

## UNARY PRIMITIVE RECURSIVE FUNCTIONS

DANIEL E. SEVERIN

**Abstract.** In this article, we study some new characterizations of primitive recursive functions based on restricted forms of primitive recursion, improving the pioneering work of R. M. Robinson and M. D. Gladstone. We reduce certain recursion schemes (mixed/pure iteration without parameters) and we characterize one-argument primitive recursive functions as the closure under substitution and iteration of certain optimal sets.

**§1. Introduction.**  $\text{Prim}$ , i.e. the set of *primitive recursive functions*, is the closure under substitution and primitive recursion of zero, successor and projection functions. For a detailed definition, the reader is referred to any standard work, for instance chapter 1 of [8]. A suitable subset is  $\text{Prim}(\mathbb{N}, \mathbb{N})$ , i.e. the set of *unary primitive recursive functions*. It will be one of the objects of our research.

Recursion schemes have been studied intensively during the twentieth century. In particular, R. M. Robinson[15, 16] and his wife J. Robinson[13, 14] proved that it is sufficient to consider one-argument functions because functions of several arguments can be reduced to them using pairing strategies. Later on, Gladstone[6, 7] and Georgieva[3] made improvements to the recursion schemes. At the same time as the study of recursive functions, several classifications were carried out over  $\text{Prim}(\mathbb{N}, \mathbb{N})$ . The first one was Grzegorzczuk hierarchy[9]. Since then, other hierarchies have appeared (cf. [12, 1, 4, 2, 11]). Finally, some algebraic properties of  $\text{Prim}(\mathbb{N}, \mathbb{N})$  were verified in [17]. Similar topics are covered in [5, 10].

The present paper improves the work of Robinson[15] and Gladstone[7].

The paper is organized as follows: In §2 we will give a useful symbolic notation for writing functions. In §3 we will show previous results, and the facts to be proved here. In §4 we will analyze a possible reduction in one of the recursion schemes. More precisely, mixed iteration without parameters with  $a$  fixed is as expressive as mixed iteration without parameters with  $a$  variable (the meaning of these schemes and  $a$  can be found in §3). In §5 we will do the same thing with pure iteration without parameters. And, in §6 we will characterize unary primitive recursive functions as the closure of the set including  $x \mapsto 1$  and  $x \mapsto x - \lfloor \sqrt{x} \rfloor^2$  with respect to substitution, iteration and the following operator:  $f \mapsto f + I$ , where  $I$  is the identity function on natural numbers.

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**§2. Notation.** To denote arbitrary functions we shall use letters in uppercase such as  $F$ ,  $G$  and  $H$ . To denote natural variables we shall use  $x$ ,  $y$ ,  $z \dots$ , whereas  $a$ ,  $b$ ,  $\dots$  are used to denote constants. Throughout the paper, the following functions will be used:

- Basic functions:

$$\begin{array}{ll}
 I(x) = x & \text{(identity)} \\
 \overline{n}(x) = n & \text{(constants)} \\
 S(x) = x + 1 & \text{(successor)} \\
 P(x) = x \dot{-} 1 & \text{(predecessor)} \\
 I_k^n(x_1, x_2, \dots, x_n) = x_k, \text{ for } 1 \leq k \leq n & \text{(projections)}
 \end{array}$$

- Arithmetic functions:

$$\begin{array}{ll}
 D(x) = 2x & \text{(double)} \\
 Sq(x) = x^2 & \text{(square)} \\
 Hf(x) = \lfloor x/2 \rfloor & \text{(half)} \\
 Pw(x) = 2^x & \text{(power of two)} \\
 Rt(x) = \lfloor \sqrt{x} \rfloor & \text{(integer square root)}
 \end{array}$$

- Cantor pairing functions:

$$\begin{array}{ll}
 A(x) = \lfloor (x^2 + x)/2 \rfloor & \text{($x$-th triangular number)} \\
 V(x) = \left\lfloor \frac{\lfloor \sqrt{8x+1} \rfloor - 1}{2} \right\rfloor & \text{(inverse of } A) \\
 J(x, y) = A(x + y) + x & \text{(pairing function)} \\
 K(x) = x - A(V(x)) & \text{(first inverse)} \\
 L(x) = A(V(x) + 1) - x - 1 & \text{(second inverse)}
 \end{array}$$

- Binary functions:

$$\begin{array}{ll}
 x \dot{-} y = \begin{cases} x - y & \text{if } x \geq y \\ 0 & \text{otherwise} \end{cases} & \text{(arithmetic difference)} \\
 |x - y| = \begin{cases} x - y & \text{if } x \geq y \\ y - x & \text{otherwise} \end{cases} & \text{(distance)}
 \end{array}$$

- Other functions:<sup>1</sup>

$$O(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases} \quad (\text{power of zero, cosignum})$$

$$Sgn(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{otherwise} \end{cases} \quad (\text{signum})$$

$$N(x) = x \bmod 2 \quad (\text{characteristic of odd numbers})$$

$$E(x) = x - \lfloor \sqrt{x} \rfloor^2 \quad (\text{excess over a square})$$

$$Q(x) = \begin{cases} 1 & \text{if } x \text{ is a square} \\ 0 & \text{otherwise} \end{cases} \quad (\text{characteristic of square numbers})$$

Let  $F, G, G_1, \dots, G_m$  be functions, and  $\mathfrak{x} = (x_1, x_2, \dots, x_n)$ , i.e. a  $n$ -tuple. The following operators on natural number functions will be used:

- Substitution:

$$\text{subst}(F, G_1, G_2, \dots, G_m)(\mathfrak{x}) = F(G_1(\mathfrak{x}), G_2(\mathfrak{x}), \dots, G_m(\mathfrak{x})).$$

A more special case is defined for one-argument functions,

$$(F \circ G)(\mathfrak{x}) = F(G(\mathfrak{x})),$$

$$(FG)(x) = F(G(x)).$$

- Primitive recursion:

$$\mathcal{R}[F, G](\mathfrak{x}, 0) = F(\mathfrak{x}),$$

$$\mathcal{R}[F, G](\mathfrak{x}, y + 1) = G(\mathfrak{x}, y, \mathcal{R}[F, G](\mathfrak{x}, y)).$$

- Restricted forms of primitive recursion:<sup>2</sup>

$$1) \mathcal{M}[F](0) = 0,$$

$$\mathcal{M}[F](x + 1) = F(x, \mathcal{M}[F](x)),$$

$$2) F^{\square(a)}(0) = a,$$

$$F^{\square(a)}(x + 1) = F(F^{\square(a)}(x)).$$

$$3) F^{\square}(x) = F^{\square(0)}(x).$$

- Power:

$$F^0(x) = x,$$

$$F^{n+1}(x) = F(F^n(x)).$$

- Miscellaneous:

$$1) (F + G)(x) = F(x) + G(x),$$

$$2) (F \dot{-} G)(x) = F(x) \dot{-} G(x),$$

$$3) |F - G|(x) = |F(x) - G(x)|,$$

$$4) J(F, G)(x) = J(F(x), G(x)).$$

<sup>1</sup>Some authors write  $0^x$ ,  $\overline{\text{sg}}(x)$  or  $\text{cosg}(x)$  instead of  $O(x)$ .

