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SOME MODEL THEORY OF SHEAVES OF MODULES

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Abstract. We explore some topics in the model theory of sheaves of modules. First we describe the formal language that we use. Then we present some examples of sheaves obtained from quivers. These, and other examples, will serve as illustrations and as counterexamples. Then we investigate the notion of strong minimality from model theory to see what it means in this context. We also look briefly at the relation between global, local and pointwise versions of properties related to acyclicity.

1. Introduction

Let X be a topological space, $\mathcal{O}_X$ a sheaf of rings (with 1) on X. By Mod-$\mathcal{O}_X$ we denote the (Grothendieck abelian) category of sheaves of $\mathcal{O}_X$-modules. Such sheaves of modules arise in a great variety of (geometric, analytic, algebraic) situations. Often those arising in practice satisfy some further conditions (such as being quasi-coherent) but here we work in complete generality, in part because one of the motivating examples is the representation of modules as sheaves over the Gabriel-Zariski (=dual-Ziegler) spectrum (see [8]), in which context it is not even clear what 'quasicoherent' should mean.

For unexplained terms we refer the reader to [4], [12] for sheaf theory, [5], [11] for the algebra and abelian categories and to [1], [2] for the more general category theory ([1] in particular has a good deal of information on axiomatisation of locally finitely presented categories).

If we assume that every $\mathcal{O}_X$-module is a direct limit of finitely presented $\mathcal{O}_X$-modules (i.e. that Mod-$\mathcal{O}_X$ is **locally finitely presented**) then we can develop a reasonable model theory for $\mathcal{O}_X$-modules. By a **finitely presented** object of a category $\mathcal{C}$ we mean an object M such that the representable functor $\mathcal{C}(M,-)$ (i.e. $\mathrm{Hom}_{\mathcal{C}}(M,-)$) commutes with direct limits: for every directed system $((N_\lambda)_\lambda, (g_{\lambda\mu} : N_\lambda \longrightarrow N_\mu)_{\lambda<\mu})$ with limit $(N, (g_{\lambda\infty} : N_\lambda \longrightarrow N)_\lambda)$ every morphism $f : M \longrightarrow N$ factors through some $g_{\lambda\infty}$. Usually we write simply (A, B) for the set or group of morphisms from A to B.

Suppose that $\mathcal{C}$ is locally finitely presented and that $\mathcal{G}$ is a set of finitely presented objects which is generating in the usual sense (that for any non-zero morphism $f : A \longrightarrow B$ in $\mathcal{C}$ there is a morphism $a : G \longrightarrow A$ with $G \in \mathcal{G}$ and $fa \neq 0$). For example, we might take $\mathcal{G}$ to contain a copy of every finitely presented object of $\mathcal{C}$. Set up the corresponding many-sorted, first-order, language $L^{\mathcal{G}}$ as follows.

There is a sort s_G for each object G. For each morphism $g : G \longrightarrow H$ with $G, H \in \mathcal{G}$ there is a unary function symbol from sort s_H to s_G. Each object M of $\mathcal{C}$ becomes an $L^{\mathcal{G}}$-structure in the natural way: $s_G M = \mathcal{C}(G, M)$ and the action between sorts corresponding to g as above is composition with g, that is $\mathcal{C}(g, M) : \mathcal{C}(H, M) \longrightarrow \mathcal{C}(G, M)$ by $f \mapsto fg$. We refer to the elements of $s_G M = \mathcal{C}(G, M)$ as the **elements of** M **of sort** G. This is a natural extension of the identification of

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elements of a module M over a ring R with the R-linear maps from R to M (with $f : R \longrightarrow M$ corresponding to $f1 \in M$).

We do not know necessary and sufficient conditions for Mod-$\mathcal{O}_X$ to be locally finitely presented but there is the following partial result.

Theorem 1.1. [9, 5.6] *If X has a basis of compact open sets then, for any sheaf $\mathcal{O}_X$ of rings on X, the category of all $\mathcal{O}_X$-modules is locally finitely presented. The sheaves of the form $j_!\mathcal{O}_U$ with U compact open form a generating set of finitely presented objects.*

Here $j : U \longrightarrow X$ is the inclusion, $j_!$ is the extension by zero functor (see later for the definition) and $\mathcal{O}_U$ is the restriction, $\mathcal{O}_X \mid_U$, of the structure sheaf, $\mathcal{O}_X$, to U.

Therefore, throughout the paper, we deal with categories Mod-$\mathcal{O}_X$ **where X has a basis of compact open sets**. Moreover the language that we use will always be that based on $\mathcal{G} = \{j_!\mathcal{O}_U : U \text{ is compact open}\}$. Because X has a basis of compact open sets this allows us to embed Mod-$\mathcal{O}_X$ as a full subcategory of the category of contravariant functors (i.e. "presheaves") on $\mathcal{G}$ (thought of as a full subcategory of Mod-$\mathcal{O}_X$) .

Proposition 1.2. *Suppose that X has a basis of compact open sets. Then the restricted Yoneda map which, on objects, is $F \mapsto (-, F) \restriction \mathcal{G}$, is a full and faithful functor from* Mod-$\mathcal{O}_X$ *to* $(\mathcal{G}^{\mathrm{op}}, \mathbf{Ab})$.

Proof. If we replace $\mathcal{G}$ by the full subcategory, mod-$\mathcal{O}_X$, of finitely presented objects of Mod-$\mathcal{O}_X$ then this is by, for instance, [1, 1.26].

Then note that for every $F \in$ Mod-$\mathcal{O}_X$ the functor $(-, F)$ on mod-$\mathcal{O}_X$ is determined by its restriction to $\mathcal{G}$ since $(-, F)$ is right exact and since every finitely presented object C of Mod-$\mathcal{O}_X$ has a finite presentation of the form $\bigoplus H_j \longrightarrow \bigoplus G_i \longrightarrow C \longrightarrow 0$, by objects H_j, G_i of $\mathcal{G}$. Similarly any natural transformation from $(-, F) \restriction \mathcal{G}$ to $(-, F') \restriction \mathcal{G}$ extends uniquely to a functor from $(-, F) \restriction$ mod-$\mathcal{O}_X$ to $(-, F') \restriction$ mod-$\mathcal{O}_X$ and this is enough for us to deduce that we have a full embedding. □

Proposition 1.3. *Suppose that X has a basis of compact open sets. Then the category* Mod-$\mathcal{O}_X$ *is a definable subcategory of the functor category* $(\mathcal{G}^{\mathrm{op}}, \mathbf{Ab})$.

Proof. First we need the fact (see below for a reference) that for any sheaf $F \in$ Mod-$\mathcal{O}_X$ and open set $U \subseteq X$ we have $(j_!\mathcal{O}_U, F) \simeq FU$. Since the open sets corresponding to objects of $\mathcal{G}$ are compact it is then clear that the following sentences axiomatise the sheaf property.

