Saeed Salehi
Department of Mathematics
University of Tabriz
P.O.Box 51666–17766
Tabriz, Iran

Tel: +98 (0)411 339 2905
Fax: +98 (0)411 334 2102
E-mail: /root@SaeedSalehi.ir/
/SalehiPour@TabrizU.ac.ir/
Web: http://SaeedSalehi.ir/

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Herbrand Consistency of Some Arithmetical Theories

Abstract

Gödel's second incompleteness theorem is proved for Herbrand consistency of some arithmetical theories with bounded induction, by using a technique of logarithmic shrinking the witnesses of bounded formulas, due to Z. Adamowicz [Herbrand consistency and bounded arithmetic, *Fundamenta Mathematicae* 171 (2002) 279–292]. In that paper, it was shown that one cannot always shrink the witness of a bounded formula logarithmically, but in the presence of Herbrand consistency, for theories $\mathrm{I}\Delta_0 + \Omega_m$ with $m \geqslant 2$, any witness for any bounded formula can be shortened logarithmically. This immediately implies the unprovability of Herbrand consistency of a theory $T \supseteq \mathrm{I}\Delta_0 + \Omega_2$ in T itself.

In this paper, the above results are generalized for $\mathrm{I}\Delta_0 + \Omega_1$. Also after tailoring the definition of Herbrand consistency for $\mathrm{I}\Delta_0$ we prove the corresponding theorems for $\mathrm{I}\Delta_0$. Thus the Herbrand version of Gödel's second incompleteness theorem follows for the theories $\mathrm{I}\Delta_0 + \Omega_1$ and $\mathrm{I}\Delta_0$.

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1 Introduction

By Gödel's first incompleteness theorem Truth is not the same as Provability in sufficiently strong theories. In other words, Provable is a proper subset of True, and thus True is not conservative over Provable. It is not even Π_1-conservative; i.e., there exists a Π_1-formula, in theories which can interpret enough arithmetic, which is true but unprovable in those theories. Thus one way of comparing the strength of a theory T over one of its sub-theories S is considering the Π_1-conservativeness of T over S. And Gödel's second incompleteness theorem provides such a Π_1-candidate: $\mathrm{Con}(S)$, the statement of the consistency of S. By that theorem $S \nvdash \mathrm{Con}(S)$, but if $T \vdash \mathrm{Con}(S)$ then T is not Π_1-conservative over S.

Examples abound in mathematics and logic: Zermelo-Frankel Set Theory ZFC is not Π_1-conservative over Peano's Arithmetic PA, because $\mathrm{ZFC} \vdash \mathrm{Con(PA)}$ but $\mathrm{PA} \nvdash \mathrm{Con(PA)}$. Inside PA the Σ_n-hierarchy is not a Π_1-conservative hierarchy, since $\mathrm{I}\Sigma_{\mathrm{n}+1} \vdash \mathrm{Con}(\mathrm{I}\Sigma_{\mathrm{n}})$ though $\mathrm{I}\Sigma_{\mathrm{n}} \nvdash \mathrm{Con}(\mathrm{I}\Sigma_{\mathrm{n}})$; see e.g. [7]. Then below the theory $\mathrm{I}\Sigma_1$ things get more complicated: for Π_1-separating $\mathrm{I}\Delta_0 + \mathrm{Exp}$ over $\mathrm{I}\Delta_0$ the candidate $\mathrm{Con}(\mathrm{I}\Delta_0)$ does not work, because $\mathrm{I}\Delta_0 + \mathrm{Exp} \nvdash \mathrm{Con}(\mathrm{I}\Delta_0)$. For this Π_1-separation, Paris and Wilkie [10] suggested the notion of cut-free consistency instead of usual - Hilbert style - consistency predicate. Here one can show that $\mathrm{I}\Delta_0 + \mathrm{Exp} \vdash \mathrm{CFCon}(\mathrm{I}\Delta_0)$, and then it was presumed that $\mathrm{I}\Delta_0 \nvdash \mathrm{CFCon}(\mathrm{I}\Delta_0)$, where CFCon stands for cut-free consistency. But this presumption took a rather long time to be established. Meanwhile, Pudlák in [11] established the Π_1-separation of $\mathrm{I}\Delta_0 + \mathrm{Exp}$ over $\mathrm{I}\Delta_0$ by other methods, and mentioned the unprovability of $\mathrm{CFCon}(\mathrm{I}\Delta_0)$ in $\mathrm{I}\Delta_0$ as an open problem. This problem is interesting in its own right. Indeed Gödel's second incompleteness theorem has been generalized to all consistent theories containing Robinson's Arithmetic Q, in the case of Hilbert consistency; see [7]. But for cut-free consistency it is still an open problem whether the theorem holds for Q, and its not too strong extensions. This is a double strengthening of Gödel's second incompleteness theorem: weakening the theory and weakening the consistency predicate. Let us note that since cut-free provability is stronger than usual Hilbert provability (with a super-exponential cost), then cut free consistency is a weaker notion of consistency. Indeed, proving Gödel's second incompleteness theorem for weak notions of consistencies in weak arithmetics turns out to be a difficult problem. We do not intend here to give a thorough history of this ongoing research area, let us just mention a few results:

• Z. Adamowicz was the first one to demonstrate the unprovability of cut free consistency in bounded arithmetics, by proving in an unpublished manuscript in 1999 (later appeared as a technical report [1]) that the tableau consistency of $\mathrm{I}\Delta_0 + \Omega_1$ is not provable in itself. Later with P. Zbierski (2001) she proved Gödel's second incompleteness theorem for Herbrand consistency of $\mathrm{I}\Delta_0 + \Omega_2$ (see [2]), and a bit later she gave a model theoretic proof of it in 2002; see [3].

• D. E. Willard introduced an $\mathrm{I}\Delta_0-$provable Π_1-formula V and showed that any theory whose axioms contains $Q + V$ cannot prove its own tableaux consistency. He also showed that tableaux consistency of $\mathrm{I}\Delta_0$ is not provable in itself, see [14, 15]; this proved the conjecture of Paris and Wilkie mentioned above.

• S. Salehi (see [13] Chapter 3 and also [12]) showed the unprovability of Herbrand consistency of a re-axiomatization of $\mathrm{I}\Delta_0$ in itself, the proof of which was heavily based on [2]. The re-axiomatization used PA^-, the theory of the positive fragment of a discretely ordered ring, as the base theory, instead of Q, and assumed two $\mathrm{I}\Delta_0-$derivable sentences as axioms. Also the model-theoretic proof of Z. Adamowicz in [3] was generalized to the $\mathrm{I}\Delta_0 + \Omega_1$ case in Chapter 5 of [13]. A polished and updated proof of it appears in the present paper.

• L. A. Kołodziejczyk showed in [8] that the notion of Herbrand consistency cannot Π_1-separate the hierarchy of bounded arithmetics (this Π_1-separation is still an open problem). Main results are the existence of an n for any given $m \geqslant 3$ such that $S_m \nvdash \mathrm{HCon}(S^n_m)$, and the existence of a natrual n such that $\bigcup_m S_m \nvdash \mathrm{HCon}(S^n_3)$, where HCon stands for Herbrand consistency.