<sup>2</sup>Notations  $F^{\square}$  and  $F^{\square(a)}$  are due to Szalkai[17].

TABLE 1. Precedence and associativity of operators.

Precedence	Operators	Associativity
First	$F + G, F \dot{-} G$	Left
Second	$FG, F \circ G$	Any
Third	$F^n, F^\square, F^{\square(a)}$	—

In order to decrease the size of this article and improve readability, we will give a symbolic notation for representing functions. If the definition of a new function  $F : \mathbb{N}^n \rightarrow \mathbb{N}$  is

$$F(x_1, x_2, \dots, x_n) = \text{expression}(x_1, x_2, \dots, x_n),$$

we will write

$$F \equiv \text{expression},$$

where *expression* is composed by the functions and operators previously defined. Precedence and associativity rules are shown in table 1. Here are some examples of well-formed expressions:

$$\begin{aligned} D &\equiv \mathcal{M}[S \circ S \circ I_2^2], & O &\equiv \text{subst}(\mathcal{R}[\bar{1}, P \circ I_3^3], I, I), \\ Pw &\equiv S(I + I + \bar{1})^\square, & V &\equiv Hf \ P \ Rt \ S \ D \ D \ D. \end{aligned}$$

A finite set of initial functions and of functional operators is called *basis*. We will denote with

$$\mathcal{F} = \langle F_1, F_2, \dots, F_n, F^\oplus, \dots, F \otimes G, \dots \rangle$$

the basis composed by the initial functions  $F_1, F_2, \dots, F_n$ , the unary operators  $F^\oplus, \dots$ , the binary operators  $F \otimes G, \dots$  and so on.

We will denote with  $\text{clos}\mathcal{F}$  the closure of the basis  $\mathcal{F}$ . An example is

$$\text{Prim} = \text{clos}(\bar{0}, S, I_k^n, \text{subst}, \mathcal{R}[F, G]).$$

**§3. Preliminaries.** In [15], some recursion schemes are introduced (all of them are particular cases of primitive recursion):

1. Mixed recursion with one parameter:  
 $F(x, 0) = G(x), \ F(x, y + 1) = H(x, y, F(x, y)).$
2. Pure recursion with one parameter:  
 $F(x, 0) = G(x), \ F(x, y + 1) = H(x, F(x, y)).$
3. Mixed iteration with one parameter:  
 $F(x, 0) = x, \ F(x, y + 1) = H(y, F(x, y)).$
4. Pure iteration with one parameter:  
 $F(x, 0) = x, \ F(x, y + 1) = H(F(x, y)).$
5. Mixed iteration without parameters:  
 $F(0) = a, \ F(y + 1) = H(y, F(y)).$
6. Pure iteration without parameters:  
 $F(0) = a, \ F(y + 1) = H(F(y)).$
7. Mixed iteration without parameters, and  $a = 0$ :  
 $F(0) = 0, \ F(y + 1) = H(y, F(y)).$

TABLE 2. Table of functions that must be added as initial functions.

	One Parameter		No Parameter	
	Recursion	Iteration	$a$ variable	$a = 0$ (fixed)
Mixed	– [15]	– [15]	$x + y$ [7]	$x + y, Q$ [15] $ x - y $ [15] $x + y, O$ §4
Pure	– [6]	– [7]	$ x - y $ [7] $x \dot{-} y$ [3, 11]	$x + y, E$ [15] $x + y, K$ [16] $x + y, L$ [16] $J, K$ [16] $J, L$ [16] $ x - y $ §5 $x \dot{-} y$ §5

8. Pure iteration without parameters, and  $a = 0$  (or simply called *iteration*):  
 $F(0) = 0, F(y + 1) = H(F(y))$ .

We will refer to these schemes as  $\text{rec}_1, \text{rec}_2, \dots, \text{rec}_8$  in the same order listed above. Note that schemes  $\text{rec}_1, \text{rec}_7$  and  $\text{rec}_8$  have symbolic notations:  $F \equiv H^{\square(a)}$ ,  $F \equiv \mathcal{M}[H]$  and  $F \equiv H^{\square}$ .

Robinson and Gladstone proved that the primitive recursion scheme can be replaced by one of the cases with one parameter, i.e.  $\text{rec}_1$ – $\text{rec}_4$ . They also proved that the cases without parameters, i.e.  $\text{rec}_5$ – $\text{rec}_8$ , are adequate but certain functions must be added to the initial functions. Table 2 summarizes which functions are sufficient to be included as initial functions (the symbol – denotes the null set). In this table, the references indicate where the proofs of previous results can be found and the section references indicates where are the proofs of new results. Now, the tables that appeared on p. 929 of [15] and on p. 654 of [7] can be substituted by our table.

In the cases without parameters, it is not necessary to take zero function as an initial function because it can be obtained from identity and iteration as follows:

$$\bar{0}(0) = 0, \bar{0}(x + 1) = I(\bar{0}(x)).$$

Moreover, in the pure cases without parameters, it is not necessary to take projection functions as initial functions if we are considering one-argument functions. In  $\text{Prim}(\mathbb{N}, \mathbb{N})$ , there is only one projection, the identity function, which can be obtained from successor and iteration as follows:

$$I(0) = 0, I(x + 1) = S(I(x)).$$

Notice that all constant functions belong to every case given in table 2, since they can be generated using zero and successor functions:  $\bar{n}(x) = S^n(\bar{0}(x))$ . Constant functions of more that one argument can be defined composing a one-argument

constant function with an arbitrary function of  $n$  arguments (e.g. a projection).

