

# SOME REMARKS ON ONE-BASEDNESS

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ABSTRACT. A type analysable in one-based types in a simple theory is itself one-based.

## 1. INTRODUCTION

Recall that a type  $p$  over a set  $A$  in a simple theory is *one-based* if for any tuple  $\bar{a}$  of realizations of  $p$  and any  $B \supseteq A$  the canonical base  $\text{Cb}(\bar{a}/B)$  is contained in  $\text{bdd}(\bar{a}A)$ . One-basedness implies that the forking geometry is particularly well-behaved; for instance one-based groups are bounded-by-abelian-by-bounded. Ehud Hrushovski showed in [3, Proposition 3.4.1] that for stable stably embedded types one-basedness is preserved under analyses: If  $p$  is stable stably embedded in a supersimple theory, and analysable (in the technical sense defined in the next section) in one-based types, then  $p$  is itself one-based. Zoé Chatzidakis then gave another proof for supersimple structures [1, Theorem 3.10], using semi-regular analyses. We shall give an easy direct proof of the theorem stated in the abstract, thus removing the hypotheses of stability, stable embedding, or supersimplicity; it is similar to Hrushovski's proof, but does not use germs of definable functions (which work less well in simple unstable theories), and has to deal with non-stationarity of types. While we are at it, we shall also generalize the notion of bounded closure and one-basedness to  $\Sigma$ -closure and  $\Sigma$ -basedness, where  $\Sigma$  is an  $\emptyset$ -invariant collection of partial types (thought of as small). This may for instance be applied to consider one-basedness *modulo types of finite SU-rank*, or *modulo superstable types*.

Our notation is standard and follows [5]. Throughout the paper, the ambient theory will be simple, and we shall be working in  $\mathfrak{M}^{heq}$ , where

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$\mathfrak{M}$  is a sufficiently saturated model of the ambient theory. Thus tuples are tuples of hyperimaginaries, and  $\text{dcl} = \text{dcl}^{heq}$ .

## 2. $\Sigma$ -CLOSURE

In this section  $\Sigma$  will be an  $\emptyset$ -invariant family of partial types. We first recall the notions of internality and analysability.

**Definition 1.** Let  $\pi$  be a partial type over  $A$ . Then  $\pi$  is

- (*almost, resp.*) *internal* in  $\Sigma$ , or (*almost, resp.*)  $\Sigma$ -*internal*, if for every realization  $a$  of  $\pi$  there is  $B \downarrow_A a$  and  $\bar{b}$  realizing types in  $\Sigma$  based on  $B$ , such that  $a \in \text{dcl}(B\bar{b})$  (or  $a \in \text{bdd}(B\bar{b})$ , respectively).
- *analysable* in  $\Sigma$ , or  $\Sigma$ -*analysable*, if for any  $a \models \pi$  there are  $(a_i : i < \alpha) \in \text{dcl}(A, a)$  such that  $\text{tp}(a_i/A, a_j : j < i)$  is  $\Sigma$ -internal for all  $i < \alpha$ , and  $a \in \text{bdd}(A, a_i : i < \alpha)$ .

A type  $\text{tp}(a/A)$  is *foreign* to  $\Sigma$  if  $a \not\downarrow_{AB} \bar{b}$  for all  $B \downarrow_A a$  and  $\bar{b}$  realizing types in  $\Sigma$  over  $B$ .

**Definition 2.** The  $\Sigma$ -closure  $\Sigma\text{cl}(A)$  of a set  $A$  is the collection of all hyperimaginaries  $a$  such that  $\text{tp}(a/A)$  is  $\Sigma$ -analysable.

We think of  $\Sigma$  as a family of small types. For instance, if  $\Sigma$  is the family of all bounded types, then  $\Sigma\text{cl}(A) = \text{bdd}(A)$ . Other possible choices might be the family of all types of  $SU$ -rank  $< \omega^\alpha$ , for some ordinal  $\alpha$ , or the family of all superstable types. If  $P$  is an  $\emptyset$ -invariant family of types, and  $\Sigma$  is the family of all  $P$ -analysable types to which all types in  $P$  are foreign, then  $\Sigma\text{cl}(A) = \text{cl}_P(A)$  as defined in [5, Definition 3.5.1]; if  $P$  consists of a single regular type  $p$ , this in turn is the  $p$ -closure from [2] (see also [4, p. 265]).

**Remark 1.** In general  $\text{bdd}(A) \subseteq \Sigma\text{cl}(A)$ ; if the inequality is strict, then  $\Sigma\text{cl}(A)$  has the same cardinality as the ambient monster model, and hence violates the usual conventions. However, this is usually harmless. Note that  $\Sigma\text{cl}(\cdot)$  is a closure operator.