• Z. Adamowicz and K. Zdanowski have obtained some results on the unprovability of the relativized notion of Herbrand consistency in theories containing $I\Delta_0 + \Omega_1$; see [4]. Their paper contains some insightful ideas about the notion of Herbrand consistency.

For $I\Delta_0 + \Omega_1$ the arguments are rather smoother, in comparison to the case of $I\Delta_0$. Our proof for the main theorem on $I\Delta_0 + \Omega_1$ borrows many ideas from [3], the major difference being the coding techniques and making use of a more liberal definition of Herbrand consistency. The definition of HCon given in [2] and [3] depends on a special coding given there. For reading the present paper no familiarity with [2] is needed, but a theorem of [3] will be of critical use here (Theorem 22). We will even use a modified version of it (Theorem 37). For $I\Delta_0$ we will see that our definition of HCon is not best suited for this theory; and we will actually tailor it for $I\Delta_0$. A hint for the obstacles in tackling Herbrand consistency in $I\Delta_0$ can be found in Chapters 3 and 4 of [13].

In Section 2 we introduce the ingredients of Herbrand's theorem from the scratch, and then explain how they can be arithmetized by Gödel coding. This sets the stage for Section 3 in which we formalize the notion of Herbrand model and use it to prove our main theorem for $I\Delta_0 + \Omega_1$. Finally in Section 4 we modify our definitions and theorems to fit the $I\Delta_0$ case. After pinpointing the places where we have made an essential use of Ω_1, we do some tailoring for $I\Delta_0$, and prove our main result for $I\Delta_0$. We finish the paper with some conclusions and some open questions.

2 Basic Definitions and Arithmetizations

This section introduces the notions of Herbrand provability and Herbran consistency, and a way of formalizing and arithmetizing these concepts. The first subsection can be read by any logician. The second subsection gets more technical with Gödel coding, for which some familiarity with [7] is presumed.

2.1 Herbrand Consistency

Skolemizing a formula is usually performed on prenex normal forms (see e.g. [6]), and since prenex normalizing a formula is not necessarily done in a unique way, then one may get different Skolemized forms of a formula. For example, the tautology $F = \forall x\phi(x) \to \forall x\phi(x)$ can be prenex normalized into either $\forall x\exists y(\phi(y) \to \phi(x))$ or $\exists y\forall x(\phi(y) \to \phi(x))$. These two formulas can be Skolemized respectively as $\phi(\mathfrak{f}(x)) \to \phi(x)$ and $\phi(\mathfrak{c}) \to \phi(x)$, where $\mathfrak{f}$ is a new unary function symbol, and $\mathfrak{c}$ is a new constant symbol. Here we briefly describe a way of Skolemizing a (not-necessarily prenex normal) formula which results in a somehow unique (up to a variable renaming) formula.

A formula is in *negation normal* form when the implication symbol does not appear in it, and the negation symbol appears in front of atomic formulas only. A formula can be (uniquely) negation normalized by the following rewriting rules:

$$(A \to B) \Longmapsto (\neg A \vee B) \qquad \neg\neg A \Longmapsto A$$
$$\neg(A \vee B) \Longmapsto (\neg A \wedge \neg B) \qquad \neg(A \wedge B) \Longmapsto (\neg A \vee \neg B)$$
$$\neg\forall x A(x) \Longmapsto \exists x \neg A(x) \qquad \neg\exists x A(x) \Longmapsto \forall x \neg A(x)$$

A formula is called *rectified* if no variables appears both bound and free in it, and different quantifiers refer to different variables. A formula is called *rectified negation normal* if it is both negation normalized and rectified. Again, any formula can be rectified. Indeed, any given formula is equivalent to its rectified negation normal form (RNNF) which can be obtained from the formula in a unique (up to a variable renaming) way (see e.g. [5]).

Now we introduce Skolem functions for existential formulas: for any (not necessarily RNNF) formula of the form $\exists x A(x)$, let $\mathfrak{f}_{\exists x A(x)}$ be a new $m-$ary function symbol where m is the number of the free variables of $\exists x A(x)$. When $m = 0$ then $\mathfrak{f}_{\exists x A(x)}$ will obviously be a new constant symbol (cf. [6]).

Definition 1 Let φ be an RNNF formula. Define φ^{S} by induction:

- $\varphi^{\mathsf{S}} = \varphi$ for atomic or negated-atomic φ;
- $(\varphi \circ \psi)^{\mathsf{S}} = \varphi^{\mathsf{S}} \circ \psi^{\mathsf{S}}$ for $\circ \in \{\wedge, \vee\}$ and RNNF formulas φ, ψ;
- $(\forall x \varphi)^{\mathsf{S}} = \forall x \varphi^{\mathsf{S}}$;
- $(\exists x \varphi)^{\mathsf{S}} = \varphi^{\mathsf{S}}[\mathfrak{f}_{\exists x \varphi(x)}(\overline{y})/x]$ where $\overline{y}$ are the free variables of $\exists x \varphi(x)$ and the formula $\varphi^{\mathsf{S}}[\mathfrak{f}_{\exists x \varphi(x)}(\overline{y})/x]$ results from the formula φ^{S} by replacing all the occurrences of the variable x with the term $\mathfrak{f}_{\exists x \varphi(x)}(\overline{y})$.

The Skolemized form of any (not necessarily RNNF) formula ψ is obtained in the following way: using the above rewriting rules we negation normalize ψ and then rename the repetitive variables (if any) to get a rectified negation normal form of ψ, say φ. Then we get φ^{S} by the above definition, and remove all the (universal) quantifiers in it (together with the variables next to them). We denote thus resulted Skolemized form of ψ by ψ^{Sk}. ⎔

Note that ψ^{Sk} can be obtained from ψ in a unique (up to a variable renaming) way, and it is an open (quantifier-less) formula. For the above example F, assuming that ϕ is atomic, we get

$$F^{\mathsf{S}} = (\exists x \neg \phi(x) \vee \forall x \phi(x))^{\mathsf{S}} = \neg \phi(\mathfrak{c}) \vee \forall x \phi(x),$$

and thus $F^{\mathrm{Sk}} = \neg \phi(\mathfrak{c}) \vee \phi(x) \equiv \phi(\mathfrak{c}) \to \phi(x)$.

Definition 2 An Skolem instance of the formula ψ is any formula resulted from substituting the free variables of ψ^{Sk} with some terms. So, if $x_1, \ldots, x_n$ are the free variables of ψ^{Sk} (thus written as $\psi^{\mathrm{Sk}}(x_1, \ldots, x_n)$) then an Skolem instance of ψ is $\psi^{\mathrm{Sk}}[t_1/x_1, \cdots, t_n/x_n]$ where $t_1, \ldots, t_n$ are terms (which could be constructed from the Skolem functions symbols).

