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THE LACZKOVICH–KOMJÁTH PROPERTY FOR COANALYTIC EQUIVALENCE RELATIONS

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Abstract. Let E be a coanalytic equivalence relation on a Polish space X and $(A_n)_{n\in\omega}$ a sequence of analytic subsets of X. We prove that if $\limsup_{n\in K} A_n$ meets uncountably many E-equivalence classes for every $K \in [\omega]^\omega$, then there exists a $K \in [\omega]^\omega$ such that $\bigcap_{n\in K} A_n$ contains a perfect set of pairwise E-inequivalent elements.

§1. Introduction. Let $(A_n)_{n\in\omega}$ be a sequence of sets and $K \in [\omega]^\omega$ an infinite subset of ω. The limit superior $\limsup_{n\in K} A_n$ is the set of all elements which belong to A_n for infinitely many $n \in K$. Laczkovich [6] showed that for every sequence $(A_n)_{n\in\omega}$ of Borel sets in a Polish space, if $\limsup_{n\in K} A_n$ is uncountable for every $K \in [\omega]^\omega$, then there exists a $K \in [\omega]^\omega$ such that $\bigcap_{n\in K} A_n$ is uncountable. Komjáth [5] generalized this result to the case where the sets $(A_n)_{n\in\omega}$ are analytic. Note that by the perfect set property of analytic sets, if $\bigcap_{n\in K} A_n$ is uncountable, then it contains a perfect set.

Balcerzak and Głąb [1] extended these results to F_σ equivalence relations in the following way.

DEFINITION. An equivalence relation E on a Polish space X has the *Laczkovich-Komjáth property* if for every sequence $(A_n)_{n\in\omega}$ of analytic subsets of X such that $\limsup_{n\in K} A_n$ meets uncountably many E-equivalence classes for every $K \in [\omega]^\omega$, there exists a $K \in [\omega]^\omega$ such that $\bigcap_{n\in K} A_n$ contains a perfect set of pairwise E-inequivalent elements.

In this terminology, Komjáth has shown that the identity relation $=$ has the Laczkovich–Komjáth property. Balcerzak and Głąb [1] proved that every F_σ equivalence relation has the Laczkovich–Komjáth property. In this paper, we generalize this to coanalytic equivalence relations.

THEOREM 1. *Every coanalytic equivalence relation on a Polish space has the Laczkovich–Komjáth property.*

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A fundamental result on coanalytic equivalence relations is Silver's theorem: a coanalytic equivalence relation either has only countably many equivalence classes, or else there exists a perfect set of pairwise inequivalent elements. Silver's original proof [9] used forcing. Harrington (unpublished) later gave a simpler (forcing) proof using effective descriptive set theory, which nowadays is usually cast in terms of the Gandy–Harrington topology. We will use similar methods and assume familiarity with effective descriptive set theory throughout the paper.

An introduction to effective descriptive set theory is given in [7], where the reader can also find the topological version of Harrington's proof. The review in [4] provides details on the Gandy–Harrington topology and strong Choquet games. Instead of strong Choquet games, we will make use of the set of low elements, which is a Polish space in the Gandy–Harrington topology. We will summarize the technical facts we use later on. Further details can be found in [2], which also provides another source on effective descriptive set theory.

This paper is organized as follows. In Section 2 we review a well-known coding mechanism for Π^1_1 and Δ^1_1 sets, mainly to fix notation and establish the uniformity of a diagonal intersection operator. In Section 3, we provide details on canonical cofinal sequences as developed in [3]. We use these sequences in Section 4 to prove our main technical result. Finally, we derive our main theorem in Section 5, where we also derive a parametric version of the theorem, as was done by Balcerzak and Głąb [1].

§2. Coding Π^1_1 and Δ^1_1 sets. In this section we review a well-known coding mechanism for Π^1_1 and Δ^1_1 sets, mainly to fix notation. A good introduction can be found in [4, Section 3.2], where the notion of uniformity is also discussed. We will need the uniformity of a diagonal intersection operation. Since this operation is not canonical, we provide a little more of the details.

A *product space* is any $X = X_0 \times \cdots \times X_n$ (with the product topology), where each factor is either ω or ω^ω. For every product space X there is a $U^X \subseteq \omega \times X$ such that $U^X \in \Pi^1_1$ and for any $A \subseteq X$, $A \in \Pi^1_1$ iff $\exists n\,(A = U^X_n)$. Such a set U^X is called a *universal* Π^1_1 *set*. A Π^1_1 *code* for $A \subseteq X$ is any $n \in \omega$ such that $A = U^X_n$.