**Fact 2.** The following are equivalent:

- (1)  $\text{tp}(a/A)$  is foreign to  $\Sigma$ .
- (2)  $a \not\downarrow_A \Sigma\text{cl}(A)$ .
- (3)  $a \not\downarrow_A \text{dcl}(aA) \cap \Sigma\text{cl}(A)$ .
- (4)  $\text{dcl}(aA) \cap \Sigma\text{cl}(A) \subseteq \text{bdd}(A)$ .

*Proof:* This follows immediately from [5, Proposition 3.4.12]; see also [5, Lemma 3.5.3].  $\square$

$\Sigma$ -closure is well-behaved with respect to independence.

**Lemma 3.** *Suppose  $A \downarrow_B C$ . Then  $\Sigma\text{cl}(A) \downarrow_{\Sigma\text{cl}(B)} \Sigma\text{cl}(C)$ . More precisely, for any  $A_0 \subseteq \Sigma\text{cl}(A)$  we have  $A_0 \downarrow_{B_0} \Sigma\text{cl}(C)$ , where  $B_0 = \text{dcl}(A_0 B) \cap \Sigma\text{cl}(B)$ . In particular,  $\Sigma\text{cl}(AB) \cap \Sigma\text{cl}(BC) = \Sigma\text{cl}(B)$ .*

*Proof:* Let  $B_1 = \Sigma\text{cl}(B) \cap \text{dcl}(BC)$ . Then  $C \downarrow_B A$  implies  $C \downarrow_{B_1} A$ , and  $\text{tp}(C/B_1)$  is foreign to  $\Sigma$  by Fact 2 ( $3 \Rightarrow 1$ ). Hence  $C \downarrow_{B_1} \Sigma\text{cl}(A)$ , and  $C \downarrow_{B_1} A_0$ .

Since  $\text{tp}(A_0/B_0)$  is foreign to  $\Sigma$  by Fact 2, we obtain  $A_0 \downarrow_{B_0} \Sigma\text{cl}(B_0)$ . But  $\Sigma\text{cl}(B_0) = \Sigma\text{cl}(B) \supseteq B_1$ , whence  $A_0 \downarrow_{B_0} C$  by transitivity, and finally  $A_0 \downarrow_{B_0} \Sigma\text{cl}(C)$  by foreignness to  $\Sigma$  again.  $\square$

### 3. $\Sigma$ -BASEDNESS

Again,  $\Sigma$  will be an  $\emptyset$ -invariant family of partial types.

**Definition 3.** A type  $p$  over  $A$  is  $\Sigma$ -based if  $\text{Cb}(\bar{a}/\Sigma\text{cl}(B)) \subseteq \Sigma\text{cl}(\bar{a}A)$  for any tuple  $\bar{a}$  of realizations of  $p$  and any  $B \supseteq A$ .

**Remark 4.** *Equivalently,  $p \in S(A)$  is  $\Sigma$ -based if  $\bar{a} \downarrow_{\Sigma\text{cl}(\bar{a}A) \cap \Sigma\text{cl}(B)} \Sigma\text{cl}(B)$  for any tuple  $\bar{a}$  of realisations of  $p$  and any  $B \supseteq A$ .*

**Lemma 5.** *Suppose  $\text{tp}(a/A)$  is  $\Sigma$ -based,  $A \subseteq B$ , and  $a_0 \in \Sigma\text{cl}(\bar{a}B)$ , where  $\bar{a}$  is a tuple of realizations of  $\text{tp}(a/A)$ . Then  $\text{tp}(a_0/B)$  is  $\Sigma$ -based.*

*Proof:* Let  $\bar{a}_0$  be a tuple of realizations of  $\text{tp}(a_0/B)$ , and  $C \supseteq B$ . There is a tuple  $\tilde{a}$  of realizations of  $\text{tp}(a/A)$  such that  $\bar{a}_0 \in \Sigma\text{cl}(\tilde{a}B)$ ; we may choose it such that  $\tilde{a} \downarrow_{\bar{a}_0 B} C$ . Then  $\Sigma\text{cl}(\tilde{a}B) \cap \Sigma\text{cl}(C) \subseteq \Sigma\text{cl}(\bar{a}_0 B)$  by Lemma 3.