Skolemized form of a theory T is by defintion $T^{\mathrm{Sk}} = \{\varphi^{\mathrm{Sk}} \mid \varphi \in T\}$. ⎔

We are now ready to state an important theorem discovered by Herbrand (probably by also Skolem and Gödel). This theorem has got some few names, and by now is a classical theorem in Mathematical Logic. Here we state a version of the theorem which we will need in the paper. The proof is omitted, though it is not too difficult to prove it directly (see e.g. [5]).

Theorem 3 (Herbrand) *Any theory T is equiconsistent with its Skolemized theory T^{Sk}. In other words, T is consistent if and only if every finite set of Skolem instances of T is (propositionally) satisfiable.* ⎔

We will use the above theorem, which reduces the consistency of a first-order theory to the satisfiability of a propositional theory, for the definition of Herbrand Consistency: a theory T is Herbrand consistent when every finite set of Skolem instances of T is propositionally satisfiable. One other concept is needed for formalizing Herbrand consistency of arithmetical theories: *evaluation*.

Convention 4 Throughout the paper we deal with closed (or ground) terms (i.e., terms with no variable) and for simplicity we call them "term". For this to make sense, we may assume that the language of the theory under consideration has at least one constant symbol. ⎔

Definition 5 An *evaluation* is a function whose domain is the set of all atomic formulas constructed from a given set of terms Λ and its range is the set $\{0,1\}$ such that

(i) $p\,[t{=}t]=1$ for all $t\in\Lambda$; and for any terms $t,s\in\Lambda$,

(ii) if $p\,[t{=}s]=1$ then $p\,[\varphi(t)]=p\,[\varphi(s)]$ for any atomic formula $\varphi(x)$.

The relation $\backsim_p$ on Λ is defined by $t\backsim_p s \iff p[t=s]=1$ for $t,s\in\Lambda$. ⎔

Lemma 6 *The relation $\backsim_p$ defined above is an equivalence relation.*

Proof. For $\varphi(x)\equiv(s{=}x)$ from $p\,[t{=}s]=1$ one can infer $p\,[s{=}t]=p\,[\varphi(t)]=p\,[\varphi(s)]=p\,[s{=}s]=1$. So, $t\backsim_p s$ implies $s\backsim_p t$. Also for $\phi(x)\equiv(t{=}x)$ the condition $p\,[s{=}r]=1$ implies $p\,[t{=}s]=p\,[\phi(s)]=p\,[\phi(r)]=p\,[t{=}r]$. So, $\backsim_p$ is a symmetric and transitive (also, by definition, a reflexive) relation. ⎔

Notation 7 The $\backsim_p-$class of a term t is denoted by t/p; and the set of all such $p-$classes for each $t\in\Lambda$ is denoted by Λ/p.

For simplicity, we write $p\models\varphi$ instead of $p\,[\varphi]=1$; thus $p\not\models\varphi$ stands for $p\,[\varphi]=0$. This definition of *satisfying* can be generalized to other open formulas in the usual way:

- $p\models\varphi\wedge\psi$ if and only if $p\models\varphi$ and $p\models\psi$;
- $p\models\varphi\vee\psi$ if and only if $p\models\varphi$ or $p\models\psi$;
- $p\models\neg\varphi$ if and only if $p\not\models\varphi$. ⎔

Let us note that $\backsim_p$ is a congruence relation as well. That is, for any set of terms t_i and s_i $(i=1,\ldots,n)$ and function symobl f, if $p\models t_1=s_1\wedge\cdots\wedge t_n=s_n$ then $p\models f(t_1,\ldots,t_n)=f(s_1,\ldots,s_n)$.

Definition 8 If all terms appearing in an Skolem instance of ϕ belong to the set Λ, that formula is called an Skolem instance of ϕ *available* in Λ.

An evaluation defined on Λ is called a $\phi-$*evaluation* if it satisfies all the Skolem instances of ϕ which are available in Λ.

Similarly, for a theory T, a $T-$*evaluation* on Λ is an evaluation on Λ which satisfies every Skolem instance of every formula of T which is available in Λ. ⎔

For illustrating the above concepts we now present an example.

Example 9 Take the language $\mathcal{L}=\{g,P,R,S\}$ in which g is a binary function symbol, and P is a binary predicate symbol, and R,S are unary predicate symbols. Let the theory T be axiomatized by:

$T_1:\ \forall x\exists y P(x,y)$;

$T_2:\ \forall x\big(R(x)\vee S(gx)\big)$;

$T_3:\ \forall x,y\big(\neg P(x,y)\vee\neg S(x)\big)$.

Let us, for the sake of simplicity, denote $\mathfrak{f}_{\exists yP(x,y)}$ by $\mathfrak{f}$; then the Skolemized form of the above theory is:

$T_1^{\mathrm{Sk}}:\ P(x,\mathfrak{f}x)$; $\quad T_2^{\mathrm{Sk}}:\ R(x)\vee S(gx)$; $\quad T_3^{\mathrm{Sk}}:\ \neg P(x,y)\vee\neg S(x)$.

For a constant symbol c let $\Lambda=\{c,gc,\mathfrak{f}c\}$. Then $P(c,\mathfrak{f}c)$ and $R(c)\vee S(gc)$ are Skolem instances of T (of T_1 and T_2) available in Λ, but the Skolem instance $R(gc)\vee S(ggc)$ of T_2 is not available in Λ. Let us note also that the Skolem instance $\neg P(gc,\mathfrak{f}gc)\vee\neg S(gc)$ of T_3 is not available in Λ.

Let q be an evaluation on Λ whose set of true atomic formulas is $\{P(c, \mathfrak{f}c), R(c)\}$. Then q is a $T-$evaluation. On the other hand the evaluation r on Λ whose set of true atomic formulas is $\{P(c, \mathfrak{f}c), R(c), S(c)\}$, is not a $T-$evaluation, though it satisfies all the Skolem instances of T_1 and T_2 which are available in Λ. Note that r does not satisfy the Skolem instance $\neg P(c, \mathfrak{f}c) \vee \neg S(c)$ of T_3. ⎔

By the above theorem of Herbrand, a theory T is consistent if and only if every finite set of its Skolem instances is satisfiable, if and only if for every finite set of terms Λ there is a $T-$evaluation on Λ. And for a formula φ, $T \vdash \varphi$ if and only if there exists a finite set of terms Λ such that there is no $(T+\neg\varphi)-$evaluation on Λ. We call this notion of provability, *Herbrand Provability*; note that then *Herbrand Consistency* of a theory T means the existence of a $T-$evaluation on any (finite) set of terms.

