# Associativity as Commutativity 

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#### Abstract

It is shown that coherence conditions for monoidal categories concerning associativity are analogous to coherence conditions for symmetric or braided strictly monoidal categories, where associativity arrows are identities. Mac Lane's pentagonal coherence condition for associativity is decomposed into conditions concerning commutativity, among which we have a condition analogous to naturality and a degenerate case of Mac Lane's hexagonal condition for commutativity. This decomposition is analogous to the derivation of the Yang-Baxter equation from Mac Lane's hexagon and the naturality of commutativity. The pentagon is reduced to an inductive definition of a kind of commutativity.


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## 1 Introduction

Associativity is a kind of commutativity. To see why, conceive of $(a \cdot(c \cdot b))$ as $\left(a \cdot \_\right) \circ(\ldots \cdot b)$ applied to $c$. We have

$$
\begin{aligned}
\left(\left(a \cdot \_\right) \circ(\ldots \cdot b)\right)(c) & =\left(a \cdot \_\right)((\ldots \cdot b)(c)) \\
& =\left(a \cdot \_\right)((c \cdot b)) \\
& =(a \cdot(c \cdot b)) .
\end{aligned}
$$

 applied to $c$ yields $((a \cdot c) \cdot b)$.

The purpose of this paper is to exploit this idea to show that monoidal categories may be conceived as a kind of symmetric strictly monoidal categories, where associativity arrows are identities. As a matter of fact, this analogy holds also with braided strictly monoidal categories. More precisely, we show that coherence conditions for monoidal categories concerning associativity are analogous to coherence conditions concerning commutativity (i.e. symmetry or braiding) for symmetric (see [9], [10] or [2]) or braided (see [3] and [10], second edition, Chapter IX) strictly monoidal categories. In particular, Mac Lane's pentagonal coherence condition for associativity (see Section 4 below) is decomposed into conditions concerning commutativity, among which we have a condition analogous to naturality and degenerate cases of Mac Lane's hexagonal condition for commutativity. (The hexagon becomes a triangle, because associativity arrows are identities, or a two-sided figure.) This decomposition is analogous to the derivation of the Yang-Baxter equation from Mac Lane's hexagon and the naturality of commutativity (see Section 5 below).

To achieve that, we replace the algebra freely generated with one binary operation (denoted by $\wedge$ ) by an isomorphic algebra generated with a family of partial operations we call insertion (denoted by $\triangleleft_{n}$ ); insertion is analogous to the composition $\circ$ at the beginning of this text, or to functional application. (This procedure is like Achilles' introduction of $\alpha$ in [1].) The latter algebra is more complicated, and is not free any more, but it enables us to present associativity arrows as commutativity arrows.

In the next section we state precisely these matters concerning insertion. After that we introduce a category $\boldsymbol{\Gamma}$, which in the remainder of the paper is shown isomorphic to a free monoidal category without unit. In $\boldsymbol{\Gamma}$ the associativity arrows appear as a kind of commutativity arrows, and coherence conditions for $\boldsymbol{\Gamma}$ take the form of an inductive definition of these commutativity arrows. Our decomposition of Mac Lane's pentagon is in the last section.

We work with categories where associativity is an isomorphism, because this is the standard approach, but our treatment is easily transferred to categories where associativity arrows are not necessarily isomorphisms, which in [2] (Section 4.2) are called semiassociative categories (see also [6]). As monoidal categories, semiassociative categories are coherent in Mac Lane's "all diagrams commute" sense. With semiassociative categories, the commutativity corresponding to associativity only ceases to be an isomorphism, and all the rest
remains as in the text that follows.
Among the coherence conditions for the main kinds of categories with structure, Mac Lane's pentagon seems to be more mysterious than the others. Our decomposition of the pentagon goes towards dispelling the mystery. The pentagon is reduced to an inductive definition of a kind of commutativity.

If the associativity arrows are isomorphisms, then the pentagon yields the definition of an associativity arrow complex in one of its indices in terms of associativity arrows simpler in that index, but more complex in the other two indices. There is no reduction of complexity in all the indices, and no real inductive definition. Our approach, which works also in the absence of isomorphism, as we said above, gives a real inductive definition.

## 2 Insertion

Let $\mathcal{L}_{1}$ be the set of words (finite sequences of symbols) in the alphabet $\{\square, \wedge,()$, defined inductively by

$$
\square \in \mathcal{L}_{1},
$$

$$
\text { if } X, Y \in \mathcal{L}_{1}, \text { then }(X \wedge Y) \in \mathcal{L}_{1} .
$$

Let $\mathcal{L}_{2}$ be $\mathcal{L}_{1}-\{\square\}$. The elements of $\mathcal{L}_{2}$ may be identified with finite planar binary trees with more than one node, while $\square$ is the trivial one-node tree. In this section, we use $X, Y$ and $Z$ for the members of $\mathcal{L}_{1}$. (Starting from the end of the section, we change this notation to $A, B, C, \ldots$ ) We omit the outermost pair of parentheses of the members of $\mathcal{L}_{1}$, taking them for granted. We make the same omission in other analogous situations later on.