At the end of §4 of [15], Robinson determined that  $Sq$ ,  $O$ ,  $Hf$ ,  $Rt$ , addition and subtraction<sup>3</sup> ( $x - y$ ) are sufficient to add as initial functions when we work with  $\text{rec}_5$ - $\text{rec}_8$ . We will rewrite this result in the next lemma.

LEMMA 3.1. *For  $i \in \{5, 6, 7, 8\}$ ,*

$$\text{Prim} = \text{clos}(S, I_k^n, Sq, O, Hf, Rt, +, -, \text{subst}, \text{rec}_i).$$

In some sections, we will just work with unary primitive recursive functions and the scheme of iteration. The following definition will help us.

DEFINITION 3.2. We say that a basis  $\mathcal{F}$  is *suitable* when  $\text{clos}\mathcal{F} = \text{Prim}(\mathbb{N}, \mathbb{N})$ .

LEMMA 3.3. *The basis  $\langle S, Sq, O, Hf, Rt, F + G, F - G, FG, F^\square \rangle$  is suitable.*

PROOF. It follows from Lemma 3.1 (also see Theorem 2 of [15]) and  $I \equiv S^\square$ . Due to the impossibility of introducing binary functions, we must incorporate operators such as  $F + G$  and  $F - G$ .  $\square$

Furthermore, Robinson proved that  $\text{Prim}$  can be obtained by adding projection functions, addition and the substitution operator to  $\text{Prim}(\mathbb{N}, \mathbb{N})$  (cf. §7 of [15]). We write this result as another lemma.

LEMMA 3.4. *Let  $\mathcal{F}$  be a suitable basis. Then,  $\text{Prim} = \text{clos}(\mathcal{F} + \langle I_k^n, +, \text{subst} \rangle)$ .*

Now, we will derive a list of suitable bases for the pure cases without parameters (see table 3, the format is the same as in table 2). Bases provided in §6 are simpler than Robinson's bases. In fact, the successor can be substituted by  $\bar{I}$ , and the addition operator can be substituted by a unary operator of the form  $f \mapsto f + I$ .

**§4. Mixed iteration without parameters.** In §4 of [7], Gladstone showed that  $\text{rec}_5$  is adequate if we include the addition function. Our aim is to verify that  $\text{rec}_7$  is adequate too, but we must incorporate a function that is not non-decreasing: cosignum. In order to do this, we need to follow the same steps as [7] but keeping in mind that we must use  $\text{rec}_7$ .

At the scope of this section, let  $\mathcal{F} = \langle S, I_k^n, O, +, \text{subst}, \mathcal{M}[F] \rangle$ .

LEMMA 4.1.  $P, N, D, Sq, Hf, Pw \in \text{clos}\mathcal{F}$ .

PROOF. In the first place,  $P \equiv \mathcal{M}[I_1^2]$ ,  $N \equiv \mathcal{M}[O \circ I_2^2]$  and  $D \equiv \mathcal{M}[S \circ S \circ I_2^2]$ . Furthermore, we have:

- Square:  $Sq(0) = 0, Sq(x + 1) = Sq(x) + 2x + 1$ .  
 $Sq \equiv \mathcal{M}[\text{subst}(+, S \circ I_2^2, D \circ I_1^2)]$ .
- Half:  $Hf(0) = 0, Hf(x + 1) = Hf(x) + N(x)$ .  
 $Hf \equiv \mathcal{M}[\text{subst}(+, I_2^2, N \circ I_1^2)]$ .
- Power of two: Let  $F$  be defined as follows:  $F(0) = 0, F(x + 1) = 2F(x) + 1$ .  
 Therefore,  $F(x) = 2^x - 1$  and  $Pw \equiv S \circ \mathcal{M}[S \circ D \circ I_2^2]$ .

<sup>3</sup>The notation  $x - y$  without dot or vertical bars, will always be used in an ambiguous sense, to stand for any function  $F(x, y)$  which is equal to  $x - y$  for  $x \geq y$ , regardless of its value when  $x < y$ . Any difference function, such as  $x \dot{-} y$  or  $|x - y|$ , can substitute  $x - y$ .

TABLE 3. Initial functions for characterizations of  $\text{Prim}(\mathbb{N}, \mathbb{N})$  using pure iteration.

$a$ variable	$a = 0$ (fixed)
$S,  F - G $ [7]	$S, E, F + G$ [15] $S, K, F + G$ [16] $S, L, F + G$ [16] $S, E, J(F, G)$ [16] $S, K, J(F, G)$ [16] $S, L, J(F, G)$ [16]
$S, F \dot{-} G$ [3, 11]	$S,  F - G $ §5 $S, F \dot{-} G$ §5 $\bar{1}, E, F + I$ §6 $\bar{1}, K, F + I$ §6 $\bar{1}, L, F + I$ §6

□

LEMMA 4.2. *The function  $\delta(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$  (namely Kronecker delta function) belongs to  $\text{clos}\mathcal{F}$ .*

PROOF. In Lemma 6 of §4 of [7], the following function  $f$  is defined using scheme  $\text{rec}_5$ :

$$f(0) = 2,$$

$$f(x+1) = N(z) + z + 2^{x+O(N(z))} + 2^{x+2O(N(z))},$$

where  $z = \left\lfloor \frac{f(x)}{2} \right\rfloor$ . We can *simulate* this function by transferring the index in one unit:

$$f'(0) = 0,$$

$$f'(x+1) = N(z') + z' + 2^{x+O(N(z'))-1} + 2^{x+2O(N(z'))-1} + O(O(x)),$$

where  $z' = \left\lfloor \frac{f'(x) - 1}{2} \right\rfloor$ . Thus,  $f(x) = f'(x+1) - 1$ .

Now, let  $g$  be defined as  $g(0) = 0$ ,  $g(x+1) = N\left\lfloor \frac{f(x-1)}{2} \right\rfloor$ .

According to Gladstone,<sup>4</sup>

$$g(x) = \begin{cases} 1 & \text{if } x \text{ is a power of two,} \\ 0 & \text{otherwise.} \end{cases}$$

Note that  $x = y$  iff  $2^x + 2^y$  is a power of two, so  $\delta(x, y) = g(2^x + 2^y)$ . □

LEMMA 4.3.  $Rt, - \in \text{clos}\mathcal{F}$ .