$\forall x_U (\bigwedge_{i=1}^n \mathrm{res}_{U,U_i} x_U = 0 \rightarrow x_U = 0)$

$\forall x_{U_1}, ..., x_{U_n} (\bigwedge_{i,j=1}^n \mathrm{res}_{U_i, U_i \cap U_j} x_{U_i} = \mathrm{res}_{U_j, U_i \cap U_j} x_{U_j} \rightarrow \exists x_U (\bigwedge_{i=1}^n \mathrm{res}_{U,U_i} x_U = x_{U_i}))$

where U ranges over compact open sets and $\{U_1, ..., U_n\}$ ranges over finite open covers of U by compact open sets. Here and elsewhere we use $\mathrm{res}_{U,V}^{(F)}$ for the restriction map (in F) from $(F)U$ to $(F)V$.

The paragraphs which follow should clarify why these are sentences of our language. □

It is an immediate consequence that all the usual results and machinery of the model theory of modules, including pp-elimination of quantifiers, hold in the context of Mod-$\mathcal{O}_X$ for the language described.

The form of a pp formula $\phi(v_1, ..., v_n)$ where the sort of v_i is V_i (meaning the sort corresponding to $j_!\mathcal{O}_{V_i}$) is $\exists w_1, ..., w_m \bigwedge_k \Sigma_i v_i r_{ik} + \Sigma_j w_j s_{jk} = 0$. Here, if the sort of w_j is W_j, then for each k there is a sort U_k such that $r_{ik} : j_!\mathcal{O}_{U_k} \longrightarrow j_!\mathcal{O}_{V_i}$

and $s_{jk} : j_!\mathcal{O}_{U_k} \longrightarrow j_!\mathcal{O}_{W_j}$ - so each term $v_i r_{ik}, w_j s_{kj}$ represents an element of sort U_k.

In order to understand the language beyond this formalism it is necessary to understand the morphism groups of the form $(j_!\mathcal{O}_U, j_!\mathcal{O}_V)$ where U, V are compact open. The basic fact we use is that $j_!$ is left adjoint to the restriction functor.

Let $j : U \longrightarrow X$ be the inclusion of an open set in X. Then $j_!$ is the functor from Mod-$\mathcal{O}_U$ to Mod-$\mathcal{O}_X$ defined on objects $G \in$ Mod-$\mathcal{O}_U$ by $j_!G.V = \{s \in G(V \cap U) :$ supp(s) is closed in $V\}$ for $V \subseteq X$ open (it is the sheafification of the presheaf extension by zero of G). Here the **support** of a section s is supp$(s) = \{x \in X : s_x \neq 0\}$ where s_x is the value of s in the stalk above x. Define the restriction of F to U, denoted j^*F or $F\mid_U$, by $F\mid_U .V = FV$ for $V \subseteq U$ open. Then, e.g. [4, p.108/9], $j_!$ is left adjoint to j^* : for every $G \in$ Mod-$\mathcal{O}_U$ and $F \in$ Mod-$\mathcal{O}_X$ we have $(j_!G, F) \simeq (G, F\mid_U)$.

Therefore we have the immediate re-interpretation of the "elements" of a sheaf $F \in$ Mod-$\mathcal{O}_X$ namely $s_U F = (j_!\mathcal{O}_U, F) \simeq (\mathcal{O}_U, F\mid_U) \simeq FU$. That is, the elements of F of sort U are precisely the sections of F over U, where U is compact open.

We remark that if we wished sections of F over arbitrary open sets to be considered as "elements" then an infinitary language, a multi-sorted $L_{\kappa\infty}$ where κ is the least cardinal such that every open cover of an open set has a subcover of cardinality less than κ, would be more appropriate.

How should we re-interpret the function symbols of the language? Let $r \in (j_!\mathcal{O}_U, j_!\mathcal{O}_V)$ (here "j" is used for both embeddings $U \longrightarrow X, V \longrightarrow X$ without, we hope, causing confusion). Then $r \in j_!\mathcal{O}_V.U = \{s \in \mathcal{O}_V(U \cap V) :$ supp(s) is closed in $U\} = \{s \in \mathcal{O}_X(U \cap V) :$ supp(s) is closed in $U\}$. That is, $(j_!\mathcal{O}_U, j_!\mathcal{O}_V)$ may be identified with the group of sections of the structure sheaf $\mathcal{O}_X$ over $U \cap V$ with support which is closed in U. In particular, if $U \subseteq V$ then $(j_!\mathcal{O}_U, j_!\mathcal{O}_V) = \mathcal{O}_X U$ (recall that the support of a section over an open set is always closed in that open set).

Lemma 1.4. *The action of $r \in (j_!\mathcal{O}_U, j_!\mathcal{O}_V)$ on $F \in$ Mod-$\mathcal{O}_X$, regarded as a map from FV to FU, is restriction from FV to $F(U \cap V)$, followed by multiplication by r, regarded as an element of $\mathcal{O}_X(U \cap V)$, followed by inclusion in FU.*

Proof. Note that since supp(r) (regarded as an element of $\mathcal{O}_X(U \cap V)$) is closed in U, any section of $F(U \cap V)$ which is a multiple of r can be extended to a section of FU (by defining it to be 0 on the, open, complement of supp(r) in U).

That the action is exactly as described can be deduced by following through the adjunction isomorphism at the level of stalks. □

In particular, if $U \subseteq V$ then the action of $r \in (j_!\mathcal{O}_U, j_!\mathcal{O}_V)$ is restriction to U followed by multiplication by r. Note that the element $1 \in \mathcal{O}_X U$ corresponds, for each $V \supseteq U$, to the element in $(j_!\mathcal{O}_U, j_!\mathcal{O}_V)$ whose action on any sheaf $F \in$ Mod-$\mathcal{O}_X$ is just restriction, $\mathrm{res}_{V,U}$, from V to U.

Let us look at the simplest formulas to see what they say.

Consider the annihilator formula $vr = 0$ where v has sort U, and $r \in (j_!\mathcal{O}_V, j_!\mathcal{O}_U)$ that is, $r : s_U \longrightarrow s_V$. Then $F \models ar = 0$ iff $\mathrm{res}^F_{U,U\cap V}(a).r = 0$ where in the latter equation we regard $r \in \mathcal{O}_X(U \cap V)$.

More generally if we consider the formula $\Sigma_i v_i r_i = 0$ where r_i has sort U_i and $r_i \in (j_!\mathcal{O}_V, j_!\mathcal{O}_{U_i})$ then $F \models \Sigma_i a_i r_i = 0$ iff "$\Sigma_i \mathrm{res}^F_{U_i,U_i\cap V}(a_i).r_i = 0$" where the sum may be interpreted as a sum of sections over V.