Example 10 In the previous example, let $\varphi = \forall x R(x)$. We show $T \vdash \varphi$ by Herbrand provability. Write $\neg\varphi = \exists x \neg R(x)$, and let $\mathfrak{c}$ denote the Skolem constant symbol $\mathfrak{f}_{\exists x \neg R(x)}$; so we have $(\neg\varphi)^{\mathrm{Sk}} = \neg R(\mathfrak{c})$. Put $\Lambda = \{\mathfrak{c}, g\mathfrak{c}, \mathfrak{f}g\mathfrak{c}\}$, and assume (for the sake of contradiction) that there is a $(T+\neg\varphi)-$evaluation p on Λ. Then p must satisfy the following Skolem instances of T in Λ: $P(g\mathfrak{c}, \mathfrak{f}g\mathfrak{c})$, $R(\mathfrak{c}) \vee S(g\mathfrak{c})$, and $\neg P(g\mathfrak{c}, \mathfrak{f}g\mathfrak{c}) \vee \neg S(g\mathfrak{c})$. Whence p must also satisfy $\neg S(g\mathfrak{c})$ and $R(\mathfrak{c})$. So p cannot satisfy the Skolem instance $\neg R(\mathfrak{c})$ of $\neg\varphi$ in Λ. Thus there cannot be any $(T + \neg\varphi)-$evaluation on Λ; whence $T \vdash \varphi$.

Note that finding an appropriate Λ is as complicated as finding a formal proof. For example we could not have taken Λ as $\{\mathfrak{c}, g\mathfrak{c}, \mathfrak{f}\mathfrak{c}\}$, since the evaluation q in the previous example would be a $(T+\neg\varphi)-$evaluation on that set. ⎔

The following couple of examples give a thorough illustrations for the above ideas, and they will be actually used later in the paper.

Example 11 Let Q denote Robinson's Arithmetic over the language of arithmetic $\langle 0, \mathfrak{s}, +, \cdot, \leqslant\rangle$, where 0 is a constant symbol, $\mathfrak{s}$ is a unary function symbol, $+, \cdot$ are binary function symbols, and $\leqslant$ is a binary predicate symbol, whose axioms are:

$$A_1: \ \forall x(\mathfrak{s}x \neq 0) \qquad A_2: \ \forall x\forall y(\mathfrak{s}x = \mathfrak{s}y \to x = y)$$
$$A_3: \ \forall x(x \neq 0 \to \exists y[x = \mathfrak{s}y]) \qquad A_4: \ \forall x\forall y(x \leqslant y \leftrightarrow \exists z[x + z = y])$$
$$A_5: \ \forall x(x + 0 = x) \qquad A_6: \ \forall x\forall y(x + \mathfrak{s}y = \mathfrak{s}(x + y))$$
$$A_7: \ \forall x(x \cdot 0 = 0) \qquad A_8: \ \forall x\forall y(x \cdot \mathfrak{s}y = x \cdot y + x)$$

Let $\psi = \forall x(x \leqslant 0 \to x = 0)$ and $\varphi = \forall x\forall y(x \leqslant \mathfrak{s}y \to x = \mathfrak{s}y \vee x \leqslant y)$. We can show $\mathrm{Q} \vdash \psi$ and $\mathrm{Q} \vdash \varphi$; these will be proved below by Herbrand provability. Suppose Q has been Skolemized as below:

$$A_1^{\mathrm{Sk}}: \ \mathfrak{s}x \neq 0 \qquad A_2^{\mathrm{Sk}}: \ \mathfrak{s}x \neq \mathfrak{s}y \vee x = y$$
$$A_3^{\mathrm{Sk}}: \ x = 0 \vee x = \mathfrak{s}\mathfrak{p}x \qquad A_4^{\mathrm{Sk}}: \ [x \not\leqslant y \vee x + \mathfrak{h}(x, y) = y] \wedge [x + z \neq y \vee x \leqslant y]$$
$$A_5^{\mathrm{Sk}}: \ x + 0 = x \qquad A_6^{\mathrm{Sk}}: \ x + \mathfrak{s}y = \mathfrak{s}(x + y)$$
$$A_7^{\mathrm{Sk}}: \ x \cdot 0 = 0 \qquad A_8^{\mathrm{Sk}}: \ x \cdot \mathfrak{s}y = x \cdot y + x$$

Here $\mathfrak{p}$ abbreviates $\mathfrak{f}_{\exists y(x=\mathfrak{s}y)}$ and $\mathfrak{h}$ stands for $\mathfrak{f}_{\exists z(x+z=y)}$.

For a fixed term t, put Σ_t be the following set of terms:

$$\Sigma_t = \{0, t, t+0, \mathfrak{h}(t,0), \mathfrak{p}\mathfrak{h}(t,0), \mathfrak{s}\mathfrak{p}\mathfrak{h}(t,0), t + \mathfrak{s}\mathfrak{p}\mathfrak{h}(t,0), \mathfrak{s}(t + \mathfrak{s}\mathfrak{p}\mathfrak{h}(t,0))\},$$

and suppose that p is an Q$-$evaluation on Σ_t. We show that $p \models t \not\leqslant 0 \vee t = 0$. Note that Skolemizing ψ results in $\psi^{\mathrm{Sk}} = (x \not\leqslant 0 \vee x = 0)$. If p is an evaluation and if $p \models t \leqslant 0$, then by A_4 we have

$p \models t + \mathfrak{h}(t,0) = 0$. Now, either $p \models \mathfrak{h}(t,0) = 0$ or $p \not\models \mathfrak{h}(t,0) = 0$. In the former case, we have $p \models t + 0 = t$ which by A_5 implies $p \models t = 0$. In the latter case, by A_3 we get $p \models \mathfrak{h}(t,0) = \mathfrak{sph}(t,0)$, and then $p \models 0 = t + \mathfrak{h}(t,0) = t + \mathfrak{sph}(t,0) = \mathfrak{s}(t + \mathfrak{ph}(t,0))$ by A_6, which is a contradiction with A_1. Thus we showed that if $p \models t \leqslant 0$ then necessarily $p \models t = 0$.

Now, for two fixed terms u, v define $\Gamma_{u,v}$ as

$$\Gamma_{u,v} = \{0, u, v, \mathfrak{s}v, \mathfrak{h}(u,\mathfrak{s}v), \mathfrak{ph}(u,\mathfrak{s}v), \mathfrak{sph}(u,\mathfrak{s}v), u + \mathfrak{ph}(u,\mathfrak{s}v), \\ u + \mathfrak{sph}(u,\mathfrak{s}v), \mathfrak{s}(u + \mathfrak{ph}(u,\mathfrak{s}v))\}.$$