<sup>4</sup>In his paper,  $g(x)$  returns 0 when  $x$  is a power of two, and 1 if not.

PROOF. Integer square root is computed as follows:  $Rt(0) = 0$ ,  $Rt(x+1) = Rt(x) + \delta((Rt(x)+1)^2, x+1)$ . Symbolically,

$$Rt \equiv \mathcal{M}[\text{subst}(+, I_2^2, \text{subst}(\delta, Sq \circ S \circ I_2^2, S \circ I_1^2))].$$

Let  $H$  be defined as follows:

$$\begin{aligned} H(0) &= 0, \\ H(x+1) &= H(x) + 2N(\lfloor \sqrt{x} \rfloor) \div 1. \end{aligned}$$

Hence,  $H(x) = E(x)$  when  $\lfloor \sqrt{x} \rfloor$  is an odd number (cf. part (4) of §6 of [15]), so that

$$x - y = H((2x + 2y)^2 + 5x + 3y + 1)$$

whenever  $x \geq y$ . The formula above defines the function subtraction as a functional operator. Finally,  $- \equiv I_1^2 - I_2^2$ .  $\square$

THEOREM 4.4.  $\text{Prim} = \text{clos}\langle S, I_k^n, O, +, \text{subst}, \mathcal{M}[F] \rangle$ .

PROOF. It follows from Lemma 3.1 and Lemmata 4.1-4.3.  $\square$

We will prove two theorems which explain the reason we included cosignum function in Theorem 4.4.

THEOREM 4.5. Let  $\mathcal{F}' = \mathcal{F} - \langle O \rangle$  (the result of removing  $O$  from the basis  $\mathcal{F}$ ). Every function  $F$  of one argument of  $\text{clos}\mathcal{F}'$  is non-decreasing:

$$\forall_{x \in \mathbb{N}} F(x) \leq F(x+1).$$

PROOF. We will proceed by structural induction over functions defined using one argument. The fact is trivial for identity and successor function. If  $F$  and  $G$  are non-decreasing functions, its substitution (i.e.  $F \circ G$ ) and its addition (i.e.  $\text{subst}(+, F, G)$ ) are non-decreasing too.

Now, let  $F$  be defined as

$$F(0) = 0, F(x+1) = G(x, F(x)).$$

Clearly,  $G$  is a function written in terms of  $I_1^2, I_2^2$  and non-decreasing functions. So,  $G$  satisfy the following property:

$$\forall_{a,b,x,y \in \mathbb{N}} G(x, y) \leq G(x+a, y+b).$$

Suppose that  $F(x) \leq F(x+1)$ . Then,

$$G(x, F(x)) \leq G(x+1, F(x+1)).$$

Therefore,  $F(x+1) \leq F(x+2)$ .  $\square$

THEOREM 4.6.  $\text{Prim} = \text{clos}\langle S, I_k^n, \hat{F}, +, \text{subst}, \mathcal{M}[F] \rangle$ , where  $\hat{F}$  is not non-decreasing.

PROOF. If  $\hat{F}$  is not non-decreasing then exists a natural number  $a$  that verifies  $\hat{F}(a) > \hat{F}(a+1)$ . Let  $G$  be defined as  $G(x) = \hat{F}(x+a)$ , i.e.  $G \equiv \hat{F} \circ S^a$ . Thus,  $G(0) > G(1)$ . Let  $H$  be defined as  $H(x) = G(x) \div G(1)$ , i.e.  $H \equiv P^{G(1)} \circ G$ ,

where  $P \equiv \mathcal{M}[I_1^2]$ . Thus,  $H(0) > H(1) = 0$ .  
Next, let  $Sgn \equiv \mathcal{M}[\bar{1}]$ . It follows easily that

$$\begin{aligned} Sgn(H(Sgn(0))) &= Sgn(H(0)) = 1, \\ Sgn(H(Sgn(x+1))) &= Sgn(H(1)) = 0. \end{aligned}$$

Therefore,  $O \equiv Sgn \circ H \circ Sgn$ . And now, we can apply Theorem 4.4.  $\square$

REMARK 4.7. In this section, we fixed the value of  $a$  to zero. However, we could have fixed the value of  $a$  to another number.<sup>5</sup>  
We will show that scheme  $\text{rec}_7$  can be expressed using  $\text{rec}_5$  with  $a > 0$ . First, we define the functions below:

$$\begin{aligned} \hat{P}(0) &= a, \quad \hat{P}(x+1) = x, \quad \text{i.e. } \hat{P} \equiv \mathcal{M}_a[I_1^2], \\ \bar{a}(0) &= a, \quad \bar{a}(x+1) = \bar{a}(x), \quad \text{i.e. } \bar{a} \equiv \mathcal{M}_a[I_2^2], \\ \bar{0} &\equiv \hat{P}^a \circ \bar{a}, \\ \hat{O}(0) &= a, \quad \hat{O}(x+1) = 0, \quad \text{i.e. } \hat{O} \equiv \mathcal{M}_a[\bar{0}], \end{aligned}$$

where  $\mathcal{M}_a[F](0) = a$ ,  $\mathcal{M}_a[F](x+1) = F(x, \mathcal{M}_a[F](x))$ .  
Now, every function  $F$  which satisfies  $F(0) = 0$  and  $F(x+1) = H(x, F(x))$  will be written as follows:

$$\begin{aligned} G(0) &= a, \\ G(x+1) &= H(x, G(x) - a) + a. \end{aligned}$$

By a simple induction,  $F(x) = G(x) - a$ , and

$$\mathcal{M}[H] \equiv \hat{P}^a \circ \mathcal{M}_a[S^a \circ \text{subst}(H, I_1^2, \hat{P}^a \circ I_2^2)].$$

Note also that  $\hat{O}$  is not non-decreasing. So, applying Theorem 4.6 we prove that  $\text{Prim} = \text{clos}\langle S, I_k^n, +, \text{subst}, \mathcal{M}_a[F] \rangle$ .

**§5. Iteration and difference.** We will follow §5 of [7] (also see Lemma 1 of [11]), replacing  $\text{rec}_6$  by  $\text{rec}_8$ :  $\text{Prim}$  is generated using a difference function (may be  $|x - y|$  or  $x \dot{-} y$ ) as the unique initial function. However, we will propose an equivalent statement. Let  $\mathcal{F}$  be  $\langle S, |F - G|, FG, F^\square \rangle$  or  $\langle S, F \dot{-} G, FG, F^\square \rangle$ . Our intention is to prove that  $\mathcal{F}$  is suitable.