Consider the divisibility formula $r \mid v$, that is, $\exists w(v = wr)$, where v has sort U, $r \in (j_!\mathcal{O}_U, j_!\mathcal{O}_V)$ and w has sort V. Then $F \models r \mid a$ iff $\exists b \in FV$ such that $\mathrm{res}^F_{V,U\cap V}(b).r = \mathrm{res}^F_{U,U\cap V}(a)$.

In particular, setting $R_U = \mathcal{O}_X U$, we see that the language, L_{R_U}, of R_U-modules is a part of that of $\mathcal{O}_X$-modules whenever U is compact open. For, each function symbol r in the former language may be regarded as a function symbol in the latter language since $r \in (R_U, R_U) \simeq (\mathcal{O}_X U, \mathcal{O}_X U) \simeq \mathcal{O}_X U \simeq (j_! \mathcal{O}_U, j_! \mathcal{O}_U)$, noting that $(j_! \mathcal{O}_U, j_! \mathcal{O}_U) \simeq (\mathcal{O}_U, (j_! \mathcal{O}_U) \mid_U) \simeq ((j_! \mathcal{O}_U) \mid_U)(U) \simeq (j_! \mathcal{O}_U)(U) \simeq \mathcal{O}_U U \simeq \mathcal{O}_X U$.

Therefore we have the following.

Lemma 1.5. *If $U \subseteq X$ is compact open and $\phi(\bar{v})$ is a formula in the language of $\mathcal{O}_X U$-modules then there is a formula $\phi'(\bar{v}')$ (pp if ϕ is pp) in the language of $\mathcal{O}_X$-modules with variables (free and bound) all of sort U such that for every $F \in \mathrm{Mod}\text{-}\mathcal{O}_X$ and for every tuple $\bar{a}$ in the $\mathcal{O}_X U$-module FU we have $FU \models \phi(\bar{a})$ iff $F \models \phi'(\bar{a})$.*

Note that the "$\bar{a}$" on the two sides of this equivalence refer, strictly speaking, to different, but equivalent, objects.

Corollary 1.6. *If $U \subseteq X$ is compact open, $F \in \mathrm{Mod}\text{-}\mathcal{O}_X$ and if H is a pp-definable subgroup of the $\mathcal{O}_X U$-module FU then H is also a pp-definable subgroup of the sheaf F in sort U.*

The converse to the corollary is far from being true, as will be clear in various of the examples that we present.

2. Examples from quivers

First we obtain a useful collection of examples from representations of certain quivers. Given one of these particular quivers, one may construct a topological space and a sheaf of rings on it (corresponding to a ring structure assigned to the quiver), such that the category of sheaves of modules is equivalent to the category of representations of the quiver (that is, to the category of modules over the associated path algebra). As well as providing a rich source of examples with which to test conjectures and illustrate results, in the other direction it can be fruitful to take the viewpoint that representations are sheaves of modules (e.g. see [3]).

Example 2.1. *Let $X = \{x, y\}$ be equipped with the topology which has open sets, $\emptyset, X$ and $U = \{y\}$. Let $\rho : R \longrightarrow S$ be a homomorphism of rings. Define $\mathcal{O}_X = \mathcal{O}_{X,\rho}$ to be the presheaf of rings with $\mathcal{O}_X.X = R$, $\mathcal{O}_X.U = S$, $\mathrm{res}^{\mathcal{O}_X}_{X,U} = \rho$.*

Since no open set has a non-trivial open cover, $\mathcal{O}_X$ is a sheaf. For the same reason every presheaf over $\mathcal{O}_X$ is actually a sheaf. If $F \in \mathrm{Mod}\text{-}\mathcal{O}_X$ then FX is an R-module, FU is an S-module and $\mathrm{res}^F_{X,U} : FX \longrightarrow FU$ is a homomorphism of R-modules, where FU is regarded as an R-module via ρ. Conversely, given $M \in \mathrm{Mod}\text{-}R$, $N \in \mathrm{Mod}\text{-}S$ and an R-linear map $M \longrightarrow N_R$, we can form a sheaf from this data. It is easy to check directly that we have an equivalence of categories but we can also see this as follows.

By 1.2 $\mathcal{O}_X$ and $j_! O_U$ together generate $\mathrm{Mod}\text{-}\mathcal{O}_X$. Both of these are projective objects: $(\mathcal{O}_X, -) \simeq \Gamma(-)$, $(j_! O_U, -) \simeq \Gamma_U(-)$, where Γ denotes the global sections functor, and both these functors are exact because an exact sequence in $\mathrm{Mod}\text{-}\mathcal{O}_X$ is also an exact sequence of presheaves in this case and hence is an exact sequence at every open set.

Therefore $\mathrm{Mod}\text{-}\mathcal{O}_X$ is equivalent to the category of right modules over $\mathrm{End}(\mathcal{O}_X \oplus j_! \mathcal{O}_U)$ and this is the matrix ring $\begin{pmatrix} (\mathcal{O}_X, \mathcal{O}_X) & (\mathcal{O}_X, j_! \mathcal{O}_U) \\ (j_! \mathcal{O}_U, j_! \mathcal{O}_U) & (j_! \mathcal{O}_U, \mathcal{O}_X) \end{pmatrix}$. Now $(\mathcal{O}_X, \mathcal{O}_X) \simeq \mathcal{O}_X X \simeq R$, $(\mathcal{O}_X, j_! \mathcal{O}_U) \simeq j_! \mathcal{O}_U X = 0$, $(j_! \mathcal{O}_U, j_! \mathcal{O}_U) \simeq (\mathcal{O}_U, (j_! \mathcal{O}_U) \mid_U) \simeq (\mathcal{O}_U, \mathcal{O}_U) \simeq S$ and $(\mathcal{O}_X, j_! \mathcal{O}_U) = \Gamma(j_! \mathcal{O}_U) = 0$. Therefore $\mathrm{End}(\mathcal{O}_X \oplus j_! \mathcal{O}_U)$ is the upper triangular matrix ring $\begin{pmatrix} R & S \\ 0 & S \end{pmatrix}$. A right module over this matrix ring is given by a pair

(M_R, N_S) of modules together with an S-linear morphism $M_R \otimes_R S_S \longrightarrow N_S$ but, noting that $\mathrm{Hom}_S(M_R \otimes_R S_S, N_S) \simeq \mathrm{Hom}_R(M_R, \mathrm{Hom}_S(S_S, N_S)) \simeq \mathrm{Hom}_R(M_R, N_R)$, *we see that this is just the category of structures (M, N, ρ) appearing above.*

Note that if m is a global section of the sheaf $(M_R, N_S, f : M_R \longrightarrow N_R)$ then $m_x = m$ and $m_y = fm$.