There exists a collection $\{U^X\}$ of universal Π^1_1 sets with the following additional property: for any $m \in \omega$ and any product space X there is a recursive function $S^{m,X} : \omega^{m+1} \to \omega$ such that

$$(e, k_1, \ldots, k_m, x) \in U^{\omega^m \times X} \Leftrightarrow (S^{m,X}(e, k_1, \ldots, k_m), x) \in U^X.$$

Such a collection is called a *good universal system*. For the rest of this paper, fix a good universal system $\{U^X\}$ for Π^1_1. This good universal system can be used to code Δ^1_1 subsets, which we now describe. This coding is always relative to a particular product space X. When there is no danger of confusion, we will drop the superscript in U^X. For every $k \in \omega$, fix a recursive bijection $(n_0, \ldots, n_{k-1}) \mapsto \langle n_0, \ldots, n_{k-1}\rangle$ between ω^k and ω, and denote the recursive inverse map by $n \mapsto ((n)_0, \ldots, (n)_{k-1})$. Define

$$(\langle m, n\rangle, x) \in U_0 \Leftrightarrow (m, x) \in U,$$
$$(\langle m, n\rangle, x) \in U_1 \Leftrightarrow (n, x) \in U.$$

Then $U_0, U_1 \in \Pi^1_1$. By the reduction property for Π^1_1 sets, there are Π^1_1 sets $U_0^*, U_1^* \subseteq \omega \times X$ such that $U_0^* \cup U_1^* = U_0 \cup U_1$ and $U_0^* \cap U_1^* = \emptyset$. Let $P = U_0^*$ and $S = (\omega \times X) \setminus U_1^*$. Define

$$\langle m, n\rangle \in C \Leftrightarrow \forall x \in X\, ((\langle m, n\rangle, x) \in U_0^* \vee (\langle m, n\rangle, x) \in U_1^*).$$

Then $C \in \Pi^1_1$. For all $n \in C$,

$$P_n = S_n := D_n.$$

A Δ^1_1 *code* for $A \subseteq X$ is any $n \in C$ such that $A = D_n$. In that case, $(n)_0$ is a Π^1_1 code for A and $(n)_1$ is a Π^1_1 code for $X \setminus A$. Conversely, if $m, n \in \omega$ are Π^1_1 codes for A and $X \setminus A$, respectively, then $\langle m, n\rangle$ is a Δ^1_1 code for A. It is important that the set C of Δ^1_1 codes is Π^1_1 and that operations hold effectively in the codes, in the following way.

EXAMPLE. Given Δ^1_1 codes $m, n \in C$ for $A, B \subseteq X$, we can effectively compute a Δ^1_1 code for $A \setminus B$. To see this, define

$$(m, n, x) \in Z_0 \Leftrightarrow x \in D_m \wedge x \notin D_n,$$
$$(m, n, x) \in Z_1 \Leftrightarrow x \notin D_m \vee x \in D_n.$$

Clearly, $Z_0, Z_1 \in \Pi^1_1$. Let e_0, e_1 be their respective Π^1_1 codes. Then for $i = 0, 1$,

$$(m, n, x) \in Z_i \Leftrightarrow (e_i, m, n, x) \in U^{\omega^2 \times X} \Leftrightarrow (S^{2,X}(e_i, m, n), x) \in U^X.$$

Also, $Z_0 = (\omega^2 \times X) \setminus Z_1$. Thus,

$$\left\langle S^{2,X}(e_0, m, n), S^{2,X}(e_1, m, n)\right\rangle$$

is a Δ^1_1 code for $A \setminus B$.

Similar uniformities hold for all basic set-theoretic operations. We will need the uniformity of a diagonal intersection operator, which we define next. Recall that when $H, K \in [\omega]^\omega$, $H \subseteq^* K$ denotes that H is almost contained in K, i.e., $K \setminus H$ is finite.

DEFINITION. For a (finite or infinite) sequence (K_n) of infinite subsets of ω with $K_n \subseteq^* K_m$ for $n > m$, define $\triangle K_n$ by $m \in \triangle K_n$ iff there exists $m_0 < m_1 < \cdots < m_k = m$ such that m_0 is the least element of K_0, m_1 is the least element of $K_0 \cap K_1$ such that $m_1 > m_0, \ldots, m_k$ is the least element of $K_0 \cap \cdots \cap K_k$ such that $m_k > m_{k-1}$.

Note that $\triangle K_n \subseteq^* K_m$ for all m. To obtain the desired uniformity for this diagonal intersection operation, we need to assume that the sequence of Δ^1_1 codes for (K_n) is effective. One way to formalize this is to let $n \in C^*$ iff

1. $n \in C^\omega$,
2. D_n^ω is infinite,
3. $\forall m\, (m \in D_n^\omega \Rightarrow (m)_1 \in C^\omega)$, and
4. $\forall i\, \exists! m\, (m \in D_n^\omega \wedge (m)_0 = i)$.

Informally, $n \in C^*$ iff n is a Δ^1_1 code for an infinite subset of ω of the form $\{\langle i, n_i\rangle : i \in \omega, n_i \in C\}$. Clearly, $C^* \in \Pi^1_1$.

LEMMA 2. *There is a function* Diag: $\omega \to \omega$ *which is* Δ^1_1 *on* C^* *such that whenever* $n \in C^*$ *is a code for an infinite* Δ^1_1 *subset* $\{\langle i, n_i\rangle : i \in \omega, n_i \in C\}$ *of* ω, Diag(n) *is a* Δ^1_1 *code for* $\triangle D^\omega_{n_i}$.