As much as possible, we will try to use  $F - G$  instead of  $|F - G|$  and  $F \dot{-} G$ , but taking care of not subtracting two functions that render the expression meaningless. In the first place,

$$\begin{aligned} I &\equiv S^\square, & D &\equiv (SS)^\square, \\ \bar{0} &\equiv S - S, & \bar{1} &\equiv S\bar{0}, \\ Pw &\equiv S(SD)^\square, & Sgn &\equiv \bar{1}^\square, \\ P &\equiv I - Sgn, & O &\equiv \bar{1} - Sgn. \end{aligned}$$

<sup>5</sup>This differs from Gladstone, because he used  $\text{rec}_5$  with several values of  $a$  (more precisely, with  $a \in \{0, 1, 2\}$ ). We show that it is sufficient to choose one value for  $a$ .

Next step is to construct the addition. The following sequence of functions

$$\begin{cases} f_0 & \equiv S, \\ f_{n+1} & \equiv f_n^{\square(f_n(1))} \end{cases}$$

is a kind of Ackermann's sequences (i.e. if  $f(x, n) = f_n(x)$  then  $f$  grows faster than any primitive recursive function; nevertheless,  $f_n$  is primitive recursive). Georgieva[3] discovered a method for constructing the addition between two functions, based on this sequence.

Let  $F, G \in \text{clos}\mathcal{F}$ . According to Lemma 6 of [3], there exists  $i \in \mathbb{N}$  such that  $F(x) \leq f_i(x)$  for every  $x$  (and there exists  $j \in \mathbb{N}$  such that  $G(x) \leq f_j(x)$ ). Let  $k$  be the maximum value between  $i$  and  $j$ . Hence,  $F(x) + G(x) \leq 2f_k(x)$ . Therefore (cf. Lemma 7 of [3]),

$$F + G \equiv Df_k - ((Df_k - F) - G).$$

Now, we will explain how to construct  $f_i$  given  $F$  by means of the following recursive definition:

$$\begin{cases} \mathcal{A} : \mathcal{F} \rightarrow \mathbb{N} \\ \mathcal{A}(S) & = 0 \\ \mathcal{A}(|F - G|) & = \max(\mathcal{A}(F), \mathcal{A}(G)) \\ \mathcal{A}(F \dot{-} G) & = \mathcal{A}(F) \\ \mathcal{A}(FG) & = \max(\mathcal{A}(F), \mathcal{A}(G)) + 2 \\ \mathcal{A}(F^{\square}) & = \mathcal{A}(F) + 1 \end{cases}$$

To express  $F + G$  using  $\text{rec}_8$  instead of  $\text{rec}_6$ , we need only to generate a sequence that grows faster than  $f_n$ .

LEMMA 5.1. *The following sequence of functions*

$$\begin{cases} B_0 & \equiv S, \\ B_{n+1} & \equiv (S^{f_n(1)} B_n)^{\square} \end{cases}$$

satisfies

$$\forall x, n \in \mathbb{N} \quad B_n(x+1) \geq f_n(x).$$

PROOF. First, we will try to rewrite  $f_n$  with iterations.

$$\begin{cases} f'_0(x) & = x, \\ f'_{n+1}(0) & = 0, \\ f'_{n+1}(x+1) & = g_n(f'_{n+1}(x)) \end{cases}$$

where  $g_n(x) = f_n(1)O(x) + f'_n(x)Sgn(x)$ . Hence,  $f'_n(x+1) = f_n(x)$  (by a simple induction on  $x$  and  $n$ ). Consider the sequence

$$\begin{cases} B_0(x) & = x+1, \\ B_{n+1}(0) & = 0, \\ B_{n+1}(x+1) & = h_n(B_{n+1}(x)) \end{cases}$$

where  $h_n(x) = f_n(1) + B_n(x)$ . Clearly,  $B_{n+1}(x) \geq f'_{n+1}(x)$  if  $B_n(x) \geq f'_n(x)$  (by comparing  $g_n$  and  $h_n$ ). We conclude that  $B_n(x+1) \geq f_n(x)$ .  $\square$

LEMMA 5.2. *If  $F, G \in \text{clos}\mathcal{F}$  then  $F + G \in \text{clos}\mathcal{F}$ .*

PROOF. Remember that there exists  $i, j \in \mathbb{N}$  such that  $F(x) \leq f_i(x)$  and  $G(x) \leq f_j(x)$ . By virtue of the previous lemma,  $F(x) \leq B_i(x+1)$  and  $G(x) \leq B_j(x+1)$ . Let  $k = \max(i, j)$ , so

$$F(x) + G(x) = 2B_k(x+1) - ((2B_k(x+1) - F(x)) - G(x)).$$

In other words,

$$F + G \equiv DB_{\max(\mathcal{A}(F), \mathcal{A}(G))}S - ((DB_{\max(\mathcal{A}(F), \mathcal{A}(G))}S - F) - G).$$

□

Now, we only need to prove that  $Sq, Rt, Hf \in \text{clos}\mathcal{F}$ . We will do this in the next lemmata.

LEMMA 5.3. *The following families of functions belong to  $\text{clos}\mathcal{F}$ :*

- *Characteristic of  $n$ :*

$$O_n(x) = \begin{cases} 1 & \text{if } x = n, \\ 0 & \text{otherwise.} \end{cases}$$

- *Multiplication functions:*

$$M_n(x) = nx.$$

- *Cycle functions:*

$$C_{n+2}(x) = \begin{cases} x+1 & \text{if } x \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

- *Moduli functions:*

$$\text{Mod}_{n+2}(x) = x \bmod (n+2).$$

- *Division functions:*

$$\text{Div}_{n+2}(x) = \lfloor x/(n+2) \rfloor.$$

PROOF. We will show the formulas of each one in the same order. All of them can be proved easily by induction on  $n$ .<sup>6</sup>

Characteristic of  $n$ :

$$O_0 \equiv O, \quad O_1 \equiv O(O + P), \quad O_{n+2} \equiv O_{n+1}P.$$

Multiplication functions:

$$M_n \equiv (S^n)^\square.$$

Cycle functions:

$$C_2 \equiv O, \quad C_{n+3} \equiv C_{n+2} + M_{n+2}O_{n+1}.$$

Moduli functions:

$$\text{Mod}_{n+2} \equiv C_{n+2}^\square.$$

Division functions:

$$\text{Div}_{n+2} \equiv (S + O \text{ Mod}_{n+3} S S)^\square - I.$$

---

<sup>6</sup>Some proofs can be consulted in §5 of [7].