Let $\mathbf{N}^{+}$be the set of natural numbers greater than 0 , and let $\mathcal{L}^{\prime}$ be the set of words in the alphabet $\{\mathbf{1}, \mathbf{2}\} \cup\left\{\triangleleft_{n} \mid n \in \mathbf{N}^{+}\right\}$defined inductively by the following clauses that involve also an inductive definition of a map || from $\mathcal{L}^{\prime}$ to $\mathbf{N}^{+}$:

$$
\begin{aligned}
& \mathbf{1} \in \mathcal{L}^{\prime} \text { and }|\mathbf{1}|=1, \\
& \mathbf{2} \in \mathcal{L}^{\prime} \text { and }|\mathbf{2}|=2,
\end{aligned}
$$

if $A, B \in \mathcal{L}^{\prime}$ and $1 \leq n \leq|A|$, then

$$
\left(A \triangleleft_{n} B\right) \in \mathcal{L}^{\prime} \text { and }\left|\left(A \triangleleft_{n} B\right)\right|=|A|+|B|-1
$$

Let $\mathcal{L}^{\prime \prime}$ be defined as $\mathcal{L}^{\prime}$ save that we omit the first clause above involving 1, and we replace $\mathcal{L}^{\prime}$ by $\mathcal{L}^{\prime \prime}$ in the two remaining clauses. For the members of $\mathcal{L}^{\prime}$ we use $A, B, C, \ldots$, sometimes with indices. As we did for $\mathcal{L}_{1}$, we omit the outermost pair of parentheses of the members of $\mathcal{L}^{\prime}$.

We define the equational calculus $\mathcal{I}^{\prime \prime}$ in $\mathcal{L}^{\prime \prime}$ (i.e. a calculus whose theorems are equations between members of $\mathcal{L}^{\prime \prime}$ ) by assuming reflexivity, symmetry and transitivity of equality, the rule that if $A=B$ and $C=D$, then $A \triangleleft_{n} C=B \triangleleft_{n} D$, provided $A \triangleleft_{n} C$ and $B \triangleleft_{n} D$ are defined, and the two axioms

$$
\begin{array}{lll}
(\text { assoc 1) } & \left(A \triangleleft_{n} B\right) \triangleleft_{m} C=A \triangleleft_{n}\left(B \triangleleft_{m-n+1} C\right) & \text { if } n \leq m<n+|B|, \\
(\text { assoc 2) } & \left(A \triangleleft_{n} B\right) \triangleleft_{m} C=\left(A \triangleleft_{m-|B|+1} C\right) \triangleleft_{n} B & \text { if } n+|B| \leq m .
\end{array}
$$

Note that the condition $n \leq m<n+|B|$ in (assoc 1) follows from the legitimacy of $B \triangleleft_{m-n+1} C$. Note also that in both (assoc 1) and (assoc 2) we have $n \leq m$. The equation (assoc 2 ) could be replaced by

$$
\left(A \triangleleft_{n} B\right) \triangleleft_{m} C=\left(A \triangleleft_{m} C\right) \triangleleft_{n+|C|-1} B \quad \text { if } m<n
$$

(The equations (assoc 1) and (assoc 2) are analogous to the two associativity equations for the cut operation one finds in multicategories; see [4] and [5], Section 3. Analogous equations are also found in the definition of operad; see [11], Section 1.)

The equational calculus $\mathcal{I}^{\prime}$ in $\mathcal{L}^{\prime}$ is defined as $\mathcal{I}^{\prime \prime}$ with the additional axiom

$$
\text { (unit) } \quad \mathbf{1} \triangleleft_{1} A=A \triangleleft_{n} \mathbf{1}=A
$$

(whose analogue one also finds in multicategories). Our purpose now is to interpret $\mathcal{L}^{\prime}$ in $\mathcal{L}_{1}$. This will make clear the meaning of the axioms of $\mathcal{I}^{\prime}$.

For $X$ in $\mathcal{L}_{1}$, let $|X|$ be the number of occurrences of $\square$ in $X$. We define in $\mathcal{L}_{1}$ the partial operation of insertion $\unlhd_{n}$ by the following inductive clauses:

$$
\begin{aligned}
& \square \unlhd_{1} Z=Z, \\
& (X \wedge Y) \unlhd_{n} Z= \begin{cases}\left(X \unlhd_{n} Z\right) \wedge Y & \text { if } 1 \leq n \leq|X| \\
X \wedge\left(Y \unlhd_{n-|X|} Z\right) & \text { if }|X|<n \leq|X|+|Y| .\end{cases}
\end{aligned}
$$

We define insertion in $\mathcal{L}_{2}$ by replacing the clause $\square \unlhd_{1} Z=Z$ above by the clauses$\wedge$$\unlhd_{2} Z=$$\wedge Z$.

Insertion gets its name from the fact that $X \unlhd_{n} Z$ is obtained by inserting $Z$ at the place of the $n$-th occurrence of $\square$ in $X$, starting from the left; namely, the $n$-th leaf of the tree corresponding to $X$ becomes the root of the tree corresponding to $Z$, and the resulting tree corresponds to $X \unlhd_{n} Z$. Insertion is called grafting in [12], and particular instances of insertion, which one finds in the source and target of the arrows $\gamma_{A, B}$ in Section 3 below, are called under and over in [7] (Section 1.5).

We interpret $\mathcal{L}^{\prime \prime}$ in $\mathcal{L}_{2}$, i.e., we define a function $v$ from $\mathcal{L}^{\prime \prime}$ to $\mathcal{L}_{2}$, in the following manner:

$$
\begin{aligned}
& v(\mathbf{2})=\square \wedge \square \\
& v\left(A \triangleleft_{n} B\right)=v(A) \unlhd_{n} v(B)
\end{aligned}
$$

For this definition to be correct, we must check that $|A|=|v(A)|$, which is easily done by induction on the length of $|A|$.

We prove first the following by an easy induction on the length of derivation.
Soundness. If $A=B$ in $\mathcal{I}^{\prime \prime}$, then $v(A)=v(B)$.
Our purpose is to prove also the converse:
Completeness. If $v(A)=v(B)$, then $A=B$ in $\mathcal{I}^{\prime \prime}$.

