] we presented an algebraic treatment of a combination of untyped combinatory logic with first-order classical logic. The model construction and the completeness proof there follow essentially the same pattern as those presented here, but they are much simpler. The system in [Cza11] contains an additional constant Cond which allows for branching on formulas. It is not difficult to see that we could add such a constant to  $\mathcal{I}_{\omega}^{c}$  and our model construction would still go through.

Remark 6.4. The construction from Section 4 could also be used to show that classical many-sorted first-order logic may be faithfully embedded into  $\mathcal{I}_{\omega}^{c}$ , but we omit this proof as it is analogous to that from Section 5. We do not know whether  $\mathcal{I}_{\omega}^{c}$  is conservative over stronger systems of logic, or whether  $\mathcal{I}_{\omega}$  is conservative over intuitionistic first-order logic.

### References

- [BBD93] Henk Barendregt, Martin W. Bunder, and Wil Dekkers. Systems of illative combinatory logic complete for first-order propositional and predicate calculus. *Journal of Symbolic Logic*, 58(3):769–788, 1993.
- [BD01] Martin W. Bunder and Wil Dekkers. Pure type systems with more liberal rules. *Journal of Symbolic Logic*, 66(4):1561–1580, 2001.
- [BD05] Martin W. Bunder and Wil Dekkers. Equivalences between pure type systems and systems of illative combinatory logic. *Notre Dame Journal of Formal Logic*, 46(2):181–205, 2005.
- [CFC58] Haskell B. Curry, Robert Feys, and William Craig. *Combinatory Logic*, volume 1. North-Holland, 1958.
- [Cza11] Łukasz Czajka. A semantic approach to illative combinatory logic. In Computer Science Logic, 25th International Workshop / 20th Annual Conference of the EACSL, CSL 2011, September 12-15, 2011, Bergen, Norway, Proceedings, volume 12 of LIPIcs, pages 174–188. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2011.
- [Cza13] Łukasz Czajka. Higher-order illative combinatory logic. Journal of Symbolic Logic, 78(3):837–872, 2013.

- [DBB98a] Wil Dekkers, Martin W. Bunder, and Henk Barendregt. Completeness of the propositions-astypes interpretation of intuitionistic logic into illative combinatory logic. *Journal of Symbolic Logic*, 63(3):869–890, 1998.
- [DBB98b] Wil Dekkers, Martin W. Bunder, and Henk Barendregt. Completeness of two systems of illative combinatory logic for first-order propositional and predicate calculus. *Archive for Mathematical Logic*, 37(5-6):327–341, 1998.
- [Sel09] Jonathan P. Seldin. The logic of Church and Curry. In Dov M. Gabbay and John Woods, editors, *Logic from Russell to Church*, volume 5 of *Handbook of the History of Logic*, pages 819–873. North-Holland, 2009.