PROOF. It suffices to find Π^1_1 codes e_0 and e_1 for $\triangle D^\omega_{n_i}$ and $\omega \setminus \triangle D^\omega_{n_i}$, respectively, because $\langle e_0, e_1\rangle$ will then be a Δ^1_1 code for $\triangle D^\omega_{n_i}$. We need the following three facts:

1. There is a recursive function $u\colon \omega \to \omega$ such that whenever $n = \langle n_0, \dots, n_k\rangle$ is a finite sequence of Δ^1_1 codes, $u(n)$ is a Δ^1_1 code for $D^\omega_{n_0} \cap \cdots \cap D^\omega_{n_k}$.
2. There is a Δ^1_1 on the codes function $i\colon \omega \times \omega \to \omega$ such that whenever $n \in C^*$, $i(n, j)$ is the (unique) $m \in \omega$ such that $\langle j, m\rangle \in D^\omega_n$.
3. There is a Δ^1_1 on the codes function $\mu\colon \omega \times \omega \to \omega$ such that whenever n is a Δ^1_1 code for an infinite subset of ω, $\mu(n, j)$ is the least element of D^ω_n greater than or equal to j.

Now define

$$\begin{aligned}(n, m) \in Z_0 \Leftrightarrow n \in C^* \wedge \exists\, \langle m_0, \dots, m_k\rangle\ (&m_0 < \cdots < m_k \wedge m_k = m \wedge\\ &m_0 = \mu(u(\langle i(n, 0)\rangle), 0) \wedge m_1 = \mu(u(\langle i(n, 0), i(n, 1)\rangle), m_0 + 1) \wedge\\ &\cdots \wedge m_k = \mu(u(\langle i(n, 0), \dots, i(n, k)\rangle), m_{k-1} + 1)).\end{aligned}$$

Then $Z_0 \in \Pi^1_1$. Pick a Π^1_1 code e_0 for Z_0. Similarly, we can write down a Π^1_1 definition for $Z_1 = C^* \setminus Z_0$ and pick a Π^1_1 code e_1. The rest of the argument is as in the example. ⊣

Now that we have established the uniformity of this diagonal intersection operator, we will use it implicitly. Finally, for codes $h, k \in C^\omega$, we write $h \subseteq^* k$ iff the set coded by h is almost contained in the set coded by k. Writing out the definitions, we see that $h \subseteq^* k$ is Δ^1_1 on the set C^ω of codes.

§3. Canonical cofinal sequences. For $w \in 2^\omega$, define a binary relation $<_w$ on a subset of ω by

$$m <_w n \Leftrightarrow w(\langle m, n\rangle) = 1.$$

The domain of $<_w$ is the set

$$\mathrm{dom}(<_w) = \{n \in \omega\colon \exists m \in \omega\ (m <_w n \text{ or } n <_w m)\}.$$

Let LO denote the set of all $w \in 2^\omega$ such that $<_w$ is a linear order, and let LO^* denote the set of all $w \in \mathsf{LO}$ such that $<_w$ has a least element and every $n \in \mathrm{dom}(<_w)$ has an immediate successor $n^+_{<_w}$. For $w \in \mathsf{LO}$, let $|<_w|$ denote the order type of $<_w$. The next lemma shows that in a uniform way, we can effectively obtain a canonical cofinal sequence in $<_w$ given $w \in \mathsf{LO}^*$.

LEMMA 3 (Gao–Jackson–Laczkovich–Mauldin [3]). *There is a Δ^1_1 function*

$$\mathrm{Cof}\colon \{(w, n, j) \in \mathsf{LO}^* \times \omega^2\colon n \in \mathrm{dom}(<_w)\} \to \omega$$

such that

1. *if $w \in \mathsf{LO}^*$, $n \in \mathrm{dom}(<_w)$ and $j \in \omega$, then $\mathrm{Cof}(w, n, j) \in \mathrm{dom}(<_w)$ and $\mathrm{Cof}(w, n, j) <_w n$, unless n is the $<_w$-least element*;
2. *if $w \in \mathsf{LO}^*$ and $n \in \mathrm{dom}(<_w)$ has an immediate predecessor in $<_w$, then $\mathrm{Cof}(w, n, j)^+_w = n$ for all $j \in \omega$*;
3. *if $w \in \mathsf{LO}^*$, $n \in \mathrm{dom}(<_w)$ is not $<_w$-least and n does not have an immediate predecessor in $<_w$, then*

(a) *if* $j < j'$, *then* $\mathrm{Cof}(w, n, j) <_w \mathrm{Cof}(w, n, j')$, *and*
(b) *for any* $q \in \mathrm{dom}(<_w)$ *with* $q <_w n$ *there is a* $j \in \omega$ *such that* $q <_w \mathrm{Cof}(w, n, j)$. ⊣

We also need a variation of this lemma for Π^1_1 norms, whose proof uses the same ideas. Recall that a Π^1_1-norm on a pointset $P \in \Pi^1_1$ is a function φ from P into the ordinals On such that there exist binary relations $<^*_\varphi$ and $\leq^*_\varphi$ in Π^1_1 with the following properties:

$$x \leq^*_\varphi y \Leftrightarrow P(x) \wedge (\neg P(y) \vee \varphi(x) \leq \varphi(y)),$$
$$x <^*_\varphi y \Leftrightarrow P(x) \wedge (\neg P(y) \vee \varphi(x) < \varphi(y)).$$

Recall that WO denotes the set of all $w \in \mathsf{LO}$ such that $<_w$ is a well-order. Every Π^1_1 set $P \subseteq \omega$ admits a Π^1_1-norm $\varphi\colon P \to \omega_1^{\mathrm{CK}}$, where

$$\omega_1^{\mathrm{CK}} = \sup\{|<_w| : w \in \mathsf{WO} \text{ is recursive}\},$$

see for example [8, Section 4B].