We show that any Q−evaluation on $\Gamma_{u,v}$ must satisfy $u \not\leqslant \mathfrak{s}v \vee u = \mathfrak{s}v \vee u \leq v$. Note that the Skolemized form of φ is $\varphi^{\mathrm{Sk}} = (x \not\leqslant \mathfrak{s}y \vee x = \mathfrak{s}y \vee x \leqslant y)$. Suppose p is an Q−evaluation on $\Gamma_{u,v}$. Then either $p \models \mathfrak{h}(u,\mathfrak{s}v) = 0$ or $p \models \mathfrak{h}(u,\mathfrak{s}v) \neq 0$. In the former case, by A_4, we have $p \models u \not\leqslant \mathfrak{s}v \vee u + 0 = \mathfrak{s}v$, and then by A_5, $p \models u \not\leqslant \mathfrak{s}v \vee u = \mathfrak{s}v$. And in the latter case $p \models \mathfrak{h}(u,\mathfrak{s}v) = \mathfrak{sph}(u,\mathfrak{s}v)$ by A_3, also by A_4 we have $p \models u \not\leqslant \mathfrak{s}v \vee u + \mathfrak{sph}(u,\mathfrak{s}v) = \mathfrak{s}v$. On the other hand from A_5 we get $p \models u + \mathfrak{sph}(u,\mathfrak{s}v) = \mathfrak{s}(u + \mathfrak{ph}(u,\mathfrak{s}v))$. Whence we get $p \models u \not\leqslant \mathfrak{s}v \vee \mathfrak{s}(u + \mathfrak{ph}(u,\mathfrak{s}v)) = \mathfrak{s}v$, then by A_2, $p \models u \not\leqslant \mathfrak{s}v \vee u + \mathfrak{ph}(u,\mathfrak{s}v) = v$, which by A_4 implies $p \models u \not\leqslant \mathfrak{s}v \vee u \leqslant v$. Hence, in both cases we showed $p \models u \not\leqslant \mathfrak{s}v \vee u = \mathfrak{s}v \vee u \leqslant v$. Finally, let us note that one could present a Herbrand proof of $\mathrm{Q} \vdash \psi$ and $\mathrm{Q} \vdash \varphi$ very similarly. ⎔

Example 12 In the language of Example 11, $\langle 0, \mathfrak{s}, +, \cdot, \leqslant\rangle$, let ind_ψ be the following induction scheme for the formula $\psi(x)$:

$$\psi(0) \wedge \forall x\big(\psi(x) \to \psi(\mathfrak{s}x)\big) \to \forall x \psi(x).$$

Assume for the moment that ψ is an atomic formula. Then the Skolemization of ind_ψ results in $\mathrm{ind}_\psi^{\mathrm{Sk}}$: $\neg\psi(0) \vee \Big(\psi(\mathfrak{c}) \wedge \neg\psi(\mathfrak{sc})\Big) \vee \psi(x)$, where $\mathfrak{c}$ is the Skolem constant symbol $\mathfrak{f}_{\exists x\left(\psi(x)\wedge\neg\psi(\mathfrak{s}x)\right)}$. Then any ind_ψ−evaluation p on the set of terms $\{0, \mathfrak{c}, \mathfrak{sc}, t\}$ must satisfy one of the following:

either (1) $p \not\models \psi(0)$ or (2) $p \models \psi(\mathfrak{c}) \wedge \neg\psi(\mathfrak{sc})$ or (3) $p \models \psi(t)$.

Now take $\psi(x)$ to be the existential formula $\exists y \varphi(x,y)$ in which φ is an atomic formula. Then the Skolemized form of ind_ψ will be as

$$\mathrm{ind}_\psi^{\mathrm{Sk}} : \ \neg\varphi(0,u) \vee \Big(\varphi(\mathfrak{c}, \mathfrak{qc}) \wedge \neg\varphi(\mathfrak{sc}, v)\Big) \vee \varphi\big(x, \mathfrak{q}(x)\big),$$

where $\mathfrak{q}$ is the Skolem function symbol for the formula $\exists y\varphi(x,y)$, and $\mathfrak{c}$ is the Skolem constant symbol for the sentence $\exists x\big(\exists w\varphi(x,w) \wedge \forall v \neg\varphi(\mathfrak{s}x, v)\big)$. The variables u, v and x are free.

We will need the case of $\varphi(x,y) = \big(y \leqslant x \cdot x \wedge y = x \cdot x\big)$ in the proof of Theorem 38 below. In this case the Skolemized form of ind_ψ is

$$\begin{aligned} &(u \not\leqslant 0^2 \vee u \neq 0^2) \bigvee \\ &\Big(\big(\mathfrak{qc} \leqslant \mathfrak{c}^2 \wedge \mathfrak{qc} = \mathfrak{c}^2\big) \wedge \big(v \not\leqslant (\mathfrak{sc})^2 \vee v \neq (\mathfrak{sc})^2\big)\Big) \bigvee \\ &\big(\mathfrak{q}(x) \leqslant x^2 \wedge \mathfrak{q}(x) = x^2\big). \end{aligned}$$

The notation ϱ^2 is a shorthand for $\varrho \cdot \varrho$. Define the set of terms Υ by

$$\Upsilon = \{0, 0+0, 0^2, \mathfrak{c}, \mathfrak{c}^2, \mathfrak{c}^2 + 0, \mathfrak{sc}, \mathfrak{qc}, (\mathfrak{sc})^2, (\mathfrak{sc})^2 + 0\}$$

and suppose p is an $(Q + \mathrm{ind}_\psi)$−evaluation on the set of terms $\Upsilon \cup \{t, t^2, \mathfrak{q}(t)\}$. Then p must satisfy the

following Skolem instance ($\eth$) of ind_ψ which is available in the set $\Upsilon \cup \{t, t^2, \mathfrak{q}(t)\}$:

$$\begin{aligned} &(0 \not\leqslant 0^2 \vee 0 \neq 0^2) \bigvee \\ &\Big(\big(\mathfrak{qc} \leqslant \mathfrak{c}^2 \wedge \mathfrak{qc} = \mathfrak{c}^2\big) \wedge \big((\mathfrak{sc})^2 \not\leqslant (\mathfrak{sc})^2 \vee (\mathfrak{sc})^2 \neq (\mathfrak{sc})^2\big)\Big) \bigvee \\ &\big(\mathfrak{q}(t) \leqslant t^2 \wedge \mathfrak{q}(t) = t^2\big). \end{aligned}$$

Now since $p \models 0 \cdot 0 = 0 + 0 = 0$ then, by Q's axioms, $p \models 0 \leqslant 0^2 \wedge 0 = 0^2$, and so p cannot satisfy the first disjunct of ($\eth$). Similarly, since $p \models (\mathfrak{sc})^2 + 0 = (\mathfrak{sc})^2$ then $p \models (\mathfrak{sc})^2 \leqslant (\mathfrak{sc})^2$, thus p cannot satisfy the second disjunct of ($\eth$) either, because $p \models (\mathfrak{sc})^2 = (\mathfrak{sc})^2$. Whence, p must satisfy the third disjucnt of ($\eth$), then necessarily $p \models \mathfrak{q}(t) = t^2$ must hold. ⎔

In Example 11 we used the axioms of Robinson's Arithmetic Q to derive two sentences that will be needed later (see Lemma 25). In Example 12 we used an axiom of $\mathrm{I}\Delta_0$ to derive the existence of an squaring Skolem function symbol (see the proof of Theorem 38)