For every $A$ in $\mathcal{L}^{\prime \prime}$ we define the natural number $c(A)$ inductively as follows:

$$
\begin{aligned}
& c(\mathbf{2})=2 \\
& c\left(B \triangleleft_{n} C\right)=c(B)(c(C)+1) .
\end{aligned}
$$

Let $s(A)$ be the sum of the indices $n$ of all the occurrences of $\triangleleft_{n}$ in $A$, and let $d(A)=c(A)+s(A)$. Then we can easily check that if $A=B$ is an instance of (assoc 1) or (assoc 2), then $d(A)>d(B)$.

Let a member of $\mathcal{L}^{\prime \prime}$ be called normal when it has no part of the form of the left-hand side of (assoc 1 ) or (assoc 2), i.e. no part of the form $\left(A \triangleleft_{n} B\right) \triangleleft_{m} C$ for $n \leq m$. It can be shown that a normal member of $\mathcal{L}^{\prime \prime}$ is of one of the following forms:

$$
\left(\mathbf{2} \triangleleft_{2} A_{2}\right) \triangleleft_{1} A_{1}, \quad \mathbf{2} \triangleleft_{2} A_{2}, \quad \mathbf{2} \triangleleft_{1} A_{1}, \quad \mathbf{2},
$$

for $A_{1}$ and $A_{2}$ normal. These are the four normal types.
Then it is easy to show by applying (assoc 1) and (assoc 2) from left to right that for every $A$ in $\mathcal{L}^{\prime \prime}$ there is a normal $A^{\prime}$ such that $A=A^{\prime}$ in $\mathcal{I}^{\prime \prime}$. We can also show the following.

Auxiliary Lemma. If $A$ and $B$ are normal and $v(A)=v(B)$, then $A$ and $B$ coincide.

Proof. If $v(A)=v(B)$, then $A$ and $B$ must be of the same normal type (otherwise, clearly, $v(A) \neq v(B))$. If $A$ is $\left(\mathbf{2} \triangleleft_{2} A_{2}\right) \triangleleft_{1} A_{1}$ and $B$ is $\left(\mathbf{2} \triangleleft_{2} B_{2}\right) \triangleleft_{1} B_{1}$, then we conclude that $v\left(A_{1}\right)=v\left(B_{1}\right)$ and $v\left(A_{2}\right)=v\left(B_{2}\right)$, and we reason by induction. We reason analogously for the second and third normal type, and the normal type 2 provides the basis of the induction.

To prove Completeness, suppose $v(A)=v(B)$. Let $A=A^{\prime}$ and $B=B^{\prime}$ in $\mathcal{I}^{\prime \prime}$ for $A^{\prime}$ and $B^{\prime}$ normal. Then by Soundness we have $v\left(A^{\prime}\right)=v(A)=v(B)=v\left(B^{\prime}\right)$, and so, by the Auxiliary Lemma, $A^{\prime}$ and $B^{\prime}$ coincide. It follows that $A=B$ in $\mathcal{I}^{\prime \prime}$, which proves Completeness.

We can now also show that if $A=A^{\prime}$ and $A=A^{\prime \prime}$ in $\mathcal{I}^{\prime \prime}$ for $A^{\prime}$ and $A^{\prime \prime}$ normal, then $A^{\prime}$ and $A^{\prime \prime}$ coincide. This follows from Soundness and the Auxiliary Lemma. (One could show this uniqueness of normal form directly in $\mathcal{L}^{\prime \prime}$, without proceeding via $v$ and $\mathcal{L}_{2}$, by relying on confluence techniques, as in the lambda calculus or term-rewriting systems. In such a proof, diagrams analogous to Mac Lane's pentagon and the Yang-Baxter equation would arise.)

We interpret $\mathcal{L}^{\prime}$ in $\mathcal{L}_{1}$ by extending the definition of $v$ from $\mathcal{L}^{\prime \prime}$ to $\mathcal{L}_{2}$ with the clause $v(\mathbf{1})=\square$. Then we can prove Soundness and Completeness with $\mathcal{I}^{\prime \prime}$ replaced by $\mathcal{I}^{\prime}$. In reducing a member of $\mathcal{L}^{\prime}$ to a normal member of $\mathcal{L}^{\prime \prime}$ or to $\mathbf{1}$ we get rid first of all superfluous occurrences of $\mathbf{1}$, by relying on the equations (unit). Otherwise, the proof proceeds as before.

We can factorize $\mathcal{L}^{\prime}$ through the smallest equivalence relation such that the equations of $\mathcal{I}^{\prime}$ are satisfied, and obtain a set of equivalence classes isomorphic to $\mathcal{L}_{1}$. For the equivalence classes $[A]$ and $[B]$ we define $\wedge$ by

$$
[A] \wedge[B]={ }_{d f}\left[\left(\mathbf{2} \triangleleft_{2} B\right) \triangleleft_{1} A\right],
$$

and the isomorphism $i$ from $\mathcal{L}_{1}$ to $\mathcal{L}^{\prime}$ is defined by

$$
\begin{aligned}
& i(\square)=[\mathbf{1}] \\
& i(X \wedge Y)=i(X) \wedge i(Y)
\end{aligned}
$$

The inverse $i^{-1}$ of $i$ is defined by $i^{-1}([A])=v(A)$. (To verify that $i$ and $i^{-1}$ are inverse to each other, we rely on the fact that every $[A]$ is equal to $\left[A^{\prime}\right]$ for $A^{\prime}$ being normal or 1.)

We designate the equivalence class $[A]$ by $A$, and to designate the elements of $\mathcal{L}_{1}$ we can then use the notation introduced for $\mathcal{L}^{\prime}$. This means that we can write $A, B, C, \ldots$ for $X, Y, Z, \ldots$, we can write 2 for $\square \wedge \square$, and we can write $\triangleleft_{n}$ for $\unlhd_{n}$. We have for $\mathcal{L}_{1}$ the equation

$$
A \wedge B=\left(\mathbf{2} \triangleleft_{2} B\right) \triangleleft_{1} A
$$

We will write howeverinstead of $\mathbf{1}$, and reserve $\mathbf{1}$ with a subscript for the name of an arrow.