Put  $X = \text{Cb}(\tilde{a}/\Sigma\text{cl}(C))$ . By  $\Sigma$ -basedness of  $\text{tp}(a/A)$  we have

$$X \subseteq \Sigma\text{cl}(\tilde{a}A) \cap \Sigma\text{cl}(C) \subseteq \Sigma\text{cl}(\bar{a}_0 B).$$

As  $\tilde{a} \downarrow_X \Sigma\text{cl}(C)$  we get  $\tilde{a}B \downarrow_{XB} \Sigma\text{cl}(C)$ , and hence  $\bar{a}_0 \downarrow_Y \Sigma\text{cl}(C)$  by Lemma 3, where  $Y = \Sigma\text{cl}(XB) \cap \text{dcl}(\bar{a}_0 XB)$ . As  $Y \subseteq \Sigma\text{cl}(C)$ , we have

$$\text{Cb}(\bar{a}_0/\Sigma\text{cl}(C)) \subseteq Y \subseteq \Sigma\text{cl}(XB) \subseteq \Sigma\text{cl}(\bar{a}_0 B). \quad \square$$

**Lemma 6.** *If  $\text{tp}(a)$  and  $\text{tp}(b)$  are  $\Sigma$ -based, so is  $\text{tp}(ab)$ .*

*Proof:* Let  $\bar{a}$  and  $\bar{b}$  be tuples of realizations of  $\text{tp}(a)$  and  $\text{tp}(b)$ , respectively, and consider a set  $A$  of parameters. We add  $\Sigma\text{cl}(\bar{a}\bar{b}) \cap \text{Cb}(\bar{a}\bar{b}/\Sigma\text{cl}(A))$  to the language. By  $\Sigma$ -basedness of  $\text{tp}(a)$  we get

$$\text{Cb}(\bar{a}/\Sigma\text{cl}(A)) \subseteq \Sigma\text{cl}(\bar{a}) \cap \text{Cb}(\bar{a}\bar{b}/\Sigma\text{cl}(A)) = \text{dcl}(\emptyset),$$

whence  $\bar{a} \perp \Sigma\text{cl}(A)$ ; similarly  $\bar{b} \perp \Sigma\text{cl}(A)$ .

Put  $b_1 = \text{Cb}(\bar{b}/\Sigma\text{cl}(\bar{a}A))$ , and choose  $\bar{a}'A' \models \text{tp}(\bar{a}A/b_1)$  with  $\bar{a}'A' \perp_{b_1} \bar{a}\bar{b}A$ . Then  $b_1 \in \Sigma\text{cl}(\bar{a}'A')$ ; by  $\Sigma$ -basedness of  $\text{tp}(a)$  and Lemma 5 applied to  $\bar{a}b_1 \in \Sigma\text{cl}(\bar{a}\bar{a}'A')$  we have  $\text{Cb}(\bar{a}b_1/\Sigma\text{cl}(AA')) \subseteq \Sigma\text{cl}(\bar{a}b_1A')$ .

If  $Y = \Sigma\text{cl}(\emptyset) \cap \text{dcl}(b_1)$ , then  $A \perp_Y b_1$  by Lemma 3, as  $b_1 \in \Sigma\text{cl}(\bar{b})$  by  $\Sigma$ -basedness of  $\text{tp}(b)$  and because  $\bar{b} \perp \Sigma\text{cl}(A)$ ; since  $\text{tp}(A'/b_1) = \text{tp}(A/b_1)$  we also have  $A' \perp_Y b_1$ , whence  $A' \perp_Y \bar{a}b_1A$ , and  $A' \perp_{YA} \bar{a}b_1$ . As  $\Sigma\text{cl}(YA) = \Sigma\text{cl}(A)$ , Lemma 3 implies

$$\begin{aligned} \text{Cb}(\bar{a}b_1/\Sigma\text{cl}(A)) &= \text{Cb}(\bar{a}b_1/\Sigma\text{cl}(AA')) \subseteq \Sigma\text{cl}(\bar{a}b_1A') \cap \Sigma\text{cl}(A) \\ &\subseteq \Sigma\text{cl}(\bar{a}b_1) \subseteq \Sigma\text{cl}(\bar{a}\bar{b}), \end{aligned}$$

by Lemma 3 since  $A' \perp_{\bar{a}b_1Y} A$ . On the other hand, put  $C = \text{Cb}(\bar{a}b_1/\Sigma\text{cl}(A))$ . Then  $\bar{b} \perp_{b_1} \Sigma\text{cl}(\bar{a}A)$  by definition of  $b_1$ , whence  $\bar{a}\bar{b} \perp_{\bar{a}b_1} \Sigma\text{cl}(A)$ ; as  $\bar{a}b_1 \perp_C \Sigma\text{cl}(A)$  we get  $\bar{a}\bar{b} \perp_C \Sigma\text{cl}(A)$ , whence  $\text{Cb}(\bar{a}\bar{b}/\Sigma\text{cl}(A)) \subseteq C$ . So