Remark 13 The arguments of the above two examples can be generalized as follows: if $T \vdash \forall \overline{x}\theta(\overline{x})$ where θ is an open (quantifier-less) RNNF formula, then $(\neg\forall \overline{x}\theta(\overline{x}))^{\mathrm{Sk}} = \neg\theta(\overline{c})$ in which $\overline{c}$ is a sequence of Skolem constant symbols. There exists a set of terms Γ (constructed from the Skolem function and constant symbols of T with $\overline{c}$) such that there exists no $(T + \neg\forall \overline{x}\theta(\overline{x}))-$evluation on Γ. So, for any sequnece of terms $\overline{t}$, if $\Gamma(\overline{t})$ is the set of terms which result from the terms of Γ by substituting $\overline{c}$ with $\overline{t}$, then any $T-$evaluation on $\Gamma(\overline{t})$ must satisfy the formula $\theta(\overline{t})$. ⎔

2.2 Arithmetization

Fix $\mathcal{L}_A$ to be our language of arithmetic; one can set $\mathcal{L}_A = \langle 0, 1, +, \cdot, < \rangle$ as e.g. in [9] or $\mathcal{L}_A = \langle 0, \mathfrak{s}, +, \cdot, \leqslant \rangle$ as e.g. in [7]. Later it will be clear that choosing this fixed language is not of much importance.

Peano's arithmetic PA is the first-order theory that extends Q (see Example 11) by the following induction schema for any arithmetical formula $\varphi(x)$: $\varphi(0) \wedge \forall x\big(\varphi(x) \to \varphi(x+1)\big) \to \forall x\varphi(x)$. Fragments of PA are extensions of Q with the induction schema restricted to a class of formulas. A formula is called bounded if its every quantifier is bounded, i.e., is either of the form $\forall x \leqslant t(\ldots)$ or $\exists\, x \leqslant t(\ldots)$ where t is a term; they are read as $\forall x(x \leqslant t \to \ldots)$ and $\exists x(x \leqslant t \wedge \ldots)$ respectively. It is easy to see that bounded formulas are decidable. The theory $\mathrm{I}\Delta_0$, also called bounded arithmetic, is axiomatized by Q plus the induction schema for bounded formulas. The exponentiation function exp is defined by $\exp(x) = 2^x$; the formula Exp expresses its totality: $(\forall x \exists y[y = \exp(x)])$. The converse of exp is denoted by log which is formally defined as $\log x = \min\{y \mid x \leqslant \exp(y)\}$; and the cut ℓog consists of the logarithms of all elements: $\ell og = \{x \mid \exists y[\exp(x) = y]\}$. The superscripts above the function symbols indicate the iteration of the functions: $\exp^2(x) = \exp(\exp(x))$, $\log^2 x = \log\log x$; similarly the cut ℓog^n is $\{x \mid \exists y[\exp^n(x) = y]\}$. Let us recall that Exp is not provable in $\mathrm{I}\Delta_0$; and sub-theories of $\mathrm{I}\Delta_0 + \mathrm{Exp}$ are called weak arithmetics. Between $\mathrm{I}\Delta_0$ and $\mathrm{I}\Delta_0 + \mathrm{Exp}$ a hierarchy of theories is considered in the literature, which has close connections with computational complexity. Define the function ω_m to be $\omega_m(x) = \exp^m\big((\log^m x)\cdot(\log^m x)\big)$. It is customary to define this function by induction: $\omega_0(x) = x^2$ and $\omega_{\mathrm{n}+1}(x) = \exp(\omega_{\mathrm{n}}(\log x))$. Let Ω_{m} express the totality of ω_{m} (i.e., $\Omega_{\mathrm{m}} \equiv \forall x \exists y[y = \omega_{\mathrm{m}}(x)]$).

By Gödel's coding method, we are now rest assured that the concepts introduced in the pervious section all can be formalized (and arithmetized) in the language of arithmetic. But we need just a bit more; and that is an "effective" coding, suitable for bounded arithmetic. The one we adopt here is taken from Chapter V of [7]. For convenience, and shortening the computations, we introduce the $\mathcal{P}$ notation.

Definition 14 We say x is of $\mathcal{P}(y)$, when the code of x is bounded above by a polynomial of y; and we write this as $\ulcorner x \urcorner \leqslant \mathcal{P}(y)$, meaning that for some n the inequality $\ulcorner x \urcorner \leqslant y^n + n$ holds. ⎔

Let us note that $X \leqslant \mathcal{P}(Y)$ is equivalent to the old (more familiar) $O-$notation "$\log X \in \mathcal{O}(\log Y)$". Here we collect some very basic facts about this fixed efficient coding that will be needed later.

Remark 15 Let A be a set or a sequence of terms, and let $|A|$ denote the cardinality of A, when A is a set, and the same $|A|$ denote the length of A, when A is a sequence. Then

- $\ulcorner \langle \alpha \rangle \urcorner \leqslant 9(\ulcorner \alpha \urcorner + 1)^2$ (Lemma 3.7.2 page 297 of [7]);
- $\ulcorner A \frown B \urcorner \ (\ulcorner A \cup B \urcorner) \leqslant 64 \cdot (\ulcorner A \urcorner \cdot \ulcorner B \urcorner)$ (Proposition 3.29 page 311 of [7]);
- $(|A|) \leqslant (\log \ulcorner A \urcorner)$ (Definition 3.27 and Section (e) pages 304–310 of [7]);

where $\ulcorner A \frown B \urcorner$ is the concatenation of (the sequences) A and B. ⎔

If we let $\mathcal{L}_A^{\mathrm{Sk}}$ to be the closure of $\mathcal{L}_A$ under Skolem function and constant symbols, i.e., let $\mathcal{L}_A^{\mathrm{Sk}}$ be the smallest set that contains $\mathcal{L}_A$ and for any $\mathcal{L}_A^{\mathrm{Sk}}-$formula $\exists x \phi(x)$ we have $\mathfrak{f}_{\exists x \phi(x)} \in \mathcal{L}_A^{\mathrm{Sk}}$, then this new countable language can also be re-coded, and this recoding can be generalized to $\mathcal{L}_A^{\mathrm{Sk}}-$terms and $\mathcal{L}_A^{\mathrm{Sk}}-$formulas. We wish to compute an upper bound for the codes of evaluations on a set of terms Λ. For a given Λ, all the atomic formulas, in the language $\mathcal{L}_A$, constructed from terms of Λ are either of the form $t = s$ or of the form $t \leqslant s$ for some $t, s \in \Lambda$. And every member of an evaluation p on Λ is an ordered pair like $\langle t = s, i \rangle$ or $\langle t \leqslant s, i \rangle$ for some $t, s \in \Lambda$ and $i \in \{0, 1\}$. Thus the code of any member of p is a constant multiple of $(\ulcorner t \urcorner \cdot \ulcorner s \urcorner)^2$, and so the code of p is bounded above by $\mathcal{P}(\prod_{t,s \in \Lambda} \ulcorner t \urcorner \cdot \ulcorner s \urcorner)$.