## 3 The category $\Gamma$

The objects of the category $\boldsymbol{\Gamma}$ are the elements of $\mathcal{L}_{1}$. To define the arrows of $\boldsymbol{\Gamma}$, we define first inductively the arrow terms of $\boldsymbol{\Gamma}$ in the following way:

$$
\begin{aligned}
& \mathbf{1}_{A}: A \rightarrow A, \\
& \gamma_{A, B}: A \triangleleft_{|A|} B \rightarrow B \triangleleft_{1} A, \\
& \gamma_{A, B}: B \triangleleft_{1} A \rightarrow A \triangleleft_{|A|} B
\end{aligned}
$$

are arrow terms of $\boldsymbol{\Gamma}$ for all objects $A$ and $B$; if $f: A \rightarrow B$ and $g: C \rightarrow D$ are arrow terms of $\boldsymbol{\Gamma}$, then $g \circ f: A \rightarrow D$ is an arrow term of $\boldsymbol{\Gamma}$, provided $B$ is $C$, and $f \triangleleft_{n} g: A \triangleleft_{n} C \rightarrow B \triangleleft_{n} D$ is an arrow term of $\boldsymbol{\Gamma}$, provided $1 \leq n \leq|A|$ and $1 \leq n \leq|B|$. Note that for all arrow terms $f: A \rightarrow B$ of $\boldsymbol{\Gamma}$ we have $|A|=|B|$; we write $|f|$ for $|A|$, which is equal to $|B|$.

The arrows of $\boldsymbol{\Gamma}$ are equivalence classes of arrow terms of $\boldsymbol{\Gamma}$ (cf. [2], Section $2.3)$ such that the following equations are satisfied:
(cat 1) $\quad \mathbf{1}_{B} \circ f=f \circ \mathbf{1}_{A}=f, \quad$ for $f: A \rightarrow B$,
(cat 2) $\quad(h \circ g) \circ f=h \circ(g \circ f)$,
(bif 1) $\quad \mathbf{1}_{A} \triangleleft_{n} \mathbf{1}_{B}=\mathbf{1}_{A \triangleleft_{n} B}$,

$$
\begin{equation*}
\left(f_{2} \circ f_{1}\right) \triangleleft_{n}\left(g_{2} \circ g_{1}\right)=\left(f_{2} \triangleleft_{n} g_{2}\right) \circ\left(f_{1} \triangleleft_{n} g_{1}\right) \tag{bif2}
\end{equation*}
$$

for $1 \leq n \leq|f|$ and $n \leq m \leq|f|+|g|-1$

$$
\begin{align*}
& (\text { assoc } 1 \rightarrow) \quad\left(f \triangleleft_{n} g\right) \triangleleft_{m} h=f \triangleleft_{n}\left(g \triangleleft_{m-n+1} h\right) \quad \text { if } n \leq m<n+|g|, \\
& (\text { assoc } 2 \rightarrow) \quad\left(f \triangleleft_{n} g\right) \triangleleft_{m} h=\left(f \triangleleft_{m-|g|+1} h\right) \triangleleft_{n} g \quad \text { if } n+|g| \leq m, \\
& (\text { unit } \rightarrow) \quad \mathbf{1}_{\square} \triangleleft_{1} f=f \triangleleft_{n} \mathbf{1}_{\square}=f, \\
& (\gamma \text { nat }) \quad \gamma_{B, D} \circ\left(f \triangleleft_{|A|} g\right)=\left(g \triangleleft_{1} f\right) \circ \gamma_{\overrightarrow{A, C}}, \\
& \gamma_{A, B}^{\overleftarrow{ }} \circ \gamma_{\overrightarrow{A, B}}=\mathbf{1}_{A \triangleleft|A|} B, \quad \gamma_{A, B} \circ \gamma_{A, B}=\mathbf{1}_{B \triangleleft_{1} A}, \\
& \gamma_{\square, A}=\gamma_{\overrightarrow{A, \square}}=\mathbf{1}_{A}, \\
& \gamma_{A \triangleleft|A|} B, C=\left(\gamma_{A, C} \triangleleft_{|A|} \mathbf{1}_{B}\right) \circ\left(\mathbf{1}_{A} \triangleleft_{|A|} \gamma_{\overrightarrow{B, C}}\right),  \tag{hex1}\\
& (\text { hex } 1 a) \quad \gamma_{A_{\triangleleft_{n} B, C}}=\gamma_{A, C} \triangleleft_{n} \mathbf{1}_{B} \quad \text { if } 1 \leq n<|A| \text {, } \\
& \text { (hex 2) } \quad \gamma_{C, A \triangleleft_{1} B}=\left(\mathbf{1}_{A} \triangleleft_{1} \gamma_{C, B}\right) \circ\left(\gamma_{C, A} \triangleleft_{|C|} \mathbf{1}_{B}\right) \text {, } \\
& \text { (hex 2a) } \quad \gamma_{C, A \triangleleft_{n} B}=\gamma_{C, A} \triangleleft_{n+|C|-1} \mathbf{1}_{B} \quad \text { if } 1<n \leq|A| \text {. }
\end{align*}
$$

We also assume besides reflexivity, symmetry and transitivity of equality that if $f=g$ and $h=j$, then for $\alpha$ being $\circ$ or $\triangleleft_{n}$ we have $f \alpha h=g \alpha j$, provided $f \alpha h$ and $g \alpha j$ are defined.