In the particular case where $\rho : R \longrightarrow S$ is $\mathrm{id} : R \longrightarrow R$ we have $\text{Mod-}\mathcal{O}_X \simeq \text{Mod-}R(A_2)$ *- the category of representations of the quiver A_2 in* Mod-R, *where A_2 is the quiver shown.*

$$\bullet \longrightarrow \bullet$$

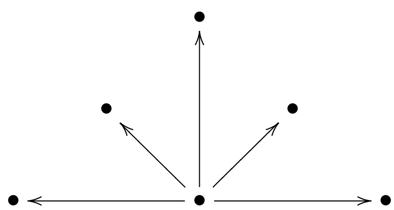
An example of such a ringed space $\mathcal{O}_X$, from algebraic geometry, is given by taking C to be an irreducible curve, x a closed point of C and y the generic point of C. Then the local ring $\mathcal{O}_{C,x}$ is such a space, with $X = \{x, y\}$ and with $\rho : R \longrightarrow S$ being $k[C]_{(x)} \longrightarrow k(C)$ where $k[C]$ is the coordinate ring of the curve C, $k[C]_{(x)}$ is its localisation at the ideal, (x), generated by x and $k(C)$ is the field of quotients of $k[C]$.

Example 2.2. *For another example, take $X = \{x, y_1, ..., y_5\}$ with open sets $\emptyset, X, U_i = \{y_i\}$ and the unions of the U_i. Take rings and ring homomorphisms R_i, $i = 0, ..., 5$, $\rho_i : R_0 \longrightarrow R_i$, $i = 1, ..., 5$ and define the presheaf $\mathcal{O}_X$ on X by:*

$\mathcal{O}_X.X = R_0$, $\mathcal{O}_X(U_{i_1} \cup ... \cup U_{i_t}) = R_{i_1} \times ... \times R_{i_t}$ for $i_1, ..., i_t$ distinct elements of $\{1, ..., 5\}$, $\mathrm{res}^{\mathcal{O}_X}_{X, U_{i_1} \cup ... \cup U_{i_t}} = \rho_{i_1} \times ... \times \rho_{i_t}$ and restriction between unions of the U_i are the projection maps. One may check that $\mathcal{O}_X$ is in fact a sheaf.

Let F be a presheaf over $\mathcal{O}_X$. Then FX is an R_0-module, $FU_i \in$ Mod-R_i *and res^F_{X, U_i} is an R_0-linear map from FX to FU_i where the latter is regarded as an R_0-module via $\rho_i : R_0 \longrightarrow R_i$. If $\tilde{F}$ denotes the sheafification of F (in this example not every presheaf is a sheaf) then we have $\tilde{F}(U_{i_1} \cup ... \cup U_{i_t}) = FU_{i_1} \times ... \times FU_{i_t}$, $\tilde{F}X = FX$ and the restriction maps in $\tilde{F}$ are the obvious ones. Observe that it can happen that 0 is the only global section yet $\tilde{F} \neq 0$.*

In particular if $R_i = R$ for each i and if all the ρ_i are id_R then $\text{Mod-}\mathcal{O}_X \simeq \text{Mod-}R(\tilde{\tilde{D}}_4)$ *where $\tilde{\tilde{D}}_4$ is the quiver shown.*



Example 2.3. *Let $X = \{x, y, z\}$ with open sets $\emptyset, X, U = \{y, z\}, V = \{z\}$. Take rings and homomorphisms $\rho : R \longrightarrow S$, $\sigma : S \longrightarrow T$ and define a presheaf $\mathcal{O}_X$ on X by $\mathcal{O}_X.X = R$, $\mathcal{O}_X.U = S$, $\mathcal{O}_X.V = T$ and with ρ, σ giving the restriction maps. Clearly this is a sheaf and also every presheaf is a sheaf.*

In particular if $R = S = T$ and $\rho = \sigma = id_R$ then $\text{Mod-}\mathcal{O}_X \simeq \text{Mod-}R(A_3)$ *where A_3 is the quiver shown.*

$$\bullet \longrightarrow \bullet \longrightarrow \bullet$$

3. Strongly minimal sheaves

First we consider the notion of strongly minimal sheaf of modules, comparing it to that of strongly minimal module. At first sight, in view of the definition for the latter, we might say that an $\mathcal{O}_X$-module is strongly minimal if it is not finite and every definable subset is finite or has finite complement. (In the case of modules it is equivalent to require that every pp-definable subgroup is finite or is the whole module.) Here, however, we have to take account of the fact that we are working with a many-sorted structure. If there are infinitely many sorts then, since every variable is sorted, the whole structure (the union of sorts) is not definable. Also, there is the question of what we should mean by a finite structure.

In a one-sorted language a structure is finite iff it has no proper elementary extensions and it is the latter which is the property which matters, so let us say that $F \in \text{Mod-}\mathcal{O}_X$ is **finite** if each sort of F is finite. Clearly this is equivalent to F having no proper elementary extensions. Then we say that an $\mathcal{O}_X$-module F is **strongly minimal** if it is not finite and if for every pp formula ϕ with one free variable, of sort U say, $\phi(F)$ is either finite or equal to $F(U)$. Example 3.1 below shows that the resulting notion of strongly minimal is language-dependent. Of course even if we start with a 1-sorted structure which is strongly minimal then the expansion of this structure obtained by adding the sort of ordered pairs is no longer strongly minimal in the above sense. This suggests that the "correct" definition of strongly minimal structure should be something along the following lines: the category of sorts should be generated by a set of sorts which are strongly minimal (in the usual sense) and which are non-orthogonal. For example, the category of (all imaginary) sorts based on the two-sorted language (with no extra structure) is generated by two strongly minimal sets which are orthogonal, so should surely not be regarded as a strongly minimal structure but, if a definable bijection (or, perhaps more convincingly, a finite-to-finite relation) between these two generating sets is added, then the resulting structure should be regarded as strongly minimal.

Example 3.1. *Let R be the path algebra $k(A_2)$ where k is an infinite field and let $\mathcal{O}_X$ be the corresponding ringed space, as described in the previous section. Since $\mathcal{O}_X(U) = k$ for each non-empty open set U this is a strongly minimal sheaf (clearly k has no proper, non-trivial pp-definable subgroups in either language since its automorphism group is k). Yet the corresponding R-module is not strongly minimal.*

Example 3.2. *A more decisive example is given by taking the same ringed space and considering the sheaf F given by $FU = k = FX$ and having 0 for the non-trivial restriction map. This sheaf is, according to the definition above, strongly minimal yet it has two orthogonal types.*

Example 3.2 shows that the given definition of strong minimality should not be used in the general case. Nevertheless we will see that the above, naive, definition does seem to be reasonable if we assume that the space X is T_1 (see 3.7 below) and so we will make do with the above, somewhat provisional, definition in this section.

The next two examples show that our conditions do not carry over to stalks.