□

DEFINITION 5.4. The conditional operator  $F \rightarrow G$  is defined as follows

$$(F \rightarrow G)(x) = \begin{cases} G(x) & \text{if } F(x) = 0, \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 5.5. If  $F, G \in \text{clos}\mathcal{F}$  then  $F \rightarrow G \in \text{clos}\mathcal{F}$ .

PROOF. Let  $\alpha(x) = 2^{x+1+Mod_2(x)} - 2^{x+1}$ . If  $x$  is even,  $\alpha(x) = 0$ . And if  $x$  is odd,  $\alpha(x) = 2^{x+1}$ . In formal terms,

$$\alpha \equiv Pw(S + Mod_2) - Pw S.$$

Now, we will divide the proof in two cases depending on the subtraction operator which we are working:

- Distance: Let  $\beta \equiv (|\alpha - (I + Pw)| + I) - Pw$ . If  $x$  is even,  $\beta(x) = 2x$ . And if  $x$  is odd,  $\beta(x) = 0$ .
- Arithmetic difference: Let  $\beta \equiv D \dot{-} \alpha$ . If  $x$  is even,  $\beta(x) = 2x$ . And if  $x$  is odd,  $\beta(x) = 0$ .

Finally, we will observe the behavior of  $w = \beta(2z + Sgn(y))$ . When  $y$  is zero,  $w = 4z$ . And when  $y$  is positive,  $w = 0$ . So,  $w = 4 \cdot (F \rightarrow G)(x)$  if  $y = F(x)$  and  $z = G(x)$ , and

$$(F \rightarrow G) \equiv Div_4 \beta(DG + Sgn F).$$

□

LEMMA 5.6.  $Q \in \text{clos}\mathcal{F}$ .

PROOF. We follow Lemma 2.3 of [4]. Let  $W$  be defined as follows:

$$W(x) = \begin{cases} 2 & \text{if } x = 0, \\ \lfloor 3x/2 \rfloor & \text{if } x \neq 0, x \bmod 10 = 0, \\ \lfloor 2x/5 \rfloor & \text{if } x \neq 0, x \bmod 2 \neq 0, x \bmod 5 = 0, \\ \lfloor 2x/3 \rfloor & \text{if } x \neq 0, x \bmod 3 = 0, x \bmod 5 \neq 0, \\ \lfloor 15x/2 \rfloor & \text{if } x \neq 0, x \bmod 3 \neq 0, x \bmod 5 \neq 0. \end{cases}$$

For all  $x > 0$ ,  $W^\square(x) \bmod 3 \neq 0$  if and only if  $x$  is a square. To write  $W$  we must use the operator defined above (see 5.4):

$$\begin{aligned} W_1(x) &\equiv DO, \\ W_2(x) &\equiv (O + Mod_{10} \rightarrow Div_2 M_3), \\ W_3(x) &\equiv (O + O Mod_2 + Mod_5 \rightarrow Div_5 D), \\ W_4(x) &\equiv (O + Mod_3 + O Mod_5 \rightarrow Div_3 D), \\ W_5(x) &\equiv (O + O Mod_3 + O Mod_5 \rightarrow Div_2 M_{15}). \end{aligned}$$

Each  $W_i$  represents one case (one line of the definition of  $W$ ). The conditions are mutually exclusive, so  $W(x) = W_i(x)$  for some  $i$  between 1 and 5.

Thus,  $W \equiv W_1 + W_2 + W_3 + W_4 + W_5$  and

$$Q \equiv Sgn Mod_3 W^\square + O.$$

□

LEMMA 5.7.  $Sq, Rt, Hf \in \text{clos}\mathcal{F}$ .

PROOF. We follow §5 of [15]. Suppose that  $R(x) = x + 2\lfloor\sqrt{x}\rfloor$ . Then,  $R \equiv (S + DQSSSS)^\square$  and

$$Sq \equiv (SR)^\square, Rt \equiv \text{Div}_2(R - I), Hf \equiv \text{Div}_2.$$

□

THEOREM 5.8. The bases  $\langle S, |F - G|, FG, F^\square \rangle$  and  $\langle S, F \dot{-} G, FG, F^\square \rangle$  are both suitable.

PROOF. It follows from Lemma 3.3 and Lemmata 5.1-5.7. □

We conclude this section with the following theorem, which is a consequence of the previous results.

THEOREM 5.9.  $\text{Prim} = \text{clos}\langle S, I_k^n, \ominus, \text{subst}, F^\square \rangle$ , where  $\ominus(x, y)$  can be  $x \dot{-} y$  or  $|x - y|$ .

PROOF. Let  $\mathcal{F} = \langle S, F \ominus G, FG, F^\square \rangle$ . In virtue of Theorem 5.8,  $\mathcal{F}$  is suitable, and therefore the operator addition belongs to  $\text{clos}\mathcal{F}$ . Note that  $+ \equiv I_1^2 + I_2^2$ ,  $F \ominus G \equiv \text{subst}(\ominus, F, G)$  and  $FG \equiv \text{subst}(F, G)$ , so  $\text{clos}\langle S, I_k^n, \ominus, \text{subst}, F^\square \rangle \supseteq \text{clos}\langle S, I_k^n, F \ominus G, \text{subst}, F^\square \rangle \supseteq \text{clos}(\mathcal{F} + \langle I_k^n, \text{subst} \rangle) \supseteq \text{clos}(\mathcal{F} + \langle I_k^n, +, \text{subst} \rangle) = \text{Prim}$  (use Lemma 3.4). Therefore,  $\text{Prim} = \text{clos}\langle S, I_k^n, \ominus, \text{subst}, F^\square \rangle$ . □

REMARK 5.10. In this section, we used scheme  $\text{rec}_6$  with  $a = 0$ . However, we could have fixed the value of  $a$  to another number as we did in 4.7. In fact,

$$F^\square \equiv \hat{P}^a(S^a F \hat{P}^a)^{\square(a)}$$

where  $\hat{P}$  may be  $|S - \bar{2}|$  or  $S \dot{-} \bar{2}$ , and  $\bar{2} \equiv SS(S - S)$ .