The equations (cat 1) and (cat 2) make of $\boldsymbol{\Gamma}$ a category. The equations (bif 1) and (bif 2) are analogous to bifunctorial equations. The equations (assoc $1 \rightarrow$ ), (assoc $2 \rightarrow)$ and (unit $\rightarrow$ ) are analogous to naturality equations. In (assoc $1 \rightarrow$ ) the associativity arrows with respect to $\triangleleft_{n}$ are not written down because they are identity arrows, in virtue of the equation (assoc 1 ) on objects. Analogous remarks hold for (assoc $2 \rightarrow$ ) and (unit $\rightarrow$ ). The equation ( $\gamma$ nat) is analogous to a naturality equation, and the equations $(\gamma \gamma)$ say that $\gamma_{\vec{A}, B}$ is an isomorphism, with inverse $\gamma_{A, B}^{\leftarrow}$.

The equation $(\gamma \mathbf{1})$ is auxiliary, and would not be needed if we had assumed $\gamma_{A, B}$ and $\gamma_{A, B}^{\overleftarrow{ }}$ only for $A$ and $B$ different from $\square$. The equations (hex 1) and (hex 2) are analogous to Mac Lane's hexagonal equation of symmetric monoidal categories (see [9], [10], Section VII.7, or [2], Section 5.1). Here the associativity arrows with respect to $\triangleleft_{n}$ are identity arrows, in virtue of the equation
(assoc 1) on objects (and so instead of hexagons we have triangles; cf. the equation (chex 1) in Section 5 below). Finally, the equations (hex $1 a$ ) and (hex 2a), together with $(\gamma \mathbf{1})$, (hex 1) and (hex 2), enable us to define inductively $\gamma_{\vec{A}, B}$ for all $A$ and $B$ in terms of the identity arrows $\mathbf{1}_{A}$, the arrows

$$
\overrightarrow{\gamma_{2,2}}: \square \wedge(\square \wedge \square) \rightarrow(\square \wedge \square) \wedge
$$

and the operations on arrows $\circ$ and $\triangleleft_{n}$. Relying on $(\gamma \gamma)$, we can proceed analogously for $\gamma_{A, B}^{\leftarrow}$ by using instead of $\underset{\mathbf{2 , 2}}{ }$ the arrows

$$
\gamma_{\mathbf{2 , 2}}^{\leftarrow}:(\square \wedge \square) \wedge \square \rightarrow \square \wedge(\square \wedge \square)
$$

The equations (hex $1 a$ ) and (hex 2a) are also analogous to Mac Lane's hexagon mentioned above (due to the presence of (assoc 2) too, the collapse is however not any more into a triangle, but into a two-sided figure).

## 4 The category A

The category $\hat{\mathbf{A}}$ has the same objects as $\boldsymbol{\Gamma}$; namely, the elements of $\mathcal{L}_{1}$. To define the arrows of $\hat{\mathbf{A}}$, we define first inductively the arrow terms of $\hat{\mathbf{A}}$ in the following way:

$$
\begin{aligned}
& \mathbf{1}_{A}: A \rightarrow A \\
& b_{A, B, C}: A \wedge(B \wedge C) \rightarrow(A \wedge B) \wedge C \\
& b_{A, B, C}^{\overleftarrow{ }}:(A \wedge B) \wedge C \rightarrow A \wedge(B \wedge C)
\end{aligned}
$$

are arrow terms of $\hat{\mathbf{A}}$ for all objects $A, B$ and $C$; if $f: A \rightarrow B$ and $g: C \rightarrow D$ are arrow terms of $\hat{\mathbf{A}}$, then $g \circ f: A \rightarrow D$ is an arrow term of $\hat{\mathbf{A}}$, provided $B$ is $C$, and $f \wedge g: A \wedge C \rightarrow B \wedge D$ is an arrow term of $\hat{\mathbf{A}}$.

The arrows of $\hat{\mathbf{A}}$ are equivalence classes of arrow terms of $\hat{\mathbf{A}}$ such that the following equations are satisfied: (cat 1), (cat 2), (bif 1) and (bif 2) with $\triangleleft_{n}$ replaced by $\wedge$, and moreover
(b nat) $\quad b_{\overrightarrow{B, D, F}} \circ(f \wedge(g \wedge h))=((f \wedge g) \wedge h) \circ b_{\overrightarrow{A, C, E}}$,

$$
\begin{align*}
& b_{A, B, C}^{\overleftarrow{ }} \circ b_{\vec{A}, B, C}=\mathbf{1}_{A \wedge(B \wedge C)}, \quad b_{\overrightarrow{A, B, C}} \circ b_{A, B, C}^{\overleftarrow{ }}=\mathbf{1}_{(A \wedge B) \wedge C}  \tag{bb}\\
& b_{A \wedge B, C, D} \circ b_{\vec{A}, B, C \wedge D}=\left(b_{A, B, C} \wedge \mathbf{1}_{D}\right) \circ b_{A, B \wedge C, D} \circ\left(\mathbf{1}_{A} \wedge b_{B, C, D}\right) \tag{b5}
\end{align*}
$$

We also assume besides reflexivity, symmetry and transitivity of equality that if $f=g$ and $h=j$, then $f \circ h=g \circ j$, provided $f \circ h$ and $g \circ j$ are defined, and $f \wedge h=g \wedge j$.

In $\hat{\mathbf{A}}$ we have that $\wedge$ is a bifunctor, $b \rightarrow$ is a natural isomorphism in all its indices, and (b5) is Mac Lane's pentagonal equation of [9], where it is proved that the category $\hat{\mathbf{A}}$ is a preorder. Namely, for all arrows $f, g: A \rightarrow B$ of $\hat{\mathbf{A}}$ we have that $f=g$ (see also [10], Section VII.2, or [2], Section 4.3). The category $\hat{\mathbf{A}}$ is the free monoidal category without unit, i.e. free associative category in the terminology of [2] (Section 4.3), generated by a single object, this object being conceived as a trivial discrete category.