Example 3.3. *If a sheaf is finite it does not follow that every stalk is finite. Let $X = \{x_1, .x_2, ..., x_n, ..., y\}$ have for open sets all the co-initial sets $U_k = \{x_n : n \geq k\} \cup \{y\}$ together with the empty set. Note that every open set is compact. Let $\mathcal{O}_X$ be the constant sheaf $\mathbb{Z}$ and let F be the $\mathcal{O}_X$-module with $F(U_k) = \mathbb{Z}_{2^k}$ and with restriction maps being the canonical inclusions. According to our definition this is a finite sheaf but the stalk at y is the Prüfer group $\mathbb{Z}_{2^\infty}$.*

Example 3.4. *Modify the example above by taking $F(U_k) = \mathbb{Z}_{2^k}^k \oplus \mathbb{Q}$ (and inclusions for the restriction maps). Then this sheaf is strongly minimal according to our definition. But the stalk at y will be $\mathbb{Z}_{2^\infty}^{(\aleph_0)} \oplus \mathbb{Q}$, which is not strongly minimal.*

We do, however, have the following, which is an immediate consequence of 1.6.

Corollary 3.5. *If F is a strongly minimal sheaf then, for each compact open $U \subseteq X$, FU is a strongly minimal or finite $\mathcal{O}_X U$-module.*

Example 3.6. *The converse is false. With notation as in Example 2.1, take $R = S = k$ to be an infinite field and let F be the sheaf with FX 1-dimensional, FU 2-dimensional and $\mathrm{res}^F_{X,U}$ an embedding. Then the formula $\exists y_X(x = y_X \mathrm{res}_{X,U})$ defines an infinite, co-infinite subset of FU, so F is not strongly minimal. On the other hand both FX and FU are strongly minimal modules.*

The next lemma indicates why good separation properties on the space accommodate the naive definition of strong minimality.

Lemma 3.7. *Suppose that $F \in \mathrm{Mod}\text{-}\mathcal{O}_X$ and that there are disjoint open subsets U, V of X such that both $F(U)$ and $F(V)$ are non-zero and at least one of these is infinite. Then F is not strongly minimal.*

Proof. Since $U \cap V = \emptyset$ we have, by the glueing property of the sheaf F, $F(U \cup V) = F(U) \oplus F(V)$ with the restriction maps from $F(U \cup V)$ to $F(U)$ and $F(V)$ being the projections. Then the subgroups $0 \oplus F(V)$ and $F(U) \oplus 0$ of $F(U \cup V)$ are definable, being the kernels of these maps so, if F were strongly minimal, each would have to be finite or all of $F(U \cup V)$. □

Lemma 3.8. *If F is strongly minimal then for every function symbol r from sort U to sort V we have that the image of $r : F(U) \longrightarrow F(V)$ is finite or all of $F(V)$ and the kernel of this map is finite or all of $F(U)$.*

Proof. Since both the kernel and image of r are pp-definable this is immediate. □

Suppose from now on in this section that the space X is T_1 (that is, given $x \in X$ then for every $y \in X$ with $y \neq x$ there is an open set containing x and not containing y) and let F be a strongly minimal $\mathcal{O}_X$-module.

Set $\mathcal{U} = \mathcal{U}_F = \{U \text{ compact open } : FU \text{ is infinite }\}$. Since F is not finite $\mathcal{U} \neq \emptyset$.

Lemma 3.9. *For all $U, V \in \mathcal{U}$ there is $W \in \mathcal{U}$ with $W \subseteq U \cap V$.*

Proof. For every $y \in U \cap V$ choose, if possible, a compact open neighbourhood V_y of y contained in $U \cap V$ with FV_y finite and hence, by 3.8, with $\mathrm{res}_{U,V_y} = 0$. If we can do this then, by the mono property of the sheaf F we have $\mathrm{res}_{U,U\cap V} = 0$ and similarly $\mathrm{res}_{V,U\cap V} = 0$. So, by the glueing property of the sheaf F we have $F(U \cup V) = FU \oplus FV$, contradiction as in 3.7 . □

Lemma 3.10. *If $U = \bigcup_\lambda U_\lambda$ with $U \in \mathcal{U}$ and all U_λ compact open then, for some λ, FU_λ is infinite (that is $U_\lambda \in \mathcal{U}$).*

Proof. Let $s \in FU$ be non-zero. If each FU_λ were finite then, as in the proof of 3.9, we would have $\mathrm{res}_{UU_\lambda} s = 0$ but then, also as in that proof, we could deduce $s = 0$ - contradiction. □

Lemma 3.11. *Let $U \in \mathcal{U}$. Then there is $y \in U$ such that for every compact open neighbourhood V of y with $V \subseteq U$, we have FV infinite (that is, $V \in \mathcal{U}$).*

Proof. Otherwise (cf. proof of 3.9) we could cover U by compact open sets V_λ with FV_λ finite, in contradiction to 3.10. □

Lemma 3.12. $|\bigcap \mathcal{U}| \leq 1$.

Proof. Let $x \in \bigcap \mathcal{U}$ if such exists. Let $U \in \mathcal{U}$. By 3.11 there is $y \in U$ and a compact open neighbourhood V of y with $V \subseteq U$ and with $V \in \mathcal{U}$, therefore with $x \in V$. But if $y \neq x$ there exists such V with $x \notin V$. So it must be that $x = y$. This is true for any point in $\bigcap \mathcal{U}$ and so there must be at most one such point. □

Lemma 3.13. *If $x \in \bigcap \mathcal{U}$ then $\mathcal{U}$ forms a basis of open neighbourhoods of x.*

Proof. Let U' be any open neighbourhood of x and let $U \in \mathcal{U}$ contain x. Choose a compact open set V with $x \in V \subseteq U \cap U'$. Each $y \in U \setminus V$ has a compact open neighbourhood V_y not containing x, hence with FV_y finite and hence, by 3.8, with $\ker(\mathrm{res}_{UV_y}) = FU$. If FV also were finite then, since the V_y together with V cover U, we would have a contradiction, by 3.10. Therefore $V \in \mathcal{U}$, as required. □

Example 3.14. *It is possible that $\bigcap \mathcal{U} = \emptyset$. For, take X to be an infinite set and take the cofinite sets for the non-empty open sets: so X is T_1. Let k be an infinite field and take the structure sheaf to be the constant sheaf k.*

Define F by $FU = k$ if $U \neq \emptyset$ and set all restriction morphisms to be the identity map. Clearly F is strongly minimal (or see 3.17 below) and $\mathcal{U}$ consists of all non-empty open sets, but $\bigcap \mathcal{U} = \emptyset$.

The next example shows that if $U' \supseteq U$ are compact open and $U \in \mathcal{U}$ it need not be the case that $U' \in \mathcal{U}$.