$$\begin{aligned} \text{Cb}(\bar{a}\bar{b}/\Sigma\text{cl}(A)) &= \text{Cb}(\bar{a}b_1/\Sigma\text{cl}(A)) \cap \text{Cb}(\bar{a}\bar{b}/\Sigma\text{cl}(A)) \\ &\subseteq \Sigma\text{cl}(\bar{a}\bar{b}) \cap \text{Cb}(\bar{a}\bar{b}/\Sigma\text{cl}(A)) = \text{dcl}(\emptyset), \end{aligned}$$

whence  $\bar{a}\bar{b} \perp \Sigma\text{cl}(A)$ . □

**Corollary 7.** *If  $\text{tp}(a_i)$  is  $\Sigma$ -based for all  $i < \alpha$ , so is  $\text{tp}(\bigcup_{i < \alpha} a_i)$ .*

*Proof:* We use induction on  $\beta$  to show that  $\text{tp}(\bigcup_{i < \beta} a_i)$  is  $\Sigma$ -based, for  $\beta \leq \alpha$ . This is clear for  $\beta = 0$ ; it follows from Lemma 6 for successor ordinals. And if  $\beta$  is a limit ordinal, then for any set  $A$

$$\text{Cb}(\bigcup_{i < \beta} a_i/\Sigma\text{cl}(A)) = \bigcup_{i < \beta} \text{Cb}(\bigcup_{j \leq i} a_j/\Sigma\text{cl}(A)) \subseteq \Sigma\text{cl}(\bigcup_{i < \beta} a_i). \quad \square$$

**Lemma 8.** *If  $\text{tp}(a/A)$  is  $\Sigma$ -based and  $a \perp A$ , then  $\text{tp}(a)$  is  $\Sigma$ -based.*

*Proof:* Let  $\bar{a}$  be a tuple of realizations of  $\text{tp}(a)$ , and consider a set  $B$  of parameters. For every  $a_i \in \bar{a}$  choose  $A_i$  with  $\text{tp}(a_iA_i) = \text{tp}(aA)$  and  $A_i \perp_{a_i} (\bar{a}, B, A_j : j < i)$ . As  $A_i \perp_{a_i}$  we obtain  $A_i \perp (\bar{a}, B, A_j : j < i)$ , whence  $A_i \perp_{(A_j : j < i)} \bar{a}B$ , and inductively  $(A_j : j \leq i) \perp \bar{a}B$ . Put  $\bar{A} =$

$\bigcup_{a_i \in \bar{a}} A_i$ ; we just saw that  $\bar{A} \perp \bar{a}B$ . Now  $\text{tp}(a_i/\bar{A})$  is  $\Sigma$ -based for all  $a_i \in \bar{a}$ , and so is  $\text{tp}(\bar{a}/\bar{A})$  by Corollary 7. As  $\bar{a} \perp_B \bar{A}$ , Lemma 3 implies  $\text{Cb}(\bar{a}/\Sigma\text{cl}(B)) = \text{Cb}(\bar{a}/\Sigma\text{cl}(\bar{A}B)) \subseteq \Sigma\text{cl}(\bar{a}\bar{A}) \cap \Sigma\text{cl}(B) = \Sigma\text{cl}(\bar{a}) \cap \Sigma\text{cl}(B)$ , where the last equality follows from  $\bar{a}A \perp_{\bar{a}} B$  and Lemma 3.  $\square$

**Corollary 9.** *If  $p$  is almost internal in  $\Sigma$ -based types, then  $p$  is  $\Sigma$ -based.*

*Proof:* Suppose  $p = \text{tp}(a/A)$ , and choose  $B \perp_A a$  and  $\bar{b}$  such that  $a \in \text{bdd}(B\bar{b})$  and  $\text{tp}(b/B)$  is  $\Sigma$ -based for all  $b \in \bar{b}$ . Then  $\text{tp}(\bar{b}/AB)$  is  $\Sigma$ -based by Lemma 7, as is  $\text{tp}(a/AB)$  by Lemma 5, and  $\text{tp}(a/A)$  by Lemma 8.  $\square$

**Lemma 10.** *If  $\text{tp}(a)$  and  $\text{tp}(b/a)$  are  $\Sigma$ -based, so is  $\text{tp}(ab)$ .*

*Proof:* Consider a tuple  $\bar{a}\bar{b}$  of realizations of  $\text{tp}(ab)$ , and a set  $A$  of parameters. As  $\text{tp}(\bar{a})$  and  $\text{tp}(\bar{b}/\bar{a})$  are both  $\Sigma$ -based, we may suppose  $a = \bar{a}$  and  $b = \bar{b}$ . Put  $C = \text{Cb}(ab/\Sigma\text{cl}(A))$ ; again we add  $\Sigma\text{cl}(ab) \cap C$  to the language. By  $\Sigma$ -basedness of  $\text{tp}(a)$  we get  $a \perp \Sigma\text{cl}(A)$ .