## 5 The isomorphism of $\Gamma$ and $\hat{A}$

We are going to prove that the categories $\boldsymbol{\Gamma}$ and $\hat{\mathbf{A}}$ are isomorphic. We define first what is missing of the structure of $\hat{\mathbf{A}}$ in $\boldsymbol{\Gamma}$ in the following manner:

$$
\begin{aligned}
& b_{A, B, C}^{\vec{A}}={ }_{d f}\left(\left(\gamma_{\mathbf{2}, \mathbf{2}} \triangleleft_{3} \mathbf{1}_{C}\right) \triangleleft_{2} \mathbf{1}_{B}\right) \triangleleft_{1} \mathbf{1}_{A} \\
& b_{A, B, C}^{\leftarrow}={ }_{d f}\left(\left(\gamma_{\mathbf{2}, \mathbf{2}}^{\leftarrow} \triangleleft_{3} \mathbf{1}_{C}\right) \triangleleft_{2} \mathbf{1}_{B}\right) \triangleleft_{1} \mathbf{1}_{A} \\
& f \wedge g={ }_{d f}\left(\mathbf{1}_{\mathbf{2}} \triangleleft_{2} g\right) \triangleleft_{1} f
\end{aligned}
$$

It can then be checked by induction on the length of derivation that the equations of $\hat{\mathbf{A}}$ are satisfied in $\boldsymbol{\Gamma}$.

We have of course the equations (cat 1) and (cat 2), while the equations (bif 1) and (bif 2) with $\triangleleft_{n}$ replaced by $\wedge$ are easy consequences of (bif 1) and (bif 2). To derive (b nat), we have that with the help of (assoc $1 \rightarrow$ ) and (bif 1) the left-hand side is equal to

$$
\left(\left(\left(\gamma_{\mathbf{2}, \mathbf{2}} \triangleleft_{3} \mathbf{1}_{F}\right) \triangleleft_{2} \mathbf{1}_{D}\right) \triangleleft_{1} \mathbf{1}_{B}\right) \circ\left(\left(\left(\mathbf{1}_{\mathbf{2} \triangleleft_{2} \mathbf{2}} \triangleleft_{3} h\right) \triangleleft_{2} g\right) \triangleleft_{1} f\right),
$$

while with the help of (assoc $1 \rightarrow)$, (assoc $2 \rightarrow)$ and (bif 1) the right-hand side is equal to

$$
\left(\left(\left(\mathbf{1}_{\mathbf{2} \triangleleft_{1} \mathbf{2}} \triangleleft_{3} h\right) \triangleleft_{2} g\right) \triangleleft_{1} f\right) \circ\left(\left(\left(\gamma_{\mathbf{2 , 2}} \triangleleft_{3} \mathbf{1}_{E}\right) \triangleleft_{2} \mathbf{1}_{C}\right) \triangleleft_{1} \mathbf{1}_{A}\right) .
$$

Then it is enough to apply (bif 2) and (cat 1). It is trivial to derive (bb) with the help of (bif 2), $(\gamma \gamma)$ and (bif 1).

We derive finally the pentagonal equation (b5). With the help of (bif 1 ), (assoc $1 \rightarrow$ ) and (assoc $2 \rightarrow$ ) we derive that each of

$$
b_{A, B, C \wedge D}^{\overrightarrow{ }, \quad b_{A \wedge B, C, D}^{\overrightarrow{ }}, \quad \mathbf{1}_{A} \wedge b_{B, C, D}, \quad b_{A, B \wedge C, D}, \quad b_{A, B, C} \wedge \mathbf{1}_{D} .}
$$

is equal to $\left(\left(\left(f \triangleleft_{4} \mathbf{1}_{D}\right) \triangleleft_{3} \mathbf{1}_{C}\right) \triangleleft_{2} \mathbf{1}_{B}\right) \triangleleft_{1} \mathbf{1}_{A}$ for $f$ being respectively

$$
\overrightarrow{\gamma_{\mathbf{2}, \mathbf{2}}} \triangleleft_{3} \mathbf{1}_{\mathbf{2}}, \quad \overrightarrow{\gamma_{\mathbf{2}, \mathbf{2}}} \triangleleft_{1} \mathbf{1}_{2}, \quad \mathbf{1}_{2} \triangleleft_{2} \overrightarrow{\gamma_{\mathbf{2}, \mathbf{2}}}, \quad \gamma_{\mathbf{2 , 2}} \triangleleft_{2} \mathbf{1}_{\mathbf{2}}, \quad \mathbf{1}_{2} \triangleleft_{1} \overrightarrow{\gamma_{\mathbf{2}, \mathbf{2}}} .
$$

Then, by relying on (bif 2), it is enough to derive the following:

$$
\begin{aligned}
&\left(\gamma_{\mathbf{2 , 2}} \triangleleft_{1} \mathbf{1}_{\mathbf{2}}\right) \circ\left(\gamma_{\mathbf{2 , 2}} \triangleleft_{3} \mathbf{1}_{\mathbf{2}}\right)=\gamma_{\mathbf{2} \triangleleft_{1} \mathbf{2}, \mathbf{2}} \circ\left(\gamma_{\mathbf{2 , 2}} \triangleleft_{3} \mathbf{1}_{\mathbf{2}}\right), \quad \text { by }(h e x 1 a), \\
&=\left(\mathbf{1}_{\mathbf{2}} \triangleleft_{1} \overrightarrow{\gamma_{\mathbf{2 , 2}}}\right) \circ \gamma_{\mathbf{2} \triangleleft_{2} \mathbf{2 , 2}}, \quad \text { by }(\gamma n a t), \\
&=\left(\mathbf{1}_{\mathbf{2}} \triangleleft_{1} \overrightarrow{\gamma_{\mathbf{2}, \mathbf{2}}}\right) \circ\left(\gamma_{\mathbf{2 , 2}} \triangleleft_{2} \mathbf{1}_{\mathbf{2}}\right) \circ\left(\mathbf{1}_{\mathbf{2}} \triangleleft_{2} \gamma_{\mathbf{2 , 2}}\right), \quad \text { by }(\text { hex } 1) .
\end{aligned}
$$