LEMMA 4. *Let* φ *be a* Π^1_1*-norm on a* Π^1_1 *set* $P \subseteq \omega$. *There is a* Π^1_1 *function* $\mathrm{Cof}\colon \omega \to \omega$ *such that*

1. *for all* $j \in \omega$, $\mathrm{Cof}(j) \in P$;
2. *if* $j < j'$, *then* $\mathrm{Cof}(j) <^*_\varphi \mathrm{Cof}(j')$ *unless* $\mathrm{Cof}(j)$ *is* $<_\varphi$*-maximal*;
3. *for any* $q \in P$, *there is a* $j \in \omega$ *such that* $q <^*_\varphi \mathrm{Cof}(j)$ *unless* q *is* $<_\varphi$*-maximal.*

PROOF. We define the function Cof by induction on j. Let $p_0 = \mathrm{Cof}(0)$ be the least integer in P. Assume we have defined $p_j = \mathrm{Cof}(j)$. If p_j is $<_\varphi$-maximal, let $p_{j+1} = p_j$. Otherwise, let $p_{j+1} = \mathrm{Cof}(j+1)$ be the smallest integer in P such that $p_j < p_{j+1}$ and $p_j <^*_\varphi p_{j+1}$. Since $n = p_{j+1}$ iff $n \in P$ and $p_j < n$ and $p_j <^*_\varphi n$ and $\forall m\,(p_j < m < n \Rightarrow m \leq^*_\varphi p_j)$, this defines a Π^1_1 function. To see that (3) holds, let $q \in P$ be a nonmaximal element. Since the sequence $(p_j)_{j\in\omega}$ is strictly increasing in the natural order $<$ on ω, there is a least integer j such that $p_j \leq q < p_{j+1}$. Because p_{j+1} is the least integer larger than p_j such that $p_j <^*_\varphi p_{j+1}$, we cannot have $p_j <^*_\varphi q$. Hence, $q \leq^*_\varphi p_j <^*_\varphi p_{j+1}$. ⊣

§4. A completely good pair. Suppose E is a Π^1_1 equivalence relation on ω^ω. A key idea in Harrington's proof of Silver's dichotomy is to consider the set

$$W = \{x \in \omega^\omega\colon \text{there is no } \Delta^1_1 \text{ set } D \text{ such that } x \in D \subseteq [x]_E\}.$$

A computation shows that W is Σ^1_1. Moreover, when E has uncountably many equivalence classes, $W \neq \emptyset$ and every nonempty Σ^1_1 subset $X \subseteq W$ meets uncountably many E-equivalence classes. In fact, a nonempty Σ^1_1 subset $X \subseteq \omega^\omega$ meets uncountably many E-equivalence classes iff $X \cap W \neq \emptyset$.

We will establish the following corresponding result in our context.

PROPOSITION 5. *Let* E *be a* Π^1_1 *equivalence relation on* ω^ω *and* $(A_n)_{n\in\omega}$ *a sequence of uniformly* Σ^1_1 *subsets of* ω^ω. *If* $\limsup_{n\in K} A_n$ *meets uncountably many* E*-equivalence classes for every* $K \in [\omega]^\omega$, *then there exists a nonempty* Σ^1_1 *set* $V \subseteq \omega^\omega$ *and a* Δ^1_1 *set* $H \in [\omega]^\omega$ *such that for every nonempty* Σ^1_1 *set* $X \subseteq V$ *and every* Δ^1_1 *set* $K \in [H]^\omega$ *the set* $X \cap \limsup_{n\in K} A_n$ *meets uncountably many* E*-equivalence classes.*

We call such a pair (V, H) *completely good*. The rest of this section is devoted to the proof of Proposition 5 and a further refinement. In contrast with Harrington's proof, we need a recursive construction of transfinite length, in which we remove all possible 'bad pairs' one by one.

DEFINITION. We say that $n = \langle y, k\rangle \in \omega$ is a *bad pair* if the following properties hold:

1. $y \in C^{\omega^\omega}$ and $k \in C^\omega$,
2. $D^\omega_k \in [\omega]^\omega$, and
3. $D^{\omega^\omega}_y \cap \limsup_{n\in D^\omega_k} A_n$ meets only countably many E-equivalence classes, i.e., $D^{\omega^\omega}_y \cap W \cap \limsup_{n\in D^\omega_k} A_n = \emptyset$.

It is clear from this definition that the set $P \subseteq \omega$ of all bad pairs is Π^1_1. Let $\varphi\colon P \to \omega_1^{\mathrm{CK}}$ be a Π^1_1-norm on P. Define a well-order on P by

$$m <_\varphi n \Leftrightarrow \varphi(m) < \varphi(n) \vee (\varphi(m) = \varphi(n) \wedge m < n)$$

and let $\leq^*_\varphi$ be the Π^1_1 relation given by

$$m \leq^*_\varphi n \Leftrightarrow P(m) \wedge (\neg P(n) \vee \varphi(m) \leq \varphi(n)).$$

For the rest of the paper, let $\mathrm{Cof}\colon \omega \to \omega$ be the Π^1_1 function related to φ and P as given by Lemma 4.

