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FORCING AXIOMS, SUPERCOMPACT CARDINALS, SINGULAR CARDINAL COMBINATORICS

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The purpose of this communication is to present some recent advances on the consequences that forcing axioms and large cardinals have on the combinatorics of singular cardinals. I will introduce a few examples of problems in singular cardinal combinatorics which can be fruitfully attacked using ideas and techniques coming from the theory of forcing axioms and then translate the results so obtained in suitable large cardinals properties.

The first example I will treat is the proof that the proper forcing axiom PFA implies the singular cardinal hypothesis SCH, this will easily lead to a new proof of Solovay's theorem that SCH holds above a strongly compact cardinal. I will also outline how some of the ideas involved in these proofs can be used as means to evaluate the "saturation" properties of models of strong forcing axioms like MM or PFA.

The second example aims to show that the transfer principle $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_2, \aleph_1)$ fails assuming Martin's Maximum MM. Also in this case the result can be translated in a large cardinal property, however this requires a familiarity with a rather large fragment of Shelah's $\mathtt{pcf}$-theory.

Only sketchy arguments will be given, the reader is referred to the forthcoming [25] and [38] for a thorough analysis of these problems and for detailed proofs.

The singular cardinal problem. Cardinal arithmetic is a central subject in modern set theory and one of the key problems in this domain is to evaluate the gimel function $\kappa \mapsto \kappa^{\mathtt{cof}(\kappa)}$ for a singular cardinal κ. There are various reasons why this question has become so relevant. First of all it is a folklore result that the behavior of the exponential function κ^λ is completely determined by the interplay between the gimel function and the powerset function $\lambda \mapsto 2^\lambda$ restricted to the class of regular cardinals (see [15] I.5). Two standard exercises in a graduate course in set theory are to show Cantor's inequality $2^\lambda > \lambda$ for all λ and to prove that $\kappa^{\mathtt{cof}(\kappa)} > \kappa$ for all singular κ.

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On the other hand one of the first application of forcing has been the result of Easton that Cantor's inequality is the unique non trivial restriction that the powerset function can have on the class of regular cardinals [9]. What has been surprising is the richness of properties of the gimel function. The singular cardinal hypothesis SCH asserts that $\kappa^{\mathtt{cof}(\kappa)} = \kappa^+ + 2^{\mathtt{cof}(\kappa)}$ for all singular κ, i.e., has always the least possible value. In the early seventies much effort has been devoted to analyze this principle. The first relevant results have been Silver's proof that SCH holds for all singular cardinals if it holds already for all singular cardinal of countable cofinality [28], Solovay's proof that SCH holds for all singular cardinals which are above a strongly compact cardinal [29] and Jensen's covering lemma showing that the failure of SCH required the existence of $0^\sharp$ [8]. Another major result of Silver has been the proof of the consistency of $\neg$SCH relative to the existence of a supercompact cardinal (see [15] theorem 21.4). In the late seventies Magidor [20] has shown that SCH can first fail even at $\aleph_\omega$. In the eighties the works of Woodin (unpublished) and Gitik [14] have considerably reduced the large cardinals hypothesis needed to obtain the consistency of $\neg$SCH and Gitik proved that $\neg$SCH is equiconsistent with the existence of a measurable cardinal κ of Mitchell order κ^{++}, which is a rather weak large cardinal hypothesis. Thus SCH holds above a sufficiently large cardinal while the consistency of $\neg$SCH requires the existence of models of ZFC with measurable cardinals. These results linked the study of the gimel function to the theory of large cardinals. In the beginning of the eighties Shelah obtained a dramatic improvement of Silver's result and showed that

$$(\aleph_\omega)^{\aleph_0} < \aleph_{\omega_4} + (2^{\aleph_0})^+$$

holds in ZFC [26]. Shelah's proof gave rise to a variety of new techniques which are the basic ingredients of `pcf`-theory (the theory of possibles cofinalities) and which have become the key tools to attack many difficult problems concerning the combinatorics of singular cardinals.

Forcing axioms. During the eighties Shelah's refinement of the forcing techniques led to the introduction of the strongest forcing axioms: the proper forcing axiom PFA (Baumgartner and Shelah [4]) and Martin's maximum MM (Foreman, Magidor and Shelah [12]). These axioms are a strengthening of Martin's axiom MA and assert a principle of saturation of the universe of sets with respect to a large class of forcing notions: a model M of these axioms has the property that a variety of sets which will exist in an appropriate forcing extension over M actually exists in M. These axioms are sufficiently strong to settle many of the classical problems which are independent of ZFC, for example MM and PFA both decide that $2^{\aleph_0} = \aleph_2$ and that SCH holds.[1] The theory produced by these axioms is opposite to the theory of L:

[1]The proof that SCH and $2^{\aleph_0} = \aleph_2$ follow from MM already appeared in [12], where this axiom was introduced. Later on Todorčević and Veličković showed that PFA implies

while the axiom of constructibility $V = L$ is a principle of minimality (i.e., the only sets which exist are those which are necessary in the theory ZFC), forcing axioms are maximality principles in the sense that they assert the actual existence of a large number of sets which can only be shown to exist consistently in ZFC. These axioms have been thus extremely useful to obtain the consistency of many principles whose negation is known to hold in L, the first and simplest example being Suslin's hypothesis which fails in L and holds in a model of MA [30]. There are however a number of other interesting conjectures which holds in models of PFA, for example Moore has shown that PFA solves positively the "five element basis" problem, i.e., in a model of this forcing axiom there are five uncountable linear orders such that, given any other uncountable linear order, at least one of them embeds into it [23].

While the actual formulation of these axioms is rather technical and requires a sophisticated knowledge of forcing, there are a number of simple combinatorial principles which are a consequence of these axioms and which can be used to interpolate the proofs of many consequences of PFA or MM. This will be substantiated in the next section. Moreover the models of these forcing axioms are currently obtained by an appropriate forcing which collapses a supercompact cardinal to $\aleph_2$. It is a matter of fact that many of the consequences of these axioms can be translated in interesting properties of the supercompact cardinal from which a model of these axioms has been obtained. This is particularly the case in all questions concerning singular cardinals because the size of the forcing to produce a model of MM is so small with respect to the size of the combinatorial object that are under investigation that the properties of these objects are almost unaffected after the forcing. Thus a result concerning the properties of singular cardinals obtained using PFA or MM can almost certainly be translated in a theorem concerning the properties of singular cardinals above a supercompact cardinal. This will be the case in all of the problems that we will examine.

§1. PFA, SCH and the P-ideal dichotomy.