Consider a Morley sequence  $(a_i b_i : i < \omega)$  in  $\text{lstp}(ab/C)$ ; we may assume that  $(a_i b_i : i < \omega) \perp_C abA$ . Since  $(a_i : i < \omega) \perp C$  we get  $ab \perp (a_i : i < \omega)$ . Moreover, as  $\text{tp}(ab/C)$  is foreign to  $\Sigma$ , we have  $ab \perp_C \Sigma\text{cl}(a_i b_i : i < \omega)$ . On the other hand  $C \in \text{dcl}(a_i b_i : i < \omega)$ , whence

$$C = \text{Cb}(ab/\Sigma\text{cl}(a_i b_i : i < \omega)).$$

Put  $b' = \text{Cb}(ab/\Sigma\text{cl}(a, a_i b_i : i < \omega))$ . Then  $a \in b'$ , and  $b' \in \Sigma\text{cl}(ab)$  by  $\Sigma$ -basedness of  $\text{tp}(b/a)$ . Put  $X = \Sigma\text{cl}(\emptyset) \cap \text{dcl}(b')$ . Then  $b' \perp_X (a_i : i < \omega)$  by Lemma 3; as  $\text{tp}(b'/a_i : i < \omega)$  is  $\Sigma$ -based by Lemma 5 and Corollary 7 applied to  $b' \in \Sigma\text{cl}(a, a_i b_i : i < \omega)$ , so is  $\text{tp}(b'/X)$  by Lemma 8. Put  $C' = \text{Cb}(b'/\Sigma\text{cl}(a_i b_i : i < \omega))$ , then  $C' \subseteq \Sigma\text{cl}(b') \subseteq \Sigma\text{cl}(ab)$  by  $\Sigma$ -basedness.

Now  $ab \perp_{b'} \Sigma\text{cl}(a_i b_i : i < \omega)$  by definition of  $b'$ ; as  $b' \perp_{C'} \Sigma\text{cl}(a_i b_i : i < \omega)$  by definition, we get  $ab \perp_{C'} \Sigma\text{cl}(a_i b_i : i < \omega)$ , whence  $C \subseteq C'$ . We obtain

$$C = C' \cap C \subseteq \Sigma\text{cl}(ab) \cap C = \text{dcl}(\emptyset),$$

whence  $ab \perp \Sigma\text{cl}(A)$ .  $\square$

**Theorem 11.** *Let  $p$  be analysable in  $\Sigma$ -based types. Then  $p$  is  $\Sigma$ -based.*

*Proof:* Suppose  $p = \text{tp}(a/A)$ . Then there is a sequence  $(a_i : i < \alpha) \subseteq \text{dcl}(aA)$  such that  $a \in \text{bdd}(A, a_i : i < \alpha)$  and  $\text{tp}(a_i/A, a_j : j < i)$  is internal in  $\Sigma$ -based types for all  $i < \alpha$ . So  $\text{tp}(a_i/A, a_j : j < i)$  is

$\Sigma$ -based for all  $i < \alpha$  by Corollary 9; we use induction on  $i$  to show that  $\text{tp}(a_j : j < i/A)$  is  $\Sigma$ -based. This is clear for  $i = 0$  and  $i = 1$ ; by Lemma 7 it is true for limit ordinals, and by Lemma 10 it holds for successor ordinals.  $\square$

**Corollary 12.** *If  $p$  is analysable in one-based types, then  $p$  is itself one-based.*  $\square$

## REFERENCES

- [1] Zoé Chatzidakis *A note on canonical bases and modular types in supersimple theories*, preprint, September 2002.
- [2] Ehud Hrushovski. *Locally modular regular types*. In: *Classification Theory, Proceedings, Chicago 1985* (ed. John Baldwin). Springer-Verlag, Berlin, D, 1985.
- [3] Ehud Hrushovski. *The Manin-Mumford conjecture and the model theory of difference fields*, Ann. Pure Appl. Logic 112:43–115, no. 1, 2001.
- [4] Anand Pillay. *Geometric stability theory*. Oxford Logic Guides 32. Oxford University Press, Oxford, GB, 1996.
- [5] F. O. Wagner. *Simple Theories*. Mathematics and Its Applications 503. Kluwer Academic Publishers, Dordrecht, NL, 2000.

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