Lemma 16 *For a set of terms Λ and evaluation p on it, $\ulcorner p \urcorner \leqslant \mathcal{P}\left(\omega_1(\ulcorner \Lambda \urcorner)\right)$.*

Proof. It suffices, by the above remark and what was said afterward, to note that $\prod_{t,s \in \Lambda} \ulcorner t \urcorner \cdot \ulcorner s \urcorner = \prod_{t \in \Lambda} (\ulcorner t \urcorner)^{2|\Lambda|} = (\prod_{t \in \Lambda} \ulcorner t \urcorner)^{2|\Lambda|} \leqslant \mathcal{P}(\ulcorner \Lambda \urcorner)^{2 \log \ulcorner \Lambda \urcorner} \leqslant \mathcal{P}(\ulcorner \Lambda \urcorner^{\log \ulcorner \Lambda \urcorner})$ and that $\ulcorner \Lambda \urcorner^{\log \ulcorner \Lambda \urcorner} \leqslant \omega_1(\ulcorner \Lambda \urcorner)$. ⎔

Let us have another look at the above lemma, which is of great importance. For a set of terms Λ, there are $|\Lambda|$ terms in it (the cardinality of Λ). So, there are $2|\Lambda|^2$ atomic formulas constructed from the terms of Λ (atomic formulas of the form $t = s$ or $t \leqslant s$ for $t, s \in \Lambda$). And thus, there are $\exp(2|\Lambda|^2)$ different evaluations on the set Λ. And finally note that by $|\Lambda| \leqslant (\log \ulcorner \Lambda \urcorner)$ we get $\exp(2|\Lambda|^2) \leqslant \mathcal{P}\left(\exp((\log \ulcorner \Lambda \urcorner)^2)\right) \leqslant \mathcal{P}\left(\omega_1(\ulcorner \Lambda \urcorner)\right)$. So, in the presence of $\omega_1(\ulcorner \Lambda \urcorner)$ we have all the evaluations on Λ in our disposal.

All these concepts can be expressed in the language of arithmetic $\mathcal{L}_A$ by appropriate formulas. And "Herbrand Consistency of the theory T" can be arithmetized as "for every set of terms there exists an $T-$evaluation on it". Let $\mathrm{HCon}(T)$ denote the $\mathcal{L}_A-$formula "T is Herbrand consistent".

3 Herbrand Models

For a theory T, when Λ is the set of all terms (constructed from the function symbols of the language of T and also the Skolem function symbols of the formulas of T) any $T-$evaluation on Λ induces a model of T, which is called a *Herbrand model*. Here we use this notion for building up a definable inner model, which will constitute the hear of the proof of our main result for $\mathrm{I}\Delta_0 + \Omega_1$.

3.1 Arithmetically Definable Herbrand Models

In the sequel, we arithmetize Herbrand models.

Definition 17 Let $\mathcal{L}$ be a language and Λ be a set of (ground) terms (constructed by the Skolem constant and function symbols of $\mathcal{L}$).

Put $\Lambda^{\langle 0\rangle}=\Lambda$, and define inductively

$$\Lambda^{\langle k+1\rangle}=\Lambda^{\langle k\rangle}\cup\{f(t_1,\ldots,t_m)\mid f\in\mathcal{L}\ \&\ t_1,\ldots,t_m\in\Lambda^{\langle k\rangle}\}$$
$$\cup\{\mathfrak{f}_{\exists x\psi(x)}(t_1,\ldots,t_m)\mid \ulcorner\psi\urcorner\leqslant k\ \&\ t_1,\ldots,t_m\in\Lambda^{\langle k\rangle}\}.$$

Let $\Lambda^{\langle\infty\rangle}$ denote the union $\bigcup_{k\in\mathbb{N}}\Lambda^{\langle k\rangle}$.

Suppose p is an evaluation on $\Lambda^{\langle\infty\rangle}$. Define $\mathfrak{M}(\Lambda,p)=\{t/p\mid t\in\Lambda^{\langle\infty\rangle}\}$ and put the $\mathcal{L}-$structure on it by

- $f^{\mathfrak{M}(\Lambda,p)}(t_1/p,\ldots,t_m/p)=f(t_1,\ldots,t_m)/p$, and
- $R^{\mathfrak{M}(\Lambda,p)}=\{(t_1/p,\ldots,t_m/p)\mid p\models R(t_1,\ldots,t_m)\}$;

for $f,R\in\mathcal{L}$ and $t_1,\ldots,t_m\in\Lambda^{\langle\infty\rangle}$. ⎔

Lemma 18 The definition of $\mathcal{L}-$structure on $\mathfrak{M}(\Lambda,p)$ is well-defined, and when p is an $T-$evaluation on $\Lambda^{\langle\infty\rangle}$, for an $\mathcal{L}-$theory T, then $\mathfrak{M}(\Lambda,p)\models T$.

Proof. That the definitions of $f^{\mathfrak{M}(\Lambda,p)}$ and $R^{\mathfrak{M}(\Lambda,p)}$ are well-defined follows directly from the definition of an evaluation (Definition 5). By the definition of $\Lambda^{\langle\infty\rangle}$ the structure $\mathfrak{M}(\Lambda,p)$ is closed under all the Skolem functions of $\mathcal{L}$, and moreover it satisfies an atomic (or negated atomic) formula $A(t_1/p,\ldots,t_m/p)$ if and only if $p\models A(t_1,\ldots,t_m)$. Then it can be shown, by induction on the complexity of formulas, that for every RNNF formula ψ, we have $\mathfrak{M}(\Lambda,p)\models\psi$ whenever p satisfies all the available Skolem instances of ψ in $\Lambda^{\langle\infty\rangle}$. ⎔

We need an upper bound on the size (cardinal) and the code of $\Lambda^{\langle j\rangle}$.

Lemma 19 *The following inequalities hold when* $\ulcorner\Lambda\urcorner$ *and* $|\Lambda|$ *are sufficiently larger than* n*:*

(1) $|\Lambda^{\langle n\rangle}|\leqslant\mathcal{P}\left(|\Lambda|^{n!}\right)$*, and*

(2) $\ulcorner\Lambda^{\langle n\rangle}\urcorner\leqslant\mathcal{P}\Big(\big(\ulcorner\Lambda\urcorner\big)^{|\Lambda|^{(n+1)!}}\Big)$.