In [38] a number of proofs that PFA implies SCH are presented. The core of these proofs is the introduction of a family of covering properties which imply SCH and follow from at least two combinatorial principles which holds under PFA: the simplest of which being the P-ideal dichotomy PID, and the other being Moore's reflection principle MRP. The original proof of SCH from PFA factors through MRP and the key covering properties which I'm going to introduce below and in the next section have been isolated analyzing it.[2] Nonetheless here I

$2^{\aleph_0} = \aleph_2$ [34], while the proof that SCH follows from PFA is the major result of my Ph.D. thesis [36].

[2]The reader interested in MRP and its applications is referred to [22], where a number of new consequences of PFA which have been proved using this axiom are listed and references to the pertinent papers provided. Other applications of MRP appears in [5]. In [38] and [37] the reader can find proofs of SCH assuming MRP.

will only sketch a proof that PID implies SCH. The reason is that PID is an elementary combinatorial statement dealing with objects and concepts which are familiar to any mathematician and the proof that PID implies SCH is at reach for anyone able to understand the statement of theorem 1.2.

1.1. The *P*-ideal dichotomy. The P-ideal dichotomy has been introduced in its full generality by Todorčević [33] developing on previous works by himself [32] and Abraham and himself [2].

Let Z be a set and $[Z]^{\leq\aleph_0} = \{X \subseteq Z : |X| \leq \aleph_0\}$. $\mathcal{I} \subseteq [Z]^{\leq\aleph_0}$ is a P-ideal if it is an ideal and for every countable family $\{X_n\}_n \subseteq \mathcal{I}$ there is an $X \in \mathcal{I}$ such that for all n, $X_n \subseteq^* X$ (where $\subseteq^*$ is inclusion modulo finite).

Definition 1.1 (Todorčević [33]). *The P-ideal dichotomy* (PID) *asserts that for every P-ideal $\mathcal{I}$ on $[Z]^{\leq\aleph_0}$ for some fixed uncountable Z, one of the following holds*:

(i) *There is Y uncountable subset of Z such that $[Y]^{\aleph_0} \subseteq \mathcal{I}$.*
(ii) *$Z = \bigcup_n A_n$ with the property that A_n is orthogonal to $\mathcal{I}$ (i.e., $X \cap Y$ is finite for all $X \in [A_n]^{\aleph_0}$ and $Y \in \mathcal{I}$) for all n.*

In simple words (i) says that $\mathcal{I}$ is large since it is the largest possible ideal on the countable subsets of an uncountable Y, on the other hand (ii) says that $\mathcal{I}$ is small since it reduces to the Frechet ideal on every A_n i.e., $\mathcal{I} \cap [A_n]^{\leq\aleph_0} = [A_n]^{<\aleph_0}$ is the smallest possible ideal which contains all finite sets. Note that (i) and (ii) are incompatible conditions.

PID is a principle which follows from PFA and which is strong enough to rule out many of the standard consequences of $V = L$. For example Abraham and Todorčević [2] have shown that under PID there are no Souslin trees while Todorčević has shown that PID implies the failure of $\square(\kappa)$ on all regular $\kappa > \aleph_1$ [33]. Due to this latter fact the consistency strength of this principle is considerable. Another interesting result by Todorčević is that PID implies that $\mathfrak{b} \leq \aleph_2$.[3] Nonetheless in [2] and [33] it is shown that this principle is consistent with CH. Other interesting applications of PID can be found in [2], [3], [33] and [35].

PID implies SCH. First of all the problem is simplified using Silver's result that SCH holds if it holds for all singular cardinals of countable cofinality [28]. By a standard calculation the latter holds assuming the conclusion of the following:

Theorem 1.2. PID *implies $\lambda^{\aleph_0} = \lambda$ for all regular $\lambda \geq 2^{\aleph_0}$.*

A sketch of the proof: For any cardinal κ of countable cofinality,

$$\mathcal{D} = \{D(n, \beta) : n < \omega, \beta \in \kappa^+\}$$

[3]In [33] it is shown that any gap in $P(\omega)/FIN$ is either an Hausdorff gap or a (κ, ω) gap with κ regular and uncountable. By another result of Todorčević ([32] Lemma 3.10) if $\mathfrak{b} > \aleph_2$ there is an (ω_2, λ) gap in $P(\omega)/FIN$ for some regular uncountable λ. Thus PID is not compatible with $\mathfrak{b} > \aleph_2$.

is a covering matrix for κ^+ if:

(i) for all n and α, $|D(n,\alpha)| < \kappa$,
(ii) for all $\alpha \in \kappa^+$, $D(n,\alpha) \subseteq D(m,\alpha)$ for $n < m$,
(iii) for all $\alpha \in \kappa^+$, $\alpha \subseteq \bigcup_n D(n,\alpha)$,
(iv) for all $\alpha < \beta \in \kappa^+$, if $\alpha \in D(n,\beta)$, then $D(n,\alpha) \subseteq D(n,\beta)$,
(v) for all $X \in [\kappa^+]^{\aleph_0}$ there is $\gamma_X < \kappa^+$ such that for all β, there is n such that $D(m,\beta) \cap X \subseteq D(m,\gamma_X)$ for all $m \geq n$.

FACT 1.3. *For any $\kappa > 2^{\aleph_0}$ singular cardinal of countable cofinality, there is a covering matrix $\mathcal{D}$ on κ^+.*

To define such a $\mathcal{D}$, let for all β, $\phi_\beta \colon \kappa \to \beta$ be a surjection and $(\kappa_n)_n$ be a strictly increasing sequence of regular cardinals converging to κ.

Define by induction on β:

$$D(n,\beta) = \{\beta\} \cup \phi_\beta[\kappa_n] \cup \{D(n,\gamma) \colon \gamma \in \phi_\beta[\kappa_n]\}.$$

It is easy to check properties $(i), \dots, (iv)$ for $\mathcal{D}$. To check property (v) observe that for any countable $X \subseteq \kappa^+$ the map $\psi_X \colon \kappa^+ \to (P(X))^\omega$ defined by $\alpha \mapsto \langle X \cap D(n,\alpha) \colon n \in \omega \rangle$ is constant on an unbounded subset S of κ^+, since $\kappa > 2^{\aleph_0}$. Now $\gamma_X = \min(S)$ satisfies (v) for X. ⊣

We remark that here and fact 1.5 below are the unique part in the proof of theorem 1.2 in which the cardinal arithmetic assumption $\kappa > 2^{\aleph_0}$ is used. Moreover this hypothesis is inessential for fact 1.3. It is possible to prove its conclusion without any cardinal arithmetic assumption with the help of some `pcf`-techniques ([36] Lemma 4.2). We will come back to this in the next section.