Denote by C^ω_∞ the set of all $n \in C^\omega$ such that $D^\omega_n \in [\omega]^\omega$. Then C^ω_∞ is Π^1_1. Given an $h \in C^\omega_\infty$, we define the *next bad pair relative to* h to be the $<_\varphi$-least $\langle y, k\rangle \in P$ such that $k \subseteq^* h$. Set $R(h, \langle y, k\rangle)$ iff $\langle y, k\rangle$ is the next bad pair relative to h.

LEMMA 6. *The relation $R \subseteq \omega \times \omega$ is Π^1_1. Moreover, R is a Δ^1_1 function on the set $B = \{h \in \omega\colon h \in C^\omega_\infty \wedge \exists n\,(R(h, n))\}$.*

PROOF. We have $R(h, \langle y, k\rangle)$ iff

$$h \in C^\omega_\infty \wedge \langle y, k\rangle \in P \wedge k \subseteq^* h \wedge \forall y', k' \in \omega\,(\langle y, k\rangle \not\leq^*_\varphi \langle y', k'\rangle \Rightarrow k' \not\subseteq^* h).$$

This is a Π^1_1 definition. If $R(h, n)$ holds, then n is the unique such integer. Thus, for $h \in B$, $\neg R(h, n) \Leftrightarrow \exists m\,(R(h, m) \wedge n \neq m)$, which is Π^1_1. Hence, R is Δ^1_1 on B. ⊣

Similarly, given a Π^1_1 set $A \subseteq P$ we define the next bad pair in A relative to h to be the $<_\varphi$-least $\langle y, k\rangle \in A$ such that $k \subseteq^* h$. The corresponding version of Lemma 6 still holds.

Initial segments of the recursive construction can be coded by reals, as follows. Recall that $\mathsf{WO}_\alpha = \{w \in \mathsf{WO}\colon |<_w| = \alpha\}$ and for $\alpha < \omega_1^{\mathrm{CK}}$, we have $\mathsf{WO}_\alpha \in \Delta^1_1$.

DEFINITION. Let $\alpha < \omega_1^{\mathrm{CK}}$. A real $z \in \omega^\omega$ is *α-adequate* if $z = \langle w, v, h\rangle$, where $w \in 2^\omega$, $v \in \omega^\omega$, and $h \in \omega^\omega$, and the following conditions are satisfied:

1. $w \in \mathsf{WO}_\alpha$,
2. if $n \notin \mathrm{dom}(<_w)$, then $v(n) = h(n) = 0$,
3. the $<_w$-least element is the $<_\varphi$-least element,
4. if $n \in \mathrm{dom}(<_w)$ is a $<_w$-successor (say $n = m^+_{<_w}$), then the following holds:
   (a) $n = \langle y, k\rangle$ is the next bad pair relative to $h(m)$ such that $\langle y, k\rangle \notin \mathrm{dom}(<_w) \restriction n$,
   (b) $v(n)$ is a canonical code for $D^{\omega^\omega}_{v(m)} \setminus D^{\omega^\omega}_y$,
   (c) $h(n) = k$.

5. if $n \in \mathrm{dom}(<_w)$ is a $<_w$-limit, then with v' the canonical code for

$$\bigcap_{j\in\omega} D^{\omega^\omega}_{v(\mathrm{Cof}(w,n,j))}$$

and h' the canonical code for $\triangle_{j\in\omega} D^{\omega}_{h(\mathrm{Cof}(w,n,j))}$, the following holds:
(a) $n = \langle y, k\rangle$ is the next pair relative to h' such that $\langle y, k\rangle \notin \mathrm{dom}(<_w) \restriction n$,
(b) $v(n)$ is the canonical code for $D^{\omega^\omega}_{v'} \setminus D^{\omega^\omega}_{y}$, and
(c) $h(n) = k$.

Some comments on these conditions: (1) says that z represents the construction up to stage α, (2) is needed only to ensure that there can be at most one α-adequate real for every $\alpha < \omega_1^{\mathrm{CK}}$, (3), (4a), and (5a) state that $<_w$ represents the order in which the bad pairs are picked in our construction and that we pick a new bad pair at each stage, and conditions (4b,c) and (5b,c) require $v(n)$ and $h(n)$ to be codes for the correct sets whenever $n \in \mathrm{dom}(<_w)$.

We call a real *adequate* if it is α-adequate for some $\alpha < \omega_1^{\mathrm{CK}}$.

LEMMA 7. *The set of all adequate reals is* Π^1_1.

PROOF. Replace condition (1) above with condition (1)′ $w \in \mathrm{WO}_{<\omega_1^{\mathrm{CK}}}$, which is Π^1_1. Conditions (2) and (3) are arithmetical. For (4), n is a $<_w$-successor, $n = (m)^+_{<_w}$, and (4b, c) are arithmetical predicates, while (4a) is Π^1_1. Thus, (4) is Π^1_1. Similarly, (5) is Π^1_1. ⊣

It is immediate from the definition of α-adequate that for each $\alpha < \omega_1^{\mathrm{CK}}$, if there is an α-adequate real, then this real is unique; denote it by z_α.

LEMMA 8. *Every adequate real is* Δ^1_1.