REMARK 5.11. A further line of inquiry is to analyze if it is possible to rewrite the operator  $F \rightarrow G$  using the difference  $F - G$  instead of  $F \dot{-} G$  and  $|F - G|$ . For example, if we prove that  $Sq \in \text{clos}\mathcal{F}$ , then we can write

$$(F \rightarrow G) \equiv \text{Div}_2(Sq(OF + G) - (Sq \ O \ F) - (Sq \ G))$$

and replace  $F \dot{-} G$  and  $|F - G|$  by  $F - G$  in table 3.

**§6. Iteration and unary operator.** Robinson[15] proved that  $\langle S, E, F + G, FG, F^\square \rangle$  is a suitable basis. In this section, we will simplify this result, showing that  $\langle \bar{1}, E, F^+, FG, F^\square \rangle$  is suitable too. Let  $\mathcal{F} = \langle \bar{1}, E, F^+, FG, F^\square \rangle$ , where the operator  $F^+$  is defined as

$$F^+(x) = F(x) + x,$$

and its precedence is the same as in  $F^\square$ . The aim of this section is to show that it is not necessary to have a binary operator such as the addition (except for, of course, the substitution). In fact, the addition can be replaced by a unary operator.

In the first place, the following functions belong to  $\text{clos}\mathcal{F}$ :

$$\begin{aligned} S &\equiv \bar{1}^+, & Sgn &\equiv \bar{1}^\square, \\ \bar{0} &\equiv E\bar{1}, & Mod_3 &\equiv (ESS)^\square, \\ D &\equiv (\bar{0}^+)^+, & M_3 &\equiv D^+, \\ O &\equiv E D S Sgn, & Q &\equiv OE, \\ R &\equiv ((DQSSS)^+ S)^\square, & Sq &\equiv (SR)^\square. \end{aligned}$$

DEFINITION 6.1. In this section, the following operators on one argument functions will be used:

$$\begin{aligned} F^-(x) &= F(x) - x, & (F \otimes G)(x) &= F(x)G(x), \\ (F \oplus G)(x) &= F(x) + G^2(x), & (F \ominus G)(x) &= F(x) - G^2(x), \end{aligned}$$

whenever  $F(x) \geq x$  and  $G(x) \geq x$  for every  $x \in \mathbb{N}$ . In the definition of  $\ominus$ ,  $F(x) \geq G^2(x)$  must hold too.<sup>7</sup>

The precedences of  $\oplus$  and  $\ominus$  are the same as in the addition, while the precedence of  $F^-$  is the same as in  $F^+$ . The precedence of  $\otimes$  is between addition and substitution (like products in arithmetical expressions).

LEMMA 6.2. *Let  $F \in \text{clos}\mathcal{F}$ . If  $F(x) \geq x$  for every  $x$ , then  $F^- \in \text{clos}\mathcal{F}$ .*

PROOF. Robinson has proved that, if  $\alpha \geq \beta$ , then

$$\alpha - \beta = E((\alpha + \beta)^2 + 3\alpha + \beta + 1).$$

If we take  $\alpha = F(x)$  and  $\beta = x$ , the formula becomes

$$F(x) - x = E((F(x) + x)^2 + 3F(x) + x + 1).$$

The following diagram shows how to compute  $F^-$ :

$$x \xrightarrow{(Sq F^+)^+} F^+(x)^2 + x \xrightarrow{(M_3 FE)^+} F^+(x)^2 + 3F(x) + x \xrightarrow{ES} F^-(x)$$

since  $E((F(x) + x)^2 + x) = x$ . Therefore,

$$F^- \equiv ES(M_3 FE)^+(Sq F^+)^+.$$

□

LEMMA 6.3.  $Hf, Rt \in \text{clos}\mathcal{F}$ .

PROOF. Cf. §5 of [15]:

$$\begin{aligned} Hf &\equiv ((S Mod_3^+)^{\square})^-, \\ Rt &\equiv Hf R^-. \end{aligned}$$

□

LEMMA 6.4. *Let  $F, G \in \text{clos}\mathcal{F}$ . If  $F(x) \geq x$  and  $G(x) \geq x$ , then  $F \oplus G \in \text{clos}\mathcal{F}$ . If  $F(x) \geq G^2(x)$  too, then  $F \ominus G \in \text{clos}\mathcal{F}$ .*

---

<sup>7</sup> $G^2(x)$  must be read as  $G(x)G(x)$ , and not  $G(G(x))$ .

PROOF. We will use the fact that if  $G(x) \geq x$  then  $E(G^2(x) + x) = x$ . Hence,

$$\begin{aligned} x &\xrightarrow{(Sq\ G)^+} G^2(x) + x \xrightarrow{(F^- E)^+} F(x) + G^2(x), \\ x &\xrightarrow{(Sq\ G)^+} G^2(x) + x \xrightarrow{(F^+ E)^-} F(x) - G^2(x). \end{aligned}$$

Therefore,

$$\begin{aligned} F \oplus G &\equiv (F^- E)^+ (Sq\ G)^+, \\ F \ominus G &\equiv (F^+ E)^- (Sq\ G)^+. \end{aligned}$$

□

LEMMA 6.5. *Let  $F, G \in \text{clos}\mathcal{F}$ . If  $F(x) \geq x$  and  $G(x) \geq x$ , then  $F \otimes G \in \text{clos}\mathcal{F}$ .*

PROOF. Note that

$$\alpha^2 \beta^2 = \frac{[(\alpha^2 + 1)^2 + \beta^2 - \alpha^4 - 1]^2 - \beta^4}{4} - \alpha^4.$$

If we take  $\alpha = F(x)$  and  $\beta = G(x)$ , we can reach  $\alpha\beta$  with

$$F \otimes G \equiv Rt((Hf\ Hf(Sq\ \hat{P}((Sq\ S\ Sq\ F \oplus G) \ominus Sq\ F) \ominus Sq\ G)) \ominus Sq\ F),$$

where  $\hat{P} \equiv ((Sq^+)^+)^+ \ominus S$ , i.e.  $\hat{P}(x+1) = x$ . □

LEMMA 6.6. *Let  $F, G \in \text{clos}\mathcal{F}$ . So,  $F + G \in \text{clos}\mathcal{F}$ . If  $F(x) \geq G(x)$  for every  $x$ , then  $F - G \in \text{clos}\mathcal{F}$  too.<sup>8</sup>*