Let $\mathcal{D}$ be a covering matrix for κ^+. $\mathcal{D}$ covers κ^+ if there is an unbounded subset A of κ^+ such that $[A]^{\aleph_0}$ is covered by $\mathcal{D}$, i.e., for every $X \in [A]^{\aleph_0}$ there is a $Y \in \mathcal{D}$ such that $X \subseteq Y$.

DEFINITION 1.4. CP *holds if $\mathcal{D}$ covers κ^+ whenever $\mathcal{D}$ is a covering matrix for κ^+ and κ has countable cofinality.*

Now the theorem follows once the following facts are proved.

FACT 1.5. *Assume* CP. *Then $\lambda^{\aleph_0} = \lambda$, for every $\lambda \geq 2^{\aleph_0}$ of uncountable cofinality.*

This is a proof by induction which uses CP only in the inductive stage where λ is the successor of a cardinal κ of countable cofinality. In this case a covering matrix $\mathcal{D}$ on κ^+ exists by the previous fact and by CP there is an unbounded $A \subseteq \kappa^+$ such that $[A]^{\aleph_0}$ is covered by $\mathcal{D}$. Now:

$$(\kappa^+)^{\aleph_0} = |[A]^{\aleph_0}| \leq |\bigcup \{[D(n,\beta)]^{\aleph_0} \colon n \in \omega, \beta \in \kappa^+\}| = \kappa^+.$$

The latter equality holds because by property (ii) of $\mathcal{D}$ each $D(n,\beta)$ has size less than κ, now the inductive assumption can be used to obtain that each $[D(n,\beta)]^{\aleph_0}$ has size less than κ. ⊣

Lemma 1.6. PID *implies* CP.

Let κ be a cardinal of countable cofinality and $\mathcal{D}$ be a covering matrix on κ^+ and set:[4]

$$\mathcal{I} = \{X \in [\kappa^+]^{\aleph_0} \colon X \cap D(n, \alpha) \text{ is finite for all } n,\ \alpha\}.$$

Claim 1.7. *$\mathcal{I}$ is a P-ideal.*

Let $\{X_n \colon n \in \omega\} \subseteq \mathcal{I}$ we need to find an $X \in \mathcal{I}$ which contains all X_n modulo finite. Set $Y = \bigcup_n X_n$. Then Y is countable. Let γ_Y be the ordinal provided by property (v) of the covering matrix. By a standard diagonal argument find an $X \subseteq Y$ such that $X_n \subseteq^* X$ and $X \cap K(m, \gamma_Y)$ is finite for all m and n. Properties (iv) and (v) of the matrix guarantee that X has finite intersection with all $K(n, \alpha)$, so $X \in \mathcal{I}$. ⊣

Now remark that if $Z \subseteq \kappa$ is any set of ordinals of size $\aleph_1$ and $\alpha \in \kappa^+$ is larger than $\sup(Z)$, there must be an n such that $Z \cap D(n, \alpha)$ is uncountable. This means that $\mathcal{I} \not\subseteq [Z]^{\aleph_0}$, since any countable subset of $Z \cap D(n, \alpha)$ is not in $\mathcal{I}$.

This forbids $\mathcal{I}$ to satisfy the first alternative of the P-ideal dichotomy. So the second possibility must be the case, i.e., we can split κ^+ in countably many sets A_n such that $\kappa = \bigcup_n A_n$ and $[A_n]^{\aleph_0} \cap \mathcal{I} = \emptyset$ for each n. Moreover since κ^+ is regular at least one A_n is unbounded in κ^+. The following claim is proved along the same lines of the previous one:

Claim 1.8. *$[A_n]^{\aleph_0}$ is covered by $\mathcal{D}$ for every n.* ⊣

This is enough to get that PID implies CP and to complete the proof of the main theorem 1.2.

§2. A family of covering properties for forcing axioms and large cardinals.

We now present a family of covering properties $\mathsf{CP}(\kappa, \lambda)$ indexed by pairs of regular cardinals $\lambda < \kappa$. The key features of these covering properties are extracted by the analysis of the proof that PID (or MRP) implies SCH. In this section we will also briefly sketch how to use these covering properties to obtain many classical results like:

- the failure of $\square(\kappa)$ assuming that κ is above a strongly compact λ (Solovay [29]),
- the failure of $\square(\kappa)$ for all $\kappa > \aleph_1$ assuming PFA (Todorčević [31]),
- Solovay's result that SCH holds above a strongly compact cardinal (Solovay [29]).

In the next section we will present some original results which use these covering properties to analyze the "saturation" of models of ZFC with a strongly compact cardinal or of models of MM. Elaborating on the proof that PID implies SCH we generalize the notion of a covering matrix as follows:

[4]Moore first noticed that a covering property of this sort holds under PFA (in fact follows from MRP) reading a draft of [37].

DEFINITION 2.1. *For regular cardinals* $\lambda < \kappa$, $\mathcal{D} = \{D(\eta,\beta) : \eta < \lambda, \beta \in \kappa\}$ *is a* λ*-covering matrix for* κ *if*:

(i) $\beta \subseteq \bigcup_{\eta<\lambda} D(\eta,\beta)$ *for all* β,
(ii) $|D(\eta,\beta)| < \kappa$ *for all* β *and* η,
(iii) $D(\eta,\beta) \subseteq D(\xi,\beta)$ *for all* $\beta < \kappa$ *and for all* $\eta < \xi < \lambda$,
(iv) *for all* $\beta < \gamma < \kappa$ *and for all* $\eta < \lambda$, *there is* $\xi < \lambda$ *such that* $D(\eta,\beta) \subseteq D(\xi,\gamma)$.

A λ*-covering matrix* $\mathcal{D}$ *is downward coherent if for all* $\alpha < \beta < \kappa$ *and* $\eta < \lambda$, *there is* $\xi < \lambda$ *such that* $D(\eta,\beta) \cap \alpha \subseteq D(\xi,\alpha)$.

A λ*-covering matrix* $\mathcal{D}$ *is locally downward coherent if for all* $X \in [\kappa]^{\leq\lambda}$, *there is* $\gamma_X < \kappa$ *such that for all* $\beta < \kappa$ *and* $\eta < \lambda$, *there is* $\xi < \lambda$ *such that* $D(\eta,\beta) \cap X \subseteq D(\xi,\gamma_X)$.