Example 3.15. *Let $X = \mathbb{N}$ and take the open sets to be the cofinite sets. Set the structure sheaf to be the constant sheaf $\mathbb{Z}$. Define F by $FU = \mathbb{Z}_2$ if $1 \in U$ and $FU = \mathbb{Z}_{2^\infty}$ if $1 \notin U$. Define the restriction morphisms to be the identity or the inclusion of $\mathbb{Z}_2$ into $\mathbb{Z}_{2^\infty}$, as appropriate. Then one may check that F is strongly minimal and clearly $\mathcal{U}$ is not closed upwards.*

In order to proceed further it seems that we need to assume further conditions. We begin by adding the condition on F that if $U' \supseteq U$ and $U \in \mathcal{U}$ then $U' \in \mathcal{U}$ - say that $\mathcal{U}$ is **upwards closed** for short. For example, any flabby sheaf (one where each restriction map is surjective) satisfies this condition.

Define subsheaves F_0, F_1 of F as follows. Note that it is enough to define them on compact open sets since these form a basis of X.

For U compact open set $F_0U = FU$ if $U \notin \mathcal{U}$, $F_0U = 0$ if $U \in \mathcal{U}$ and set $F_1U = 0$ if $U \notin \mathcal{U}$ and $F_1U = FU$ if $U \in \mathcal{U}$. Note that F_0 is a finite sheaf.

Lemma 3.16. *Assume that $\mathcal{U} = \mathcal{U}_F$ is upwards closed. Then F_0 and F_1 are subsheaves of F and $F = F_0 \oplus F_1$.*

Proof. We just have to check that the restriction morphisms of F preserve F_0 and F_1. For F_0 this follows from the assumption that $\mathcal{U}$ is upwards closed. In the case of F_1, if $U \in \mathcal{U}$ and $V \subseteq U$ then either $V \in \mathcal{U}$ and then res^F_{UV} takes F_1U to F_1V by definition of F_1 or $V \notin \mathcal{U}$ in which case the restriction map is zero. The direct sum decomposition of F then is immediate from the definitions of F_0 and F_1. □

The example 3.15 above shows that, without some assumption such as that $\mathcal{U}$ is upwards closed, we do not have such a direct sum decomposition of F.

It follows, since F_0 is finite, that F is strongly minimal iff F_1 is strongly minimal and so we may assume (under the assumption that $\mathcal{U}$ is upwards closed) that we are dealing with a sheaf F with the property that if V is compact open and $V \notin \mathcal{U}$ then $FV = 0$.

Say that a filter base (that is, a set with the finite intersection property), $\mathcal{U}$, of compact open sets is a **maximal filter base** if for every member $U \in \mathcal{U}$ and for every (finite) open cover $(U_\lambda)_\lambda$ of U by compact open sets, there is some λ such that $U_\lambda \in \mathcal{U}$. Note that, assuming $\mathcal{U}_F$ is upwards closed, $\mathcal{U}_F$ does have this property (3.10). Given any maximal filter base, $\mathcal{U}$, of compact open sets and a module M over the direct limit ring $\varinjlim_{U \in \mathcal{U}} \mathcal{O}_X U$ define the sheaf, $F_{\mathcal{U}M}$, **with stalk** M **supported on** $\mathcal{U}$ by $F_{\mathcal{U},M}U = M$, regarded as an $\mathcal{O}_XU$-module via the map $\mathcal{O}_XU \longrightarrow \varinjlim_{V \in \mathcal{V}} \mathcal{O}_XV$, if $U \in \mathcal{U}$ and $F_{\mathcal{U}M}U = 0$ if $U \notin \mathcal{U}$ (and with all restriction maps being zero or the identity).

Lemma 3.17. *If $\mathcal{U}$ is a maximal filter base of compact open sets and if M is a strongly minimal module over $R = \varinjlim_{U \in \mathcal{U}} \mathcal{O}_X U$ then the sheaf $F_{\mathcal{U}M}$ with stalk M supported on $\mathcal{U}$ is strongly minimal.*

Proof. Let $\phi(x)$ be a pp formula with free variable x of sort $U \in \mathcal{U}$. Because the restriction maps between open sets in $\mathcal{U}$ are identity maps and all other restriction maps are zero, all bound variables and function symbols appearing in ϕ may be replaced by variables of sort U' and function symbols from sort U' to itself, respectively, where U' is a suitable member of $\mathcal{U}$ with $U \supseteq U'$, in such a way that we obtain a pp formula ϕ' in the language of $\mathcal{O}_X U'$-modules with the 'same' solution set as ϕ. Via the canonical map $\mathcal{O}_X U' \longrightarrow R$ we may regard ϕ' as a formula in the language of R-modules. By strong minimality of M the solution set of ϕ is either finite or cofinite, as required. $\square$

Returning now to our strongly minimal sheaf $F = F_1$ we make our final assumption, that $\bigcap \mathcal{U}_F \neq \emptyset$ and hence, by 3.12, $\bigcap \mathcal{U}_F$ is a singleton.

Lemma 3.18. *If $\bigcap \mathcal{U}_F \neq \emptyset$, say $\bigcap \mathcal{U}_F = \{z\}$, then $\mathcal{U}$ is upwards closed and consists exactly of those compact open sets which contain z.*

Proof. Suppose that U is compact open and that $z \in U$. By 3.13 there is a compact open $V \subseteq U$ with $z \in V$ and $V \in \mathcal{U}$. For each $y \in U$ with $y \neq z$ choose a compact open neighbourhood V_y of y contained in U and with $z \notin V_y$, hence with $V_y \notin \mathcal{U}$, hence with FV_y finite. Let $s \in FV$ and consider $\mathrm{res}_{V,V\cap V_y} s$: if this section over $V \cap V_y$ is further restricted to any compact open subset of $V \cap V_y$ it becomes 0 (since $z \notin V \cap V_y$) and so, since $V \cap V_y$ is covered by compact open sets, $\mathrm{res}_{V,V\cap V_y} s = 0$. This is so for every V_y and so, by the glueing property of the sheaf F, there is a section s' of FU with $\mathrm{res}_{U,V} s' = s$. Hence FU is infinite, as required. $\square$

Lemma 3.19. *Suppose that $\bigcap \mathcal{U}_F \neq \emptyset$. Let $U' \supseteq U$ both be in $\mathcal{U}$. Then, still assuming F is such that, for every compact open set V, FV is either infinite or zero, $\mathrm{res}^F_{U'U}$ is an isomorphism.*

Proof. Let $s \in FU'$. For each $y \in U' \setminus U$ choose a compact open neighbourhood V_y of y such that FV_y is finite and hence is zero. If we also have $\mathrm{res}_{U'U} s = 0$ then, by the mono property of F, we conclude $s = 0$. Hence $\mathrm{res}_{U'U}$ is monic. Hence, by 3.8, it also is an epimorphism. $\square$

Let $z \in X$, let $\mathcal{U}$ be the set of compact open sets containing z, let $R = \varinjlim_{U \in \mathcal{U}} \mathcal{O}_X U = \mathcal{O}_{X,z}$ be the stalk of $\mathcal{O}_X$ at z and let M be an R-module. Then the **skyscraper sheaf with stalk** M **supported at** z is given by $FU = M$ if $z \in U$, $FU = 0$ otherwise and all restriction maps are zero or the identity. Since our space X is assumed to be T_1, so points are closed, this sheaf is just the extension by zero, $j_! M$, where we regard M as a sheaf over the closed set $\{z\}$.