PROOF. Let z_α be α-adequate for some fixed $\alpha < \omega_1^{\mathrm{CK}}$. Then $z = \langle w, v, h\rangle$ equals z_α iff z satisfies conditions (1) through (5). The first 3 conditions are Δ^1_1. Conditions (4) and (5) are Π^1_1, because (4a) and (5a) contain a predicate $R(n, h)$, i.e., n is the next bad pair relative to h (where $h = h(m)$ in 4a and $h = h'$ in 5a). However, since z is given, we know that this h is an element of $B = \{h \in \omega\colon h \in C^\omega_\infty \wedge \exists n\,(R(h, n))\}$. By Lemma 6, R is Δ^1_1 on B. Thus, conditions (4) and (5) are Δ^1_1 in this case. ⊣

Finally, we define $V \subseteq \omega^\omega$ and $H \in [\omega]^\omega$ as follows. Let $x \in V$ iff

$$\forall z \in \Delta^1_1\,(z = \langle w, v, h\rangle \text{ adequate} \Rightarrow \forall n\,(n \in \mathrm{dom}(<_w) \Rightarrow x \in D^{\omega^\omega}_{v(n)}))$$

and $n \in H$ iff

$$\exists z \in \Delta^1_1\,(z = \langle w, v, h\rangle \text{ is adequate } \wedge \forall j \leq n\,(\mathrm{Cof}(j) \in \mathrm{dom}(<_w) \Rightarrow n \in \triangle_{j\leq n} h(\mathrm{Cof}(j)))).$$

Equivalently by Lemma 8, $n \in H$ iff

$$\forall z \in \Delta^1_1\,(z = \langle w, v, h\rangle \text{ is adequate} \wedge \forall j \leq n\,(\mathrm{Cof}(j) \in \mathrm{dom}(<_w) \Rightarrow n \in \triangle_{j\leq n} h(\mathrm{Cof}(j)))).$$

LEMMA 9. *$V \in \Sigma^1_1$ and $H \in \Delta^1_1$. Moreover, $V \neq \emptyset$ and $H \in [\omega]^\omega$.*

PROOF. By Kleene's restricted quantification theorem (see for example [8, Theorem 4D.3]), $V \in \Sigma^1_1$. (Note: if the construction stops below ω_1^{CK}, then V is actually Δ^1_1 but we will not need that fact.) Similarly, the first definition of H is Π^1_1 and the second definition is Σ^1_1. Therefore, $H \in [\omega]^\omega$ is Δ^1_1. We show that $V \neq \emptyset$.

Suppose towards a contradiction that $V = \emptyset$. Then for every $x \in \omega^\omega$ there is an $\alpha < \omega_1^{\mathrm{CK}}$ and a $k \in \omega$ such that for $z_\alpha = \langle w_\alpha, v_\alpha, h_\alpha \rangle$, we have $k \in \mathrm{dom}(<_{w_\alpha})$ and $x \notin D^{\omega^\omega}_{v_\alpha(k)}$. For $k \in \mathrm{dom}(<_{w_\alpha})$, denote by $y_\alpha(k)$ the code for the set removed at that stage. By assumption,

$$\omega^\omega = \bigcup_{\alpha < \omega_1^{\mathrm{CK}}} \bigcup_{k \in \mathrm{dom}(<_{w_\alpha})} D^{\omega^\omega}_{y_\alpha(k)}.$$

Since $H \subseteq^* D^\omega_{h_\alpha(k)}$ for every $k \in \mathrm{dom}(<_{w_\alpha})$,

$$\limsup_{n \in H} A_n \subseteq \limsup_{n \in D^\omega_{h_\alpha(k)}} A_n.$$

In particular for every $k \in \mathrm{dom}(<_{w_\alpha})$,

$$D^{\omega^\omega}_{y_\alpha(k)} \cap \limsup_{n \in H} A_n \subseteq D^{\omega^\omega}_{y_\alpha(k)} \cap \limsup_{n \in D^\omega_{h_\alpha(k)}} A_n.$$

Hence,

$$\begin{aligned} \limsup_{n \in H} A_n &= \bigcup_{\alpha < \omega_1^{\mathrm{CK}}} \bigcup_{k \in \mathrm{dom}(<_{w_\alpha})} D^{\omega^\omega}_{y_\alpha(k)} \cap \limsup_{n \in H} A_n \\ &\subseteq \bigcup_{\alpha < \omega_1^{\mathrm{CK}}} \bigcup_{k \in \mathrm{dom}(<_{w_\alpha})} D^{\omega^\omega}_{y_\alpha(k)} \cap \limsup_{n \in D^\omega_{h_\alpha(k)}} A_n \end{aligned}$$

meets only countably many E-equivalence classes, a contradiction. Thus, $V \neq \emptyset$. $\dashv$

We now verify that the pair (V, H) is indeed completely good. In the proof of the next lemma we use the following observation. Let $z = \langle w, v, h \rangle$ be an adequate real. If $m <_w n$, then $m, n \in P$ and $\varphi(m) < \varphi(n)$. This is the case, because whenever $\langle y, k \rangle$ is a bad pair such that $k \subseteq^* h(n)$, also $k \subseteq^* h(m)$, since $h(n) \subseteq^* h(m)$.