$\beta_{\mathcal{D}} \leq \kappa$ *is the least* β *such that for all* η *and* γ, $\mathtt{otp}(D(\eta,\gamma)) < \beta$. $\mathcal{D}$ *is trivial if* $\beta_{\mathcal{D}} = \kappa$

The previous notion of a covering matrix $\mathcal{D}$ for the successor of a κ of countable cofinality is an example of a locally downward coherent ω-covering matrix $\mathcal{D}$ for κ^+ with $\beta_{\mathcal{D}} = \kappa$. There are several means to construct covering matrices:

LEMMA 2.2 ([36] Lemma 4.2). *Assume* κ *is singular of cofinality* λ. *Then there is a* λ*-covering matrix* $\mathcal{D}$ *on* κ^+ *such that* $\mathcal{D}$ *is locally downward coherent and* $\beta_{\mathcal{D}} = \kappa$.

LEMMA 2.3 (Cummings and Schimmerling [7]). *Assume* κ *is regular. Then there is a* κ*-covering matrix* $\mathcal{D}$ *on* κ^+ *such that* $\mathcal{D}$ *is downward coherent and* $\beta_{\mathcal{D}} = \kappa$.

LEMMA 2.4 (Jensen). *Assume* $\square_\kappa$. *Then there is an* ω*-covering matrix* $\mathcal{D}$ *on* κ^+ *which is downward coherent and such that* $\beta_{\mathcal{D}} = \kappa$.

The first lemma is a generalization of fact 1.3. Its proof ties up the notion of a covering matrix with some interesting square-like principles on singular cardinals. Recall that a Jensen matrix on the successor of a singular cardinal κ of countable cofinality is an ω-covering matrix $\mathcal{D}$ on κ^+ with $\beta_{\mathcal{D}} = \kappa$ such that for all α of uncountable cofinality:

$$\bigcup_{n\in\omega} [D(n,\alpha)]^{\aleph_0} \subseteq \bigcup_{n<\omega,\beta<\alpha} [D(n,\beta)]^{\aleph_0}.$$

Jensen constructed such a matrix from square at κ and GCH. Magidor and Foreman introduced the notion of "very weak square" [11]. This is a square-like principle on the successor of a singular κ of countable cofinality which is consistent with κ being larger than a supercompact. There are two equivalent formulation of this principle: one is slightly stronger than the statement that there is a club of points of cofinality $\aleph_1$ in the approachability ideal $\mathcal{I}[\kappa^+]$, the other is that there is a Jensen matrix on κ^+. On the other hand, Shelah has shown that for any fixed uncountable cofinality less than

κ there is always a stationary set S in $\mathcal{I}[\kappa^+]$ of points with this cofinality [27]. One direction of the Magidor and Foreman's equivalence can be used to produce an ω-covering matrix $\mathcal{D}$ with $\beta_{\mathcal{D}} = \kappa$ such that for club many $\alpha \in S$:

$$\bigcup_{n\in\omega}[D(n,\alpha)]^{\aleph_0} \subseteq \bigcup_{n\in\omega,\beta<\alpha}[D(n,\beta)]^{\aleph_0}.$$

This property of $\mathcal{D}$ is enough to show that $\mathcal{D}$ is locally downward coherent and the construction is carried out in ZFC. As we've seen in the previous section, this lemma is essential in the proof that PID implies SCH.

The matrices produced by the second lemma are the key combinatorial devices to prove all the results in the next section. The matrices constructed in the third lemma are useful to obtain proofs that $\square_\kappa$ fails whenever either κ is uncountable and PFA holds or $\kappa \geq \lambda$ and λ is strongly compact.[5]

Definition 2.5. $\mathsf{CP}(\kappa, \lambda)$ *holds if there is A unbounded subset of κ such that $[A]^\lambda$ is covered by $\mathcal{D}$ whenever $\mathcal{D}$ is a locally downward coherent λ-covering matrix on κ.*

To give a flavor on how these covering properties are applied we show the following.

Theorem 2.6. *Assume* PID. *Then* $\mathsf{CP}(\kappa, \omega)$ *holds for all regular* $\kappa \geq \aleph_2$.

This is just a variation of the proof that PID implies CP in the previous section.

Theorem 2.7. *Assume $\kappa \geq \lambda > \theta$ are regular cardinals and λ is strongly compact. Then* $\mathsf{CP}(\kappa, \theta)$ *holds.*

Proof. Recall that λ is strongly compact if for every $\kappa \geq \lambda$ there is a λ-complete fine measure on $[\kappa]^{<\lambda}$. It is not hard to see that this entails that for every regular $\kappa \geq \lambda$, there is a λ-complete uniform ultrafilter $\mathcal{U}$ on κ.

Now let $\theta < \lambda \leq \kappa$ be regular cardinals with λ strongly compact and fix a θ-covering matrix $\mathcal{D} = \{D(\alpha, \beta) \colon \alpha \in \theta, \beta \in \kappa\}$ for κ and a uniform λ-complete ultrafilter $\mathcal{U}$ on κ. Let $A^\gamma_\alpha = \{\beta > \gamma \colon \gamma \in D(\alpha, \beta)\}$ and $A_\alpha = \{\gamma \in \kappa \colon A^\gamma_\alpha \in \mathcal{U}\}$. Since $\theta < \lambda$, by the λ-completeness of $\mathcal{U}$, for every $\gamma \in \kappa$, there is an $\alpha < \theta$ such that $A^\gamma_\alpha \in \mathcal{U}$. Thus $\bigcup_{\alpha<\theta} A_\alpha = \kappa$. So there is $\alpha < \theta$ such that $A_\alpha \in \mathcal{U}$. In particular A_α is unbounded. Now let X be a subset of A_α of size θ. Then $A^\gamma_\alpha \in \mathcal{U}$ for all $\gamma \in X$. Since $|X| = \theta < \lambda$, $\bigcap_{\gamma\in X} A^\gamma_\alpha \in \mathcal{U}$ and thus is non-empty. Pick β in this latter set. Then $X \subseteq D(\alpha, \beta)$. Since X is an arbitrary subset of A_α of size θ, we conclude that $[A_\alpha]^\theta$ is covered by $\mathcal{D}$. This concludes the proof.[6] $\dashv$

[5]Section 2.2.1 of [36] contains a proof by Todorčević of this result which allows for a stronger conclusion: when applied to a $\square(\kappa)$ sequence it produces an ω-covering matrix on κ which can still deny the property $\mathsf{CP}(\kappa, \omega)$ to be defined below. For the sake of simplicity we will content ourselves to the current formulation of the lemma.

[6]Remark that this proof uses only property (i) of $\mathcal{D}$.