Diagrammatically, we have


So the pentagon is decomposed into a triangle (a degenerate hexagon, corresponding to (hex 1)), a square (analogous to a naturality square, corresponding to $(\gamma$ nat $)$ ) and a two-sided diagram (corresponding to (hex $1 a)$ ).

If $c_{A, B}: A \wedge B \rightarrow B \wedge A$ is the commutativity arrow of symmetric monoidal categories, for which in strict categories of this kind, where associativity arrows are identities, we have the equations

$$
\begin{array}{ll}
(c \text { nat }) & c_{B, D} \circ(f \wedge g)=(g \wedge f) \circ c_{A, B} \\
(c \text { hex } 1) & c_{A \wedge B, C}=\left(c_{A, C} \wedge \mathbf{1}_{B}\right) \circ\left(\mathbf{1}_{A} \wedge c_{B, C}\right)
\end{array}
$$

then we derive the Yang-Baxter equation

$$
\left(c_{B, C} \wedge \mathbf{1}_{A}\right) \circ\left(\mathbf{1}_{B} \wedge c_{A, C}\right) \circ\left(c_{A, B} \wedge \mathbf{1}_{C}\right)=\left(\mathbf{1}_{C} \wedge c_{A, B}\right) \circ\left(c_{A, C} \wedge \mathbf{1}_{B}\right) \circ\left(\mathbf{1}_{A} \wedge c_{B, C}\right)
$$

in the following way:

$$
\begin{aligned}
&\left(c_{B, C} \wedge \mathbf{1}_{A}\right) \circ\left(\mathbf{1}_{B} \wedge c_{A, C}\right) \circ\left(c_{A, B} \wedge \mathbf{1}_{C}\right)=c_{B \wedge A, C} \circ\left(c_{A, B} \wedge \mathbf{1}_{C}\right), \quad \text { by }(c \text { hex } 1) \\
&=\left(\mathbf{1}_{C} \wedge c_{A, B}\right) \circ c_{A \wedge B, C}, \quad \text { by }(c \text { nat }), \\
&=\left(\mathbf{1}_{C} \wedge c_{A, B}\right) \circ\left(c_{A, C} \wedge \mathbf{1}_{B}\right) \circ\left(\mathbf{1}_{A} \wedge c_{B, C}\right), \quad \text { by }(c \text { hex } 1)
\end{aligned}
$$

This derivation is analogous to our derivation of (b5) above, where however the arrow corresponding to $\mathbf{1}_{B} \wedge c_{A, C}$ is identity, in virtue of the equation (assoc 2) on objects.

Alternatively, we derive ( $b 5$ ) by using the following:

$$
\begin{aligned}
\left(\gamma_{\mathbf{2}, \mathbf{2}} \triangleleft_{1} \mathbf{1}_{\mathbf{2}}\right) \circ\left(\gamma_{\mathbf{2}, \mathbf{2}}\right. & \left.\triangleleft_{3} \mathbf{1}_{\mathbf{2}}\right)=\left(\gamma_{\mathbf{2}, \mathbf{2}} \triangleleft_{1} \mathbf{1}_{\mathbf{2}}\right) \circ \gamma_{\mathbf{2}, \mathbf{2} \triangleleft_{2} \mathbf{2}}, \quad \text { by }(h e x 2 a), \\
& =\gamma_{\mathbf{2}, \mathbf{2} \triangleleft_{1} \mathbf{2}} \circ\left(\mathbf{1}_{\mathbf{2}} \triangleleft_{2} \overrightarrow{\gamma_{\mathbf{2}, \mathbf{2}}}\right), \quad \text { by }(\gamma n a t), \\
& =\left(\mathbf{1}_{\mathbf{2}} \triangleleft_{1} \overrightarrow{\gamma_{\mathbf{2}, \mathbf{2}}}\right) \circ\left(\gamma_{\mathbf{2}, \mathbf{2}} \triangleleft_{2} \mathbf{1}_{\mathbf{2}}\right) \circ\left(\mathbf{1}_{\mathbf{2}} \triangleleft_{2} \overrightarrow{\mathbf{2}, \mathbf{2}}\right), \quad \text { by }(\text { hex } 2) .
\end{aligned}
$$

Diagrammatically, we have


This is an alternative decomposition of the pentagon into a triangle, a square and a two-sided diagram. Hence we have in $\boldsymbol{\Gamma}$ all the equations of $\hat{\mathbf{A}}$.