LEMMA 10. *If $X \subseteq V$ is a nonempty Σ^1_1 set and $K \in [H]^\omega$ a Δ^1_1 set, then $X \cap \limsup_{n \in K} A_n$ meets uncountably many E-classes.*

PROOF. Suppose $X \cap \limsup_{n \in K} A_n$ meets only countably many E-equivalence classes, i.e., $X \cap \limsup_{n \in K} A_n \cap W = \emptyset$. By Σ^1_1-separation, there is a Δ^1_1 set $Y \subseteq \omega^\omega$ such that $X \subseteq Y$ and $Y \cap \limsup_{n \in K} A_n \cap W = \emptyset$. Let y, k be a code for Y, K, respectively. Clearly, $\langle y, k \rangle$ is a bad pair.

First, suppose the construction halted at stage $\alpha < \omega_1^{\mathrm{CK}}$. Let $z = \langle w, v, h \rangle$ be the unique α-adequate real. The construction stops only if there does not exist a next bad pair which we have not picked already. Since $\langle y, k \rangle$ is a bad pair such that $k \subseteq^* h(n)$ for every $n \in \mathrm{dom}(<_{w_\alpha})$, there must be an $n \in \mathrm{dom}(<_{w_\alpha})$ such that $n = \langle y, k \rangle$, i.e., we picked $\langle y, k \rangle$ at that stage (otherwise, we can extend the construction by picking it now). But then $D^{\omega^\omega}_{v(n)} \cap D^{\omega^\omega}_{y} = \emptyset$, which implies $V \cap Y = \emptyset$ and so $V \cap X = \emptyset$.

Second, suppose the construction continued all the way up to ω_1^{CK}. Then there exists an $\alpha < \omega_1^{\mathrm{CK}}$ such that $\alpha > \varphi(\langle y, k \rangle)$. Let $z = \langle w, v, h \rangle$ be α-adequate.

By the observation above, the pair $\langle y, k\rangle$ was considered, hence there exists an $n \in \mathrm{dom}(<_{w_\alpha})$ such that $n = \langle y, k\rangle$. Again, this implies $V \cap X = \emptyset$. ⊣

This finishes the proof of Proposition 5. We now derive a further refinement. A second key element of Harrington's proof is that E is meager on $W \times W$, when W is given the (subspace) Gandy–Harrington topology τ_{GH}. This is the topology on ω^ω generated by the Σ^1_1 sets. Although ω^ω with the Gandy–Harrington topology is not metrizable, it is strong Choquet and this enables one to redo the familiar construction of a perfect set of inequivalent elements, using a winning strategy for the second player. While this approach would also work in our case, we will use the set X_{low} of low elements instead. This makes the construction in the proof of the main theorem more transparent, at the cost of some technicalities which we now summarize.

Let $X_{\mathrm{low}} = \{x \in \omega^\omega : \omega_1^{\mathrm{CK}(x)} = \omega_1^{\mathrm{CK}}\}$. We will use the following facts about W, X_{low}, and τ_{GH}:

1. W and X_{low} are both nonempty Σ^1_1 sets,
2. X_{low} is dense in τ_{GH} and $(X_{\mathrm{low}}, \tau_{\mathrm{GH}})$ is a Polish space, and
3. a nonempty Σ^1_1 set $A \subseteq \omega^\omega$ meets uncountably many E-equivalence classes iff $A \cap W \neq \emptyset$ iff $A \cap W \cap X_{\mathrm{low}} \neq \emptyset$.

Proofs of these facts can be found in [2].

PROPOSITION 11. *Let E be a Π^1_1 equivalence relation on ω^ω and $(A_n)_{n\in\omega}$ a sequence of uniformly Σ^1_1 subsets of ω^ω. If $\limsup_{n\in K} A_n$ meets uncountably many E-equivalence classes for every $K \in [\omega]^\omega$, then there exists a completely good pair (V, H) such that V is a Polish space in the Gandy–Harrington topology τ_{GH} and E is meager on $V \times V$ (with the product topology $\tau_{\mathrm{GH}} \times \tau_{\mathrm{GH}}$).*

PROOF. Let (V, H) be the completely good pair given by Proposition 5. Using the facts stated above, it is easy to see that $(V \cap W \cap X_{\mathrm{low}}, H)$ is a completely good pair with the required additional properties. ⊣

§5. Proof of the main theorem. We now prove an effective version of Theorem 1. By the usual relativization and transfer arguments, this implies our main result.

THEOREM 12. *Let E be a Π^1_1 equivalence relation on ω^ω and $(A_n)_{n\in\omega}$ a sequence of uniformly Σ^1_1 subsets of ω^ω. If $\limsup_{n\in K} A_n$ meets uncountably many E-equivalence classes for every $K \in [\omega]^\omega$, then there exists a $K \in [\omega]^\omega$ such that $\bigcap_{n\in K} A_n$ contains a perfect set of pairwise E-inequivalent elements.*

PROOF. Let (V, H) be the completely good pair given by Proposition 11. Since E is meager on $V \times V$ in the Gandy-Harrington topology τ_{GH}, we can fix an increasing sequence $(F_n)_{n\in\omega}$ of τ_{GH}-closed nowhere dense sets such that $E \subseteq \bigcup_{n\in\omega} F_n$. We may assume that the diagonal $\{(x, x) : x \in V\}$ is contained in F_0. We will recursively define a strictly increasing sequence $j_0 < j_1 < \dots$ of natural numbers and a Cantor scheme $(X_s)_{s\in 2^{<\omega}}$ of nonempty Σ^1_1 subsets of V such that for all $s, t \in 2^{<\omega}$,

1. $\overline{X}_{s^\frown 0}, \overline{X}_{s^\frown 1} \subseteq X_s$, $\overline{X}_{s^\frown 0} \cap \overline{X}_{s^\frown 1} = \emptyset$, and $\mathrm{diam}(X_s) \leq 2^{-\mathrm{lh}(s)}$,
2. if $s \neq t \in 2^{n+1}$, then $X_s \times X_t \cap F_n = \emptyset$, and
3. if $s \in 2^n$, then $X_s \subseteq A_{j_0} \cap \dots \cap A_{j_n}$.