THEOREM 2.8. *Assume* $\mathsf{CP}(\kappa^+, \omega)$. *Then* $\square_\kappa$ *fails.*

PROOF. Assume to the contrary that $\mathsf{CP}(\kappa^+, \omega)$ and $\square_\kappa$ hold. By lemma 2.4, $\square_\kappa$ implies that there is $\mathcal{D} = \{D(n, \beta) : n \in \omega, \beta < \kappa\}$ downward coherent ω-covering matrix on κ^+ with $\beta_\mathcal{D} = \kappa$. Now by $\mathsf{CP}(\kappa^+, \omega)$ there is A unbounded subset of κ^+ such that $[A]^{\aleph_0}$ is covered by $\mathcal{D}$.

We first claim that for every β there is n such that $A \cap \beta \subseteq D(n, \beta)$. If this is not the case find a β such that $A \cap \beta \not\subseteq D(n, \beta)$ for all n. Now find X countable subset of $A \cap \beta$ such that $X \not\subseteq D(n, \beta)$ for all n. By $\mathsf{CP}(\kappa, \omega)$, X is contained in $D(n, \alpha)$ for some n and α. By the downward coherence of $\mathcal{D}$ there should be an m such that $X \subseteq D(n, \alpha) \cap \beta \subseteq D(m, \beta)$. This contradicts the very definition of X. Now find β such that $\mathtt{otp}(A \cap \beta) > \kappa$. Then, since $A \cap \beta \subseteq D(n, \beta)$ for some n, $\kappa < \mathtt{otp}(A \cap \beta) \leq \mathtt{otp}(D(n, \beta)) \leq \kappa$. This is the desired contradiction. ⊣

The combination of the last two theorems gives an alternative proof of Solovay's result that $\square_\kappa$ fails for all κ above a strongly compact cardinal. Using the preceding results it is also straightforward to obtain a proof of Solovay's theorem that SCH holds above a strongly compact cardinal.

§3. "Saturation" properties of models of strong forcing axioms.

Since forcing axioms have been able to settle many of the classical problems of set theory, we can expect that the models of a forcing axiom are in some sense categorical. There are many ways in which one can give a precise formulation to this concept. For example, one can study what kind of forcing notions can preserve PFA or MM, or else if a model V of a forcing axiom can have an interesting inner model M of the same forcing axiom. There are many results in this area, some of them very recent. First of all there are results that shows that one has to demand a certain degree of resemblance between V and M. For example assuming large cardinals it is possible to use the stationary tower forcing introduced by Woodin[7] to produce two transitive models $M \subseteq V$ of PFA (or MM or whatever is not conflicting with large cardinal hypothesis) with different ω-sequences of ordinals and an elementary embedding between them. However M and V do not compute the same way neither the ordinals of countable cofinality nor the cardinals. On the other hand, König and Yoshinobu [17, Theorem 6.1] showed that PFA is preserved by ω_2-closed forcing, while it is a folklore result that MM is preserved by ω_2-directed closed forcing. Notice however that all these forcing notions do not introduce new sets of size at most $\aleph_1$. In the other direction, in [34] Veličković used a result of Gitik to show that if MM holds and M is an inner model such that $\omega_2^M = \omega_2$, then $\mathcal{P}(\omega_1) \subseteq M$ and Caicedo and Veličković [5] showed, using the mapping reflection principle MRP introduced by Moore in [24], that if $M \subseteq V$ are models of BPFA and $\omega_2^M = \omega_2$ then $\mathcal{P}(\omega_1) \subseteq M$.

[7] [18] gives a complete presentation of this subject.

In any case all the results so far produced show that any two models $V \subseteq W$ of some strong forcing axiom and with the same cardinals have the same ω_1-sequences of ordinals. Thus it is tempting to conjecture that forcing axioms produce models of set theory which are "saturated" with respect to sets of size $\aleph_1$. One possible way to give a precise formulation to this idea may be the following:

CONJECTURE 3.1 (Caicedo, Veličković). *Assume $W \subseteq V$ are models of* MM *with the same cardinals. Then $[Ord]^{\leq\omega_1} \subseteq W$.*

The reader will find in [38] a complete and detailed presentation of the results below as well as of their proofs. They are obtained elaborating further on the ideas introduced in the previous sections.

THEOREM 3.2. *Assume* MM. *Let κ be a strong limit cardinal and W be an inner model such that κ is regular in W and $\kappa^+ = (\kappa^+)^W$. Then $\mathtt{cof}(\kappa) > \omega_1$.*

The following is the "large cardinal version" of the previous theorem:

THEOREM 3.3. *Assume $\kappa \geq \lambda$ where λ is strongly compact. Let W be an an inner model such that κ is a regular cardinal of W and such that $(\kappa^+)^W = \kappa^+$. Then $\mathtt{cof}(\kappa) \geq \lambda$.*

This shows that above a strongly compact λ one cannot change the cofinality of some regular κ to some $\theta < \lambda$ and preserve at the same time κ^+ and the strong-compactness of λ. A consequence of these theorems is that Prikry forcing on κ produces a generic extension in which MM fails and there are no strongly compact cardinals below κ.

The next proposition is a variation of the original proof by Foreman, Magidor and Shelah that MM implies $\kappa^{\aleph_1} = \kappa$ for all regular $\kappa \geq \aleph_1$ and combined with the previous theorems[8] shows that conjecture 3.1 cannot be made false by set-forcing.

PROPOSITION 3.4. *Assume* MM *and that all limit cardinals are strong limit. Moreover assume that the universe V is a set-generic extension of a class W with the same ordinals of cofinality ω and ω_1 and such that $P(\omega_1)^W \subseteq W$. Then $[Ord]^{\leq\omega_1} \subseteq W$.*

The above results suggest that another interesting form of saturation of models of MM may hold. Gitik has shown [13] that assuming suitable large cardinals it is possible to produce a model of set theory W and a generic extension V of W with the same cardinals and such that the first W-regular cardinal κ which is singular in V has an arbitrarily chosen uncountable cofinality. However the ground model W is obtained by a cardinal preserving forcing which shoots Prikry sequences on a large number of cardinals below κ. Thus this approach cannot work to disprove the following conjecture:

[8]In a model of MM where all limit cardinals are strong limit, an inner model W with the same cardinal will have the same ordinals of cofinality at most $\aleph_1$ by the above theorem and will thus satisfy the hypothesis of the proposition below.