PROOF. First, we will compute the sum  $F^+(x) + G^+(x)$  by using the following properties:  $(\alpha + \beta)^2 = 2\alpha\beta + \alpha^2 + \beta^2$ ,  $F^+(x) \geq x$ ,  $G^+(x) \geq x$ ,

$$F^+(x) + G^+(x) = \sqrt{2F^+(x)G^+(x) \oplus F^+(x) \oplus G^+(x)}.$$

Now, see that  $F(x) + G(x) = F^+(x) + G^+(x) - 2x$ . Therefore,

$$F + G \equiv ((Rt((D(F^+ \otimes G^+) \oplus F^+) \oplus G^+))^-)^-.$$

To compute  $F - G$ , we can use the same trick as in Lemma 6.2. Finally,

$$F - G \equiv ES(Sq(F + G) + M_3F + G).$$

□

THEOREM 6.7.  $\langle \bar{1}, E, F^+, FG, F^\square \rangle$  is a suitable basis.

PROOF. It follows from Lemma 3.3 and Lemmata 6.2-6.6. □

Now, we will prove that  $\mathcal{F}$  is suitable if we use  $K$  (or  $L$ ) instead of  $E$ . Let  $\mathcal{F}' = \langle \bar{1}, K, F^+, FG, F^\square \rangle$ . Following p. 664 of [16],

$$\begin{aligned} S &\equiv \bar{1}^+, & Sgn &\equiv \bar{1}^\square, \\ \bar{0} &\equiv K\bar{1}, & D &\equiv (\bar{0}^+)^+, \\ Y &\equiv ((Sgn\ K)^+ S)^\square, & Z &\equiv (SSK^+)^\square, \end{aligned}$$

<sup>8</sup>The formula of  $F - G$  works well even when  $F(x) < G(x)$  for some values of  $x$ . We can use them regardless of the values which render the formula meaningless.

where  $Y(x) = 2x - \lfloor \sqrt{x} \rfloor$  and  $Z(x) = x(x+3)/2$ .  
Let  $F \in \text{clos } \mathcal{F}'$ . Using the fact that

$$K((\alpha + \beta)(\alpha + \beta + 3)/2 + 2\alpha + 3) = \alpha - \beta, \text{ if } \alpha \geq \beta,$$

we see that

$$I - F \equiv KSSS((ZF^+)^+)^+, \text{ if } x \geq F(x),$$

and that

$$D - F \equiv KSSS((((Z(F^+)^+)^+)^+)^+), \text{ if } 2x \geq F(x).$$

Thus,

$$Rt \equiv D - Y, Sq \equiv (((\bar{1}^+)^+ Rt)^+)^{\square}, E \equiv I - Sq \ Rt.$$

The application of Theorem 6.7 makes  $\mathcal{F}'$  suitable. If  $\mathcal{F}'' = \langle \bar{1}, L, F^+, FG, F^{\square} \rangle$ , we may define  $K$  by the formula  $K \equiv L\bar{1}^+((\bar{1}^+)^+ L)^+$ , and  $\mathcal{F}''$  results to be suitable too.

**THEOREM 6.8.**  $\langle \bar{1}, K, F^+, FG, F^{\square} \rangle$  and  $\langle \bar{1}, L, F^+, FG, F^{\square} \rangle$  are suitable bases.

**REMARK 6.9.** In this section, we used scheme  $\text{rec}_6$  with  $a = 0$ . However, we could have fixed the value of  $a$  to another number as we did in 5.10. We only need a suitable predecessor  $\hat{P}$ , i.e.  $\hat{P}(x+1) = x$ .

For  $a = 1$ , we can proceed as follows:

(I) Let  $\mathcal{F} = \langle E, F^+, FG, F^{\square(1)} \rangle$ . Then,

$$\begin{aligned} O &\equiv E^{\square(1)}, & \bar{0} &\equiv EO, \\ \bar{1} &\equiv O\bar{0}, & S &\equiv \bar{1}^+, \\ D &\equiv (\bar{0}^+)^+, & Q &\equiv OE, \\ G &\equiv (((DQSS)^+ S)^{\square(1)})^{\square(1)}, & \hat{P} &\equiv ESS((G^+)^+)^+, \end{aligned}$$

where  $G(x) = (x+1)^2$ . Now, iteration can be defined with

$$F^{\square} \equiv \hat{P}(SF\hat{P})^{\square(1)}.$$

In virtue of Theorem 6.7,  $\mathcal{F}$  is suitable.

(II) Let  $\mathcal{F} = \langle K, F^+, FG, F^{\square(1)} \rangle$ . Then,

$$\begin{aligned} O &\equiv K^{\square(1)}, & \bar{0} &\equiv KO, \\ \bar{1} &\equiv O\bar{0}, & S &\equiv \bar{1}^+, \\ H &\equiv S(SK^+ S)^{\square(1)}, & \hat{P} &\equiv KSSS(H^+)^+, \end{aligned}$$

where  $H(x) = (x+1)(x+4)/2$ . Now, iteration can be defined as in (I). In virtue of Theorem 6.8,  $\mathcal{F}$  is suitable.

(III) Let  $\mathcal{F} = \langle L, F^+, FG, F^{\square(1)} \rangle$ . Then,

$$\bar{1} \equiv L^{\square(1)}, \quad K \equiv L\bar{1}^+((\bar{1}^+)^+ L)^+,$$

and by using (II) we prove that  $\mathcal{F}$  is suitable.

For  $a > 1$  it is not known if  $\langle X, F^+, FG, F^{\square(a)} \rangle$ , for  $X \in \{E, K, L\}$ , is suitable. This question will be studied in the future.

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FACULTAD DE CIENCIAS EXACTAS, INGENIERIA Y AGRIMENSURA  
 UNIVERSIDAD NACIONAL DE ROSARIO  
 ROSARIO, SANTA FE, ARGENTINA  
 URL: <http://www.fceia.unr.edu.ar/~daniel>  
 E-mail: [daniel@fceia.unr.edu.ar](mailto:daniel@fceia.unr.edu.ar)