Note that in (1), the closures and diameter are relative to (V, τ_{GH}). Once this construction is completed, let $K = \{j_0, j_1, \dots\}$ and

$$P = \bigcup_{\sigma \in 2^\omega} \bigcap_{n \in \omega} X_{\sigma \restriction n}.$$

It is easy to see that $P \subseteq \bigcap_{n \in K} A_n$ is nonempty perfect set of pairwise E-inequivalent elements.

Without loss of generality we may assume that $A_0 = \omega^\omega$. Start the construction with $j_0 = 0$ and $X_\emptyset = \omega^\omega$. Suppose we have defined natural numbers $j_0 < \cdots < j_n$ and nonempty Σ^1_1 sets $X_s \subseteq A_{j_0} \cap \cdots \cap A_{j_n}$ for $s \in 2^n$ satisfying the requirements above. By intersecting with sufficiently small basic open neighborhoods, we can split each X_s into disjoint nonempty Σ^1_1 sets $X_{s^\frown 0}$ and $X_{s^\frown 1}$ satisfying requirement (1). Since F_n is closed nowhere dense, given any pair $s \neq t \in 2^{n+1}$ we can shrink X_s and X_t so that $X_s \times X_t \cap F_n = \emptyset$. After finitely many iterations, we have defined X_s for $s \in 2^{n+1}$ satisfying requirements (1) and (2).

Claim. *There is a $j > j_n$ such that $X_s \cap A_j \neq \emptyset$ for all $s \in 2^{n+1}$.*

Proof. Suppose towards a contradiction that for every $j > j_n$ there is an $s \in 2^{n+1}$ such that $X_s \cap A_j = \emptyset$. Define a binary relation $R \subseteq \omega \times 2^{n+1}$ by $R(j, s) \Leftrightarrow X_s \cap A_j = \emptyset$. Since R is Π^1_1, there is a Δ^1_1 uniformizing function $f : \omega \to 2^{n+1}$. By the pigeonhole principle, there is an $s \in 2^{n+1}$ such that $\{j \in \omega : f(j) = s\} \cap H$ is infinite. Pick such an $s \in 2^{n+1}$. Then $K = \{j \in \omega : j \in H \text{ and } f(j) = s\}$ is Δ^1_1, $K \in [H]^\omega$, and $X_s \cap \bigcup_{n \in K} A_n = \emptyset$. This implies that $X_s \cap \limsup_{n \in K} A_n = \emptyset$, contradicting the fact that (V, H) is a completely good pair. ⊣

To complete this step in the construction, let $j_{n+1} = j$ and intersect each X_s with $A_{j_{n+1}}$. This finishes the proof of Theorem 12. ⊣

The following parametric version of the Laczkovich–Komjáth property was also considered by Balcerzak and Głąb.

Definition. An equivalence relation E on a Polish space Y has the *parametric Laczkovich–Komjáth property* if for every uncountable Polish space X and every sequence $(A_n)_{n \in \omega}$ of analytic subsets of $X \times Y$, if $\limsup_{n \in K} A_n(x)$ meets uncountably many E-equivalence classes for every $x \in X$ and $K \in [\omega]^\omega$, then there exists a $K \in [\omega]^\omega$ and a perfect set $P \subseteq X$ such that $\bigcap_{n \in K} A_n(x)$ meets perfectly many E-equivalence classes for each $x \in P$.

Theorem 13 (Balcerzak–Głąb [1]). *If E has the Laczkovich–Komjáth property and for every analytic set $A \subseteq X \times X$, the set*

$$\{x \in X : A_x \text{ meets uncountably many } E\text{-equivalence classes}\}$$

is analytic, then E has the parametric Lackovich–Komjáth property. ⊣

Proposition 14. *Every coanalytic equivalence relation on a Polish space has the parametric Laczkovich–Komjáth property.*

Proof. Let E be a coanalytic equivalence relation on a Polish space X and $A \subseteq X \times X$ an analytic subset. Without loss of generality we may assume E is a Π^1_1 equivalence relation on $X = \omega^\omega$ and $A \subseteq \omega^\omega \times \omega^\omega$ is Σ^1_1. Since A is Σ^1_1, each section A_x is Σ^1_1 as well. Hence, A_x meets uncountably many E-equivalence classes

iff $A_x \cap W \neq \emptyset$. Thus,

$$\{x \in \omega^\omega : A_x \text{ meets uncountably many } E\text{-equivalence classes}\}$$

is Σ^1_1. Hence, E has the parametric Laczkovich–Komjáth property by Theorem 1 and Theorem 13. ⊣

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