CONJECTURE 3.5. *Assume $W \subseteq V$ are models of* MM *with the same cardinals. Then they have the same cofinalities.*

§4. MM implies $(\aleph_{\omega+1}, \aleph_\omega) \not\twoheadrightarrow (\aleph_2, \aleph_1)$. Recall that the Chang conjecture $(\lambda, \kappa) \twoheadrightarrow (\theta, \nu)$ holds for $\lambda > \kappa \geq \theta > \nu$ if for every structure $\langle Y, \lambda, \kappa, ...\rangle$ with predicates for λ and κ there is $X \prec Y$ such that $|X \cap \lambda| = \theta$ and $|X \cap \kappa| = \nu$. It is known that $(\aleph_2, \aleph_1) \twoheadrightarrow (\aleph_1, \aleph_0)$ as well as many other Chang conjectures are consistent relative to appropriate large cardinals assumptions. For example $(\aleph_2, \aleph_1) \twoheadrightarrow (\aleph_1, \aleph_0)$ holds in models of MM and is consistent relative to the existence of a measurable cardinal [26]. Turning to Chang conjectures for larger cardinals, it is possible to see that $(j(\kappa^{+\theta}), j(\kappa^{+\gamma})) \twoheadrightarrow (\kappa^{+\theta}, \kappa^{+\gamma})$ whenever κ is the critical point of a 2-huge embedding and $\gamma < \theta < \kappa$. Developing on this, Levinsky, Magidor and Shelah in [19] showed that $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_1, \aleph_0)$ is consistent relative to the existence of a 2-huge cardinal. However all the known examples of a consistent $(\kappa^+, \kappa) \twoheadrightarrow (\theta^+, \theta)$ where κ is singular and θ regular are such that $\theta = \mathtt{cof}(\kappa)$. Thus a folklore problem in this field is the following:

PROBLEM 4.1. *Is it consistent that $(\kappa^+, \kappa) \twoheadrightarrow (\theta^+, \theta)$ for some regular θ and singular κ of cofinality different than θ?*

First of all such Chang conjectures affect cardinal arithmetic. A simple fact is the following:

FACT 4.2. *Assume $(\kappa^+, \kappa) \twoheadrightarrow (\theta^+, \theta)$ for some singular κ. Then $\theta^+ \leq \theta^{\mathtt{cof}(\kappa)}$.*

PROOF. Notice that $\kappa^{\mathtt{cof}(\kappa)} > \kappa$. Now assume $(\kappa^+, \kappa) \twoheadrightarrow (\theta^+, \theta)$. Fix $\lambda > \kappa^+$ regular and large enough and let $H(\lambda)$ denotes the family of sets whose transitive closure has size less than θ. Fix $M \prec \langle H(\lambda), \kappa^+, \kappa, ...\rangle$ with $|M \cap \kappa^+| = \theta^+$ and $|M \cap \kappa| = \theta$. Pick a family $\{X_\alpha : \alpha < \kappa^+\} \in M$ of distinct elements of $[\kappa]^{\mathtt{cof}(\kappa)}$. By elementarity of M, $X_\alpha \cap M \neq X_\beta \cap M$ for all $\alpha, \beta \in M \cap \kappa^+$. Thus we have a family of θ^+ distinct elements of $[M \cap \kappa]^{M \cap \mathtt{cof}(\kappa)}$. Now $|M \cap \kappa| = \theta$ and $|M \cap \mathtt{cof}(\kappa)| \leq \mathtt{cof}(\kappa)$. Thus $\theta^+ \leq |[M \cap \kappa]^{M \cap \mathtt{cof}(\kappa)}| \leq \theta^{\mathtt{cof}(\kappa)}$. ⊣

Cummings in [6] has shown that these Chang conjectures can be studied by means of pcf-theory as developed by Shelah[9] and has obtained several other restrictions on the combinatorics of the singular cardinals κ which may satisfy an instance of the above problem. For example he has shown that these Chang conjectures have a much stronger effect on cardinal arithmetic than fact 4.2 and subsume the existence of very strong large cardinals, i.e., out of the scope of analysis of the current inner model theory: it can be argued by the analysis brought up in [6] that if κ has countable cofinality and $(\kappa^+, \kappa) \twoheadrightarrow (\theta^+, \theta)$, $\square_\kappa$ fails and SCH holds at κ. Following the pattern of Cummings' paper it is possible to apply some pcf-theory to obtain further

[9] [1] is a good introduction to the subject.

constraints on the possible scenarios under which $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_2, \aleph_1)$ holds. By the above fact 4.2, this Chang conjecture is the first instance of problem 4.1 which is possibly consistent with $\kappa = \aleph_\omega$ and the failure of the continuum hypothesis. The main result we can achieve is that a mild reflection principle for stationary subsets of $\aleph_2$ denies the consistency of $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_2, \aleph_1)$. Let S^κ_λ denote the subset of λ of points of cofinality κ. A stationary subset of λ reflects on α if it intersects all the closed and unbounded subsets of α.

DEFINITION 4.3. *Let λ be a regular cardinal. $S(\lambda)$ holds if every family of less than λ-many stationary subsets of $S^{\aleph_0}_\lambda$ jointly reflects on an ordinal of cofinality less than λ. $S^*(\lambda)$ holds if for every family $\{S_\alpha : \alpha < \lambda\}$ of stationary subsets of $S^{\aleph_0}_\lambda$ there are stationary many $\delta < \lambda$ such that S_α reflects on δ for all $\alpha < \delta$.*

It is clear that $S^*(\lambda)$ implies $S(\lambda)$. Moreover $S^*(\lambda)$ holds if λ is weakly compact[10] and $S^*(\aleph_2)$ follows from a suitable fragment of MM. The exact consistency strength of $S(\aleph_2)$ and $S^*(\aleph_2)$ has been given by Magidor [21] who showed that both these principles are equiconsistent with $\aleph_2$ being a weakly compact cardinal[11] in L. Another scenario suggested by Foreman to obtain $S^*(\lambda)$ is the following: assume that $\mathcal{I}$ is a λ-complete, fine ideal which concentrates on $[\kappa]^{<\lambda}$ and such that $P_\mathcal{I} = P([\kappa]^{<\lambda})/\mathcal{I}$ is a proper forcing. Then $S^*(\lambda)$ holds. Here is a sketchy argument: First of all $\mathcal{I}$ is precipitous since $P_\mathcal{I}$ is proper ([10] Proposition 4.10). Let G be a generic filter for $P_\mathcal{I}$. Then the ultrapower $M = V^{([\kappa]^{<\lambda})} \cap V/G$ defined in $V[G]$ is well-founded. Let $j : V \to M$ be the associated generic elementary embedding. Since $\mathcal{I}$ is λ-complete and fine, we have that the critical point of j is λ. Now let $\{S_\alpha : \alpha < \lambda\} \in V$ be a family of stationary subsets of $S^{\aleph_0}_\lambda$. It is clear that M models that $j(\{S_\alpha : \alpha < \lambda\}) = \{T_\alpha : \alpha < j(\lambda)\}$ is a family of stationary subset of $S^{\aleph_0}_{j(\lambda)}$. Now $T_\alpha = j(S_\alpha)$ and $j(S_\alpha) \cap \lambda = S_\alpha$ for all $\alpha < \lambda$. Since P is proper, S_α is a stationary subset of λ in $V[G]$ so it is certainly a stationary subset of λ in M. Then M models that $j(S_\alpha)$ reflects on λ for all $\alpha < \lambda$. Now the argument to show that $S^*(\lambda)$ holds in V is routine.[12] The main result is the following:

THEOREM 4.4. *$S(\aleph_2)$ implies $(\kappa^+, \kappa) \not\twoheadrightarrow (\aleph_2, \aleph_1)$ for all singular κ of countable cofinality.*

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[10] This is a routine fact if we use the following definition of weak compactness: λ is weakly compact if for every transitive model M of ZFC minus the powerset axiom such that M has size λ and $H(\lambda) \subseteq M$, there is an elementary embedding of M into a transitive structure N with critical point λ.

[11] I thank Assaf Sharon for pointing out this latter result to me and Paul Larson for a proof that $S^*(\aleph_2)$ follows from MM.

[12] Notice that we've hidden a (possibly very) large cardinal assumption in the requirement that P is proper. See [10] for a presentation of generic large cardinals.

We introduce some basic concepts of pcf-theory. Let κ be a singular cardinal. Shelah has shown that there is an increasing sequence of regular cardinals $\{\kappa_\xi : \xi < \mathtt{cof}(\kappa)\}$ converging to κ and a family $\mathcal{F} = \{f_\alpha : \alpha < \kappa^+\} \subseteq \prod_{\xi<\mathtt{cof}(\kappa)} \kappa_\xi$ which is strictly increasing and cofinal under the ordering of eventual dominance $<^*$ (where $f <^* g$ if the set of $\eta < \mathtt{cof}(\kappa)$ such that $f(\eta) \geq g(\eta)$ is bounded). Such an $\mathcal{F}$ is called a scale.

DEFINITION 4.5. *An ordinal*[13] $\delta < \kappa^+$ *of cofinality larger than* $\mathtt{cof}(\kappa)$ *is a good point for a scale* $\mathcal{F} = \{f_\alpha : \alpha < \kappa^+\}$ *on* $\prod_{\xi<\mathtt{cof}(\kappa)} \kappa_\xi$ *iff there is* X *unbounded subset of* δ *and* $\xi < \mathtt{cof}(\kappa)$ *such that* $f_\alpha(\eta) < f_\beta(\eta)$ *for all* $\alpha < \beta \in X$ *and all* $\eta > \xi$.

The following fact can be proved:

FACT 4.6. *Assume* κ *is singular of countable cofinality. Then* $(\kappa^+, \kappa) \twoheadrightarrow (\theta^+, \theta)$ *implies that the set of* δ *of cofinality* θ^+ *which are not good for a scale* $\mathcal{F}$ *is stationary in* κ^+.

The main result follow from the above fact and theorem 4.7:

THEOREM 4.7. *Assume* $S(\lambda)$. *Let* κ *be singular of cofinality less than* λ. *Let* $\mathcal{F} = \{f_\alpha : \alpha < \kappa^+\} \subseteq \prod_{\xi<\mathtt{cof}(\kappa)} \kappa_\xi$ *be a scale. Then the set of good points for* $\mathcal{F}$ *is a club in* $S^{\lambda}_{\kappa^+}$.

A comment is in order. Sharon has first exploited the idea of using joint reflection of stationary subsets of $\aleph_n$ to investigate $S^{\aleph_n}_{\aleph_{\omega+1}}$ and has proved the following:[14]

THEOREM 4.8 (Sharon). *Assume* $2^{\aleph_n} < \aleph_\omega$ *and* $S^*(\aleph_n)$. *Then the set of good points for a scale in* $S^{\aleph_n}_{\aleph_{\omega+1}}$ *equals modulo a club the set of approachable points.*

From the two theorems above it can be inferred:

COROLLARY 4.9. *Assume* MM. *Then the set of approachable points is a club in* $S^{\aleph_2}_{\aleph_{\omega+1}}$.

Remark that Magidor (unpublished) has shown that MM implies that the set of non-good points for a scale in $S^{\aleph_1}_{\aleph_{\omega+1}}$ is stationary. The interested reader is referred to the forthcoming [25] for a proof of the above theorems.

§5. Some open problems.

First of all there are the two conjectures:

CONJECTURE 5.1 (Caicedo, Veličković). *Assume* $W \subseteq V$ *are models of* MM *with the same cardinals. Then* $[Ord]^{\leq\omega_1} \subseteq W$.

CONJECTURE 5.2. *Assume* $W \subseteq V$ *are models of* MM *with the same cardinals. Then they have the same cofinalities.*

A problem which seems to be related to the first conjecture is the following:

[13] We adopt standard terminology, see again [6] for a fast review of these concepts.

[14] The reader can find a definition of approachable point in [1] or even in [6]. I do not introduce it here for the sake of simplicity.

PROBLEM 5.3. *Does* MM *imply that* $\aleph_\omega$ *is not a Jónsson cardinal*?

Recall that κ is Jónsson if for every $f : [\kappa]^{<\omega} \to \kappa$ there is X proper subset of κ of size κ which is f-closed, i.e., $f[[X]^{<\omega}] \subseteq X$.

A link between problem 5.3 and conjecture 5.1 is the following observation: if $W \subseteq V$ are counterexamples to the conjecture at $\aleph_\omega$ and $[\aleph_n]^{\aleph_n} \subseteq W$ for all n, then W models that $\aleph_\omega$ is Jónsson. Moreover König has shown that MM is consistent with $\aleph_\omega$ not being Jónsson [16].

These problems concern the properties of singular cardinals in models of strong forcing axioms. On the other hand a question which calls for a solution in ZFC is suggested by the results of the previous section:

PROBLEM 5.4. *Is the Chang conjecture* $(\aleph_{\omega+1}, \aleph_\omega) \twoheadrightarrow (\aleph_2, \aleph_1)$ *consistent*?

REFERENCES

[1] U. ABRAHAM and M. MAGIDOR, *Cardinal arithmetic*, ***Handbook of set theory*** (M. Foreman, A. Kanamori, and M. Magidor, editors), North Holland, to appear.

[2] U. ABRAHAM and S. TODORČEVIĆ, *Partition Properties of ω_1 Compatible with CH*, ***Fundamenta Mathematicae***, vol. 152 (1997), pp. 165–180.