To define what is missing of the structure of $\boldsymbol{\Gamma}$ in $\hat{\mathbf{A}}$, we have first the following inductive definition of $\triangleleft_{n}$ on arrows:

$$
\begin{aligned}
& \text { if } A^{\prime} \wedge\left(B^{\prime} \wedge C^{\prime}\right)=(A \wedge(B \wedge C)) \triangleleft_{n} D, \\
& \qquad \begin{array}{l}
\quad b_{A, B, C} \triangleleft_{n} \mathbf{1}_{D}=b_{A^{\prime}, B^{\prime}, C^{\prime}}, \quad b_{A, B, C}^{\leftarrow} \triangleleft_{n} \mathbf{1}_{D}=b_{A^{\prime}, B^{\prime}, C^{\prime}}^{\leftarrow}, \\
(g \circ f) \triangleleft_{n} \mathbf{1}_{D}=\left(g \triangleleft_{n} \mathbf{1}_{D}\right) \circ\left(f \triangleleft_{n} \mathbf{1}_{D}\right), \\
(f \wedge g) \triangleleft_{n} \mathbf{1}_{D}= \begin{cases}\left(f \triangleleft_{n} \mathbf{1}_{D}\right) \wedge g & \text { if } 1 \leq n \leq|f| \\
f \wedge\left(g \triangleleft_{n-|f|} \mathbf{1}_{D}\right) & \text { if }|f|<n \leq|f|+|g|,\end{cases} \\
\mathbf{1}_{\square} \triangleleft_{1} f=f, \\
\mathbf{1}_{A \wedge B} \triangleleft_{n} f= \begin{cases}\left(\mathbf{1}_{A} \triangleleft_{n} f\right) \wedge \mathbf{1}_{B} & \text { if } 1 \leq n \leq|A| \\
\mathbf{1}_{A} \wedge\left(\mathbf{1}_{B} \triangleleft_{n-|A|} f\right) & \text { if }|A|<n \leq|A|+|B|,\end{cases} \\
f \triangleleft_{n} g=\left(\mathbf{1}_{B} \triangleleft_{n} g\right) \circ\left(f \triangleleft_{n} \mathbf{1}_{C}\right) .
\end{array}
\end{aligned}
$$

We define $\gamma_{\overrightarrow{A, B}}$ and $\gamma_{A, B}^{\leftarrow}$ by stipulating

$$
\gamma_{\mathbf{2 , 2}}^{\overrightarrow{2}}={ }_{d f} b_{\square, \square, \square}, \quad \gamma_{\mathbf{2}, \mathbf{2}}^{\leftarrow}={ }_{d f} b_{\square, \square, \square}^{\overleftarrow{\square}, \square}
$$

and by using $(\gamma \mathbf{1})$, (hex 1), (hex $1 a)$, (hex 2) and (hex $2 a)$ as clauses in an inductive definition.

The equations of $\boldsymbol{\Gamma}$ certainly hold in $\hat{\mathbf{A}}$ for this defined structure because $\hat{\mathbf{A}}$ is a preorder, as we said above. To finish the proof that $\boldsymbol{\Gamma}$ and $\hat{\mathbf{A}}$ are isomorphic categories, it remains only to check that the clauses of the inductive definitions of $\triangleleft_{n}, \gamma_{\overrightarrow{A, B}}$ and $\gamma_{A, B}^{\overleftarrow{ }}$ hold as equations in $\boldsymbol{\Gamma}$ for $b_{A, B, C}, b_{A, B, C}^{\overleftarrow{ }}$ and $\wedge$ defined as they are defined in $\boldsymbol{\Gamma}$. This is done by using essentially (assoc $1 \rightarrow$ ) and $($ assoc $2 \rightarrow)$. So $\boldsymbol{\Gamma}$ is isomorphic to $\hat{\mathbf{A}}$, and is hence a preorder.

If we have instead of $\hat{\mathbf{A}}$ the free monoidal category without unit, i.e. the free associative category, $\hat{\mathbf{A}}^{\prime}$ generated by an arbitrary nonempty set of objects $\mathcal{P}$, conceived as a discrete category, then, instead of $\boldsymbol{\Gamma}$, the analogous category $\boldsymbol{\Gamma}^{\prime}$ isomorphic to $\hat{\mathbf{A}}^{\prime}$ would have as generators $\mathcal{P} \cup\{\mathbf{2}\}$. Every object of $\boldsymbol{\Gamma}^{\prime}$ different from a member of $\mathcal{P}$ can be written in the form

$$
\left(\ldots\left(C \triangleleft_{n} p_{n}\right) \ldots \triangleleft_{2} p_{2}\right) \triangleleft_{1} p_{1}
$$

for $C$ an object of $\boldsymbol{\Gamma}$ (more precisely, a member of $\mathcal{L}_{2}$ ), $n=|C|$ and $p_{1}, \ldots, p_{n}$ members of $\mathcal{P}$. For the arrows $\gamma_{A, B}$ and $\gamma_{A, B}^{\leftarrow}$ we would assume that $p_{|A|}$ in $A$ coincides with $p_{1}$ in $B$, and the equations $(\gamma \mathbf{1})$ and (unit $\rightarrow$ ) would have to be adapted.

We have seen above how Mac Lane's pentagon arises from a Yang-Baxter hexagon by collapsing, according to (assoc 2), the vertices corresponding to $\left(\mathbf{2} \triangleleft_{1} \mathbf{2}\right) \triangleleft_{3} \mathbf{2}$, i.e. $B \wedge A \wedge C$, and $\left(\mathbf{2} \triangleleft_{2} \mathbf{2}\right) \triangleleft_{1} \mathbf{2}$, i.e. $B \wedge C \wedge A$, into a single vertex corresponding to $(\square \wedge \square) \wedge(\square \wedge \square)$. We can apply this collapsing procedure based on (assoc 2) to the three-dimensional permutohedron (whose vertices correspond to permutations of four letters and edges to transpositions of adjacent letters) in order to obtain the three-dimensional associahedron (whose vertices correspond to planar binary trees with five leaves and edges to arrow terms of $\hat{\mathbf{A}}$ with a single $b^{\rightarrow}$ ), and afterwards we can proceed to higher dimensions. The function that corresponds to our procedure is described in [13]. Our paper provides a motivation for that function.

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