Proposition 3.20. *Let X be any space with a basis of compact open sets and let $\mathcal{O}_X$ be any sheaf of rings on X. Let M be a strongly minimal $\mathcal{O}_{X,x}$-module. Then the skyscraper sheaf at x with value M is a strongly minimal $\mathcal{O}_X$-module.*

Proof. The skyscraper sheaf, F say, referred to is defined by $FU = M$ if $x \in U$ and $FU = 0$ if $x \notin U$. For any (compact) open $U \subseteq X$ we have the adjunction $(j_! \mathcal{O}_U, F) \simeq (\mathcal{O}_U, F\mid_U) \simeq FU$. This isomorphism is functorial so we have, for every compact open $U \subseteq X$ with $x \in U$, a canonical identification of the corresponding sort of F with M (and if $x \notin U$ the corresponding sort is 0). In a similar way the function symbols of the language of $\mathcal{O}_X$-modules can be associated in a canonical way to elements of the ring $\mathcal{O}_{X,x}$. Therefore, to every pp formula in the language of $\mathcal{O}_X$-modules, we can associate, in a canonical way, a pp formula in the language of $\mathcal{O}_{X,x}$-modules in such a way that pp-definable subgroups of F correspond to pp-definable subgroups of M, and the result then follows. □

Proposition 3.21. *With assumptions as above, let F be a strongly minimal sheaf and suppose that $\bigcap \mathcal{U} \neq \emptyset$, where $\mathcal{U} = \mathcal{U}_F$. Then $F = F_0 \oplus F_1$ where F_0 is a finite sheaf and F_1 is a skyscraper sheaf with stalk a strongly minimal module. Every sheaf which has this form is strongly minimal.*

Proof. Split F as $F_0 \oplus F_1$ as above. Applying lemma 3.19 to F_1 we deduce that F_1 is a sheaf with constant value, which clearly must be strongly minimal, supported on $\mathcal{U}$. Since $\mathcal{U}$ is just the filter base of all compact opens which contain the unique point in $\bigcap \mathcal{U}$ it follows that F must be a skyscraper sheaf. The converse is clear. □

It would be nice to remove the assumption on $\mathcal{U}_F$ and have a formulation which allows a non-split extension by a finite sheaf: see example 3.15 above but also the following example.

Example 3.22. *Let $X = \mathbb{N}$, take the cofinite sets for the open sets and take the structure sheaf to be the constant sheaf $\mathbb{Z}$. Define F by $FU = \mathbb{Z}_{2^\infty}$ for all U and define the restriction map from U to V to be the identity unless $1 \in U$ and $1 \notin V$, in which case define it to be the multiplication by 2 map. One may check that F is strongly minimal.*

If we assume that (X is T_1 and) for every filter base $\mathcal{U}$ of compact open sets we have $\bigcap \mathcal{U} \neq \emptyset$ then X must be Hausdorff. To see this suppose that the Hausdorff property fails, say x, y are distinct points of X such that every open containing x intersects every open containing y. Then the set, $\mathcal{U}$, of compact open sets containing x or y forms a filter but then if $z \in \bigcap \mathcal{U}$ we would have that x is in the closure of $z \neq x$, contradicting that X is T_1.

Proposition 3.23. *Suppose that X is Hausdorff and that $F \in \text{Mod-}\mathcal{O}_X$. Then F is strongly minimal iff F is a skyscraper sheaf with strongly minimal stalk.*

Proof. Suppose that F is strongly minimal. First we show that in this Hausdorff case the finite part, F_0, is zero. Suppose otherwise and choose V compact open with FV finite and non-zero. Let $U \in \mathcal{U}$. By the argument of 3.11 there is $y \in V$ such that for every compact open $V' \subseteq V$ with $y \in V'$ we have FV' non-zero (and finite). For each $x \in U$ choose disjoint compact open neighbourhoods, U_x of x and V_x of y. Since U is compact finitely many of these suffice to cover U so let V_0 be the intersection of the corresponding neighbourhoods V_x of y. Let V' be any compact open neighbourhood of y contained in V_0. Since, by construction, V' and U are disjoint we have, as in 3.7, $F(U \cup V) = FU \oplus FV$, contradicting that F is strongly minimal, as required.

Since (3.9) $\mathcal{U}$ has the finite intersection property and since each member of $\mathcal{U}$ is compact, hence (because X is Hausdorff) closed, $\bigcap \mathcal{U} \neq \emptyset$. Therefore 3.21 applies and so, with notation as there, $F = F_1$. □

Note that the Hausdorff condition is rather strong in the presence of our global assumption that there is a basis of compact open sets: it implies that the space is totally disconnected (of course, this does include the important example of the Pierce sheaf representation of modules over commutative regular rings [6]).

Finally in this section, we remark that an analysis of sheaves of U-rank 1 may be carried out along the lines above, with only minor changes.

4. Comparison of some global, local and pointwise properties

In this section we present mainly examples which show that there can be little connection between global, local and pointwise properties. The properties we consider are related to purity (which is fundamental in the model theory of locally finitely presented abelian categories) and/or relate to algebraic or sheaf-theoretic homological triviality. We assume that the reader is acquainted with notions around purity and injectivity either in the abelian context or in the more general categorical context (as in, for instance, [1]).

Example 4.1. *Here is an example of a sheaf F which is not absolutely pure yet which is such that, for each open set $U \subseteq X$, FU is an injective $\mathcal{O}_X U$-module and which is such that for every $x \in X$ the stalk F_x is an injective module over the ring $\mathcal{O}_{X,x}$.*

Let $X = \{x, y\}$ with open sets $\emptyset, X, U = \{y\}$. Let $\mathcal{O}_X$ be the ringed space with $\mathcal{O}_X X = k = \mathcal{O}_X U$ where k is some field. Let $F \in \mathrm{Mod}\text{-}\mathcal{O}_X \simeq \mathrm{Mod}\text{-}k(A_2)$.
1. For any open set $V \neq \emptyset$, $\mathcal{O}_X V = k$ and hence FV is an injective $\mathcal{O}_X V$-module.
2. $\mathcal{O}_{X,x} = k$ and $\mathcal{O}_{X,y} = k$ and so F_z is injective in $\mathrm{Mod}\text{-}\mathcal{O}_{X,z}$ for each $z \in X$.