[3] B. BALCAR, T. JECH, and T. PAZÁK, *Complete ccc Boolean algebras, the order sequential topology, and a problem of von Neumann*, ***Bulletin of the London Mathematical Society***, vol. 37 (2005), pp. 885–898.

[4] J. BAUMGARTNER, *Applications of the proper forcing axiom*, ***Handbook of set-theoretic topology*** (K. Kunen and J. E. Vaughan, editors), North-Holland, Amsterdam, 1984, pp. 913–959.

[5] A. CAICEDO and B. VELIČKOVIĆ, *Bounded proper forcing axiom and well orderings of the reals*, ***Mathematical Research Letters***, vol. 13 (2006), no. 2–3, pp. 393–408.

[6] J. CUMMINGS, *Collapsing successors of singulars*, ***Proceedings of the American Mathematical Society***, vol. 125 (1997), no. 9, pp. 2703–2709.

[7] J. CUMMINGS and E. SCHIMMERLING, *Indexed squares*, ***Israel Journal of Mathematics***, vol. 131 (2002), pp. 61–99.

[8] K. DEVLIN and R. B. JENSEN, *Marginalia to a theorem of Silver*, $\models$ ***ISILC Logic Conference*** (G. H. Müller, A. Oberschelp, and K. Potthoff, editors), Lecture Notes in Mathematics, vol. 499, Springer, 1975, pp. 115–142.

[9] W. B. EASTON, *Powers of regular cardinals*, ***Annals of Mathematical Logic***, vol. 1 (1970), pp. 139–178.

[10] M. FOREMAN, *Ideals and Generic Elementary Embeddings*, ***Handbook of set theory*** (M. Foreman, A. Kanamori, and M. Magidor, editors), North Holland, to appear.

[11] M. FOREMAN and M. MAGIDOR, *A very weak square principle*, ***The Journal of Symbolic Logic***, vol. 1 (1997), pp. 175–196.

[12] M. FOREMAN, M. MAGIDOR, and S. SHELAH, *Martin's Maximum, saturated ideals and nonregular ultrafilters*, ***Annals of Mathematics (2)***, vol. 127 (1988), no. 1, pp. 1–47.

[13] M. GITIK, *Changing cofinalities and the non-stationary ideal*, ***Israel Journal of Mathematics***, vol. 56 (1986), no. 3, pp. 280–314.

[14] ———, *The strength of the failure of SCH*, ***Annals of Pure and Applied Logics***, vol. 51 (1991), no. 3, pp. 215–240.

[15] T. JECH, ***Set theory***, The Third Millennium Edition ed., Springer, 2002, revised and expanded.

[16] B. KÖNIG, *Forcing indestructibility of set theoretic axioms*, ***The Journal of Symbolic Logic***, vol. 72 (2007), pp. 349–360.

[17] B. KÖNIG and Y. YOSHINOBU, *Fragments of Martin's Maximum in generic extensions*, ***Mathematical Logic Quarterly***, vol. 50 (2004), no. 3, pp. 297–302.

[18] P. B. LARSON, ***The Stationary Tower: Notes on a Course by W. Hugh Woodin***, AMS, 2004.

[19] J. LEVINSKI, M. MAGIDOR, and S. SHELAH, *Chang's conjecture for* $\aleph_\omega$, ***Israel Journal of Mathematics***, vol. 69 (1990), no. 2, pp. 161–172.

[20] M. MAGIDOR, *On the singular cardinal problem I*, ***Israel Journal of Mathematics***, vol. 28 (1977), pp. 1–31.

[21] ———, *Reflecting stationary sets*, ***The Journal of Symbolic Logic***, vol. 4 (1982), pp. 755–771.

[22] J. T. MOORE, *Proper forcing, the continuum, and uncountable linear orders*, this BULLETIN, vol. 11 (2005), no. 1, pp. 51–60.

[23] ———, *A five element basis for the uncountable linear orders*, ***Annals of Mathematics***, vol. 163 (2006), no. 2, pp. 669–688.

[24] ———, *The proper forcing axiom, Prikry forcing, and the singular cardinals hypothesis*, ***Annals of Pure and Applied Logic***, vol. 140 (2006), no. 1–3, pp. 128–132.

[25] A. SHARON and M. VIALE, *A mild reflection principle on* $\aleph_2$ *implies that all points in* $S^{\aleph_2}_{\aleph_{\omega+1}}$ *are approachable*, in preparation.

[26] S. SHELAH, ***Proper Forcing***, Springer, 1982.

[27] ———, ***Cardinal Arithmetic***, Oxford University Press, 1994.

[28] J. H. SILVER, *On the singular cardinal problem*, ***Proceedings of the International Congress of Mathematicians, Vancouver, B.C., 1974***, vol. 1, 1975, pp. 265–268.

[29] R. M. SOLOVAY, *Strongly compact cardinals and the GCH*, ***Proceedings of the Symposium in Pure Mathematics*** (L. Henkin et al., editors), vol. XXV, University of California, Berkeley, CA, 1971, pp. 365–372.

[30] R. M. SOLOVAY and S. TENNEBAUM, *Iterated Cohen extensions and Souslin's problem*, ***Annals of Mathematics***, vol. 94 (1971), no. 2, pp. 201–245.

[31] S. TODORČEVIĆ, *A note on the proper forcing axiom*, ***Axiomatic set theory (Boulder, Colorado, 1983)***, Contemporary Mathematics, vol. 31, American Mathematical Society, 1984, pp. 209–218.

[32] ———, ***Partitions problems in topology***, Contemporary mathematics, vol. 84, American Mathematical Society, 1989.

[33] ———, *A dichotomy for P-ideals of countable sets*, ***Fundamenta Mathematicae***, vol. 166 (2000), no. 3, pp. 251–267.

[34] B. VELIČKOVIĆ, *Forcing axioms and stationary sets*, ***Advances in Mathematics***, vol. 94 (1992), no. 2, pp. 256–284.

[35] ———, *CCC Forcing and Splitting Reals*, ***Israel Journal of Mathematics***, vol. 147 (2005), pp. 209–220.

[36] M. VIALE, ***Application of the proper forcing axiom to cardinal arithmetic***, Ph.D. thesis, Université Paris 7-Denis Diderot, 2006.

[37] ———, *The Proper Forcing Axiom and the Singular Cardinal Hypothesis*, ***The Journal of Symbolic Logic***, vol. 71 (2006), no. 2, pp. 473–479.

[38] ———, *A family of covering properties*, ***Mathematical Research Letters***, to appear.

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