On the other hand, there are modules in $\mathrm{Mod}\text{-}k(A_2)$ which are not absolutely pure (=injective for this, noetherian, ring), such as the (simple projective) module $0 \longrightarrow k$. The sheaf F corresponding to this has $FX = 0$, $FU = k$.

Example 4.2. *Here is an example of $X, \mathcal{O}_X$ and an $\mathcal{O}_X$-module F which is injective but for which there is an open set $U \subseteq X$ such that FU is not an absolutely pure $\mathcal{O}_X U$-module and for which there is a point $x \in X$ such that F_x is not an absolutely pure $\mathcal{O}_{X,x}$-module.*

Let $X = \{x, y\}$ with open sets $\emptyset, X, U = \{y\}$ and let $\mathcal{O}_X$ be given by $\mathcal{O}_X X = \overline{\mathbb{Z}_{(2)}}$ (the ring of 2-adic integers), $\mathcal{O}_X U = \mathbb{Z}_2 \simeq \overline{\mathbb{Z}_{(2)}}/2\overline{\mathbb{Z}_{(2)}}$ and $\mathrm{res}^F_{X,U}$ the natural map.

So $\mathrm{Mod}\text{-}\mathcal{O}_X$ is equivalent to the category of representations of A_2 with the source a $\overline{\mathbb{Z}_{(2)}}$-module, the sink a $\mathbb{Z}_2$-module and the map from the first to the second being reduction modulo the maximal ideal (2).

Let $F \in \mathrm{Mod}\text{-}\mathcal{O}_X$ be given by $FX = \mathbb{Z}_2 = FU$, $\mathrm{res}^F_{X,U} = id$. Note that $FX = \mathbb{Z}_2 \in \mathrm{Mod}\text{-}\mathcal{O}_X X = \mathrm{Mod}\text{-}\overline{\mathbb{Z}_{(2)}}$ is not absolutely pure, nor is $F_x = \mathbb{Z}_2 \in \mathrm{Mod}\text{-}\mathcal{O}_{X,x} = \mathrm{Mod}\text{-}\overline{\mathbb{Z}_{(2)}}$. But, we claim, $F \in \mathrm{Mod}\text{-}\mathcal{O}_X$ is injective.

To see this suppose that $\alpha : F \longrightarrow G$ is an embedding in $\mathrm{Mod}\text{-}\mathcal{O}_X$, so we have monomorphisms $\alpha_X : FX \longrightarrow GX$, that is $\alpha_X : \overline{\mathbb{Z}_{(2)}} \longrightarrow GX$, and $\alpha_U : FU = \mathcal{Z}_2 \longrightarrow GU$ and the commutative diagram shown.

$$\begin{tikzcd}
\overline{\mathbb{Z}_{(2)}} \arrow[r, "\alpha_X"] \arrow[d, "\mathrm{res}"'] & GX \arrow[d, "\mathrm{res}"] \\
\mathbb{Z}_2 \arrow[r, "\alpha_U"'] & GU
\end{tikzcd}$$

Suppose that the pp-type of $\alpha_X 1$ in GX strictly contains the pp-type of 1 in $\overline{\mathbb{Z}_{(2)}}$. Then, by the characterisation of purity for the ring $\overline{\mathbb{Z}_{(2)}}$ (e.g. see [7, §$\mathbb{Z}$]), it must be that 2 divides $\alpha_X 1$ in GX and hence 2 divides $\mathrm{res}^G_{X,U}\alpha_X 1 = \alpha_U \mathrm{res}^F_{X,U} 1$ in GU. But that implies $\alpha_U \mathrm{res}^F_{X,U} 1 = 0$ so $\mathrm{res}^F_{X,U} 1 = 0$ - contradiction.

Therefore α_X is a pure hence, since $\overline{\mathbb{Z}_{(2)}}$ is pure-injective, split embedding, say $GX = \alpha_X \overline{\mathbb{Z}_{(2)}} \oplus H$. Since GU is a $\mathbb{Z}_2$-module, $\mathrm{res}(\alpha_X \overline{\mathbb{Z}_{(2)}})$ is itself a direct summand of GU. So αF is a direct summand of G and we conclude that F is indeed injective in $\mathrm{Mod}\text{-}\mathcal{O}_X$.

A sheaf F is **flabby** if for every pair $U \supseteq V$ of open sets the restriction map from FU to FV is surjective. Given a basis $\mathcal{B}$ we say that F is **flabby with respect to** $\mathcal{B}$ if the surjectivity condition holds for all pairs of open sets $U \supseteq V$ in $\mathcal{B}$. Injective sheaves are cohomologically trivial (in the sense that their sheaf cohomology groups $\mathrm{H}^i(X, F)$ are zero for $i > 0$) and, more generally, the same is true of flabby sheaves though for very different reasons (consider the case where X is a one-point space - every sheaf ($=\mathcal{O}_X$-module) has trivial sheaf cohomology but certainly need not have trivial cohomology in the algebraic sense (i.e. need not be injective)). We compare this with absolute purity.

Example 4.3. *A sheaf F which is flabby with respect to a basis but which is not flabby. Let X be a three-point space with open sets $\emptyset$, $\{x\}$, $\{y\}$ and X. Let F be the sheafification of the constant presheaf $\mathbb{Z}$ on X, so $FU = \mathbb{Z}^n$ where n is the number of connected components of U. This is not flabby: consider X - a connected open non-empty set and $V = \{x\}, W = \{y\}$ - disjoint open subsets of X. Then $\mathrm{res}^F_{X,V\cup W}$ is the diagonal map $\mathbb{Z} \longrightarrow \mathbb{Z}\oplus\mathbb{Z}$ which is not epi. But the connected open sets form a basis and clearly F is flabby with respect to this basis.*

We remark that F is not absolutely pure: for we can embed F into a flabby sheaf E. Then E satisfies the condition that every section over $V\cup W$, in particular every element of $F(V\cup W)$, is the restriction of a section over X and this can be expressed by pp formulas but fails to be true in F.

Example 4.4. *A flabby sheaf need not be absolutely pure. If X is discrete (for example, a one-point space) then every sheaf is flabby but in general it is far from being the case that every sheaf=module over $\prod_{x\in X}\mathcal{O}_{X,x}$ is absolutely pure.*

Proposition 4.5. *([10, 4.7.2]) If F is absolutely pure then F is flabby with respect to inclusions of compact open sets.*

Proof. Fix an embedding $f : F \longrightarrow E$ with E flabby. If $U \supseteq V$ are compact open and $a \in FV$ then we have $E \models \exists y_U(\mathrm{res}_{U,V}\, y = fa)$ and hence, since F is absolutely pure, $F \models \exists y_U(\mathrm{res}_{U,V}\, y = a)$. Therefore the restriction map $\mathrm{res}^F_{U,V}$ is onto. □

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