Proof. Denote $\ulcorner\Lambda^{\langle k\rangle}\urcorner$ by λ_k (thus $\ulcorner\Lambda\urcorner=\lambda_0=\lambda$) and $|\Lambda^{\langle k\rangle}|$ by σ_k (and thus $|\Lambda|=\sigma_0=\sigma$). We first note that $\sigma_{k+1}\leqslant\sigma_k+M\sigma_k^M+k\sigma_k^k$ for a fixed M. Thus $\sigma_{k+1}\leqslant\mathcal{P}(\sigma_k^{k+1})$, and then, by an inductive argument, we have $\sigma_n\leqslant\mathcal{P}(\sigma^{n!})$. For the second statement, we first compute an upper bound for the code of the Cartesian power A^m for a set A. By an argument similar to that of the proof of Lemma 16, we have $\ulcorner A^{k+1}\urcorner\leqslant\mathcal{P}\big(\prod_{t\in A^k\&s\in A}\ulcorner t\urcorner\cdot\ulcorner s\urcorner\big)\leqslant\mathcal{P}\big(\ulcorner A^{k}\urcorner\cdot\ulcorner A^{|A|^k}\urcorner\big)$, and thus $\ulcorner A^m\urcorner\leqslant\mathcal{P}\big(\ulcorner A^{|A|^m}\urcorner\big)$ can be shown by induction on m. Now we have $\lambda_{k+1}\leqslant\mathcal{P}\big(\ulcorner\Lambda^{\langle k\rangle}\urcorner\cdot\ulcorner(\Lambda^{\langle k\rangle})^M\urcorner\cdot\ulcorner(\Lambda^{\langle k\rangle})^k\urcorner\big)$ for a fixed M. So, $\lambda_{k+1}\leqslant\mathcal{P}\big(\lambda_k^{\sigma_k{}^k}\big)$ and finally our desired conclusion $\lambda_m\leqslant\mathcal{P}\big(\lambda^{\sigma^{(m+1)!}}\big)$ follows by induction. ⎔

Stating the above fact as a lemma, despite of the fact that it is indeed a crucial tool for our arguments, let us state the following corollary of it as a theorem, and later on we will use the theorem and will leave the lemma right here.

Theorem 20 *If for a set of terms Λ with non-standard $\ulcorner\Lambda\urcorner$ the value $\omega_2(\ulcorner\Lambda\urcorner)$ exists, then for a non-standard j the value $\ulcorner\Lambda^{\langle j\rangle}\urcorner$ will exist.*

Proof. There must exist a non-standard j such that $j \leqslant \log^4(\ulcorner\Lambda\urcorner)$. Thus $2(j+1)! \leqslant 2^{2^j} \leqslant \log^2\ulcorner\Lambda\urcorner$. Now, by Lemma 19 we can write $\ulcorner\Lambda^{\langle j\rangle}\urcorner \leqslant \mathcal{P}\left((\ulcorner\Lambda\urcorner)^{|\Lambda|^{(j+1)!}}\right) \leqslant \mathcal{P}\left((2^{2\log\ulcorner\Lambda\urcorner})^{(\log\ulcorner\Lambda\urcorner)^{(j+1)!}}\right) \leqslant$
$\mathcal{P}\left(\exp((\log\ulcorner\Lambda\urcorner)^{2(j+1)!})\right) \leqslant \mathcal{P}\left(\exp(\omega_1(\log\ulcorner\Lambda\urcorner))\right)$, or in other words $\ulcorner\Lambda^{\langle j\rangle}\urcorner \leqslant \mathcal{P}\left(\omega_2(\ulcorner\Lambda\urcorner)\right)$. ⊚

The reason that Theorem 20 is stated for non-standard Λ is that the set $\Lambda^{\langle\infty\rangle}$, needed for constructing the model $\mathfrak{M}(\Lambda, p)$, is not definable in $\mathcal{L}_A$. But the existence of the definable $\Lambda^{\langle j\rangle}$ for a non-standard j can guarantee the existence of $\Lambda^{\langle\infty\rangle}$ and thus of $\mathfrak{M}(\Lambda, p)$. This non-standard j exists for non-standard $\ulcorner\Lambda\urcorner$.

3.2 The Main Theorem for $\mathrm{I}\Delta_0 + \Omega_1$

Two interesting theorems were proved by Z. Adamowicz in [3] about Herbrand Consistency of the theories $\mathrm{I}\Delta_0 + \Omega_\mathrm{m}$ for $m \geqslant 2$:

Theorem 21 (Z. Adamowicz [3]) *For a bounded formula $\theta(\overline{x})$ and $m \geqslant 2$, if the theory $(\mathrm{I}\Delta_0 + \Omega_\mathrm{m}) + \exists\overline{x}\in\ell og^{m+1}\theta(\overline{x}) + \mathrm{HCon}_{\ell og^{m-2}}(\mathrm{I}\Delta_0 + \Omega_\mathrm{m})$ is consistent, then so is the theory $(\mathrm{I}\Delta_0 + \Omega_\mathrm{m}) + \exists\overline{x}\in\ell og^{m+2}\theta(\overline{x})$, where $\mathrm{HCon}_{\ell og^{m-2}}$ is the relativization of HCon to the cut ℓog^{m-2}.* ⎔

Theorem 22 (Z. Adamowicz [3]) *For any natural $m, n \geqslant 0$ there exists a bounded formula $\eta(\overline{x})$ such that $(\mathrm{I}\Delta_0 + \Omega_\mathrm{m}) + \exists\overline{x}\in\ell og^{n}\eta(\overline{x})$ is consistent, but $(\mathrm{I}\Delta_0 + \Omega_\mathrm{m}) + \exists\overline{x}\in\ell og^{n+1}\eta(\overline{x})$ is not consistent.* ⎔

These two theorems (by putting $n = m + 1$ for $m \geqslant 2$) imply together that for any $m \geqslant 2$:

$$\mathrm{I}\Delta_0 + \Omega_\mathrm{m} \nvdash \mathrm{HCon}_{\ell og^{m-2}}(\mathrm{I}\Delta_0 + \Omega_\mathrm{m}).$$

Here we extend Theorem 21 for $\mathrm{I}\Delta_0 + \Omega_1$, namely we show that

Theorem 23 *For any bounded formula $\theta(x)$, the consistency of the theory $(\mathrm{I}\Delta_0 + \Omega_1) + \exists x\in\ell og^2\theta(x) + \mathrm{HCon}(\mathrm{I}\Delta_0 + \Omega_1)$ implies the consistency of the theory $(\mathrm{I}\Delta_0 + \Omega_1) + \exists x\in\ell og^3\theta(x)$.*

The rest of this section is devoted to proving this theorem. Let us note that Theorem 22 holds already for $\mathrm{I}\Delta_0 + \Omega_1$, and below we reiterate the part that we need here:

Theorem 24 (Z. Adamowicz [3]) *There exists a bounded formula $\eta(\overline{x})$ such that the arithmetical theory $(\mathrm{I}\Delta_0 + \Omega_1) + \exists\overline{x}\in\ell og^2\eta(\overline{x})$ is consistent, but $(\mathrm{I}\Delta_0 + \Omega_1) + \exists\overline{x}\in\ell og^3\eta(\overline{x})$ is not consistent.* ⎔

Having proved the main theorem (23), we can immediately infer that

$$\mathrm{I}\Delta_0 + \Omega_1 \nvdash \mathrm{HCon}(\mathrm{I}\Delta_0 + \Omega_1).$$

As the proof of Theorem 23 is long, we will break it into a few lemmas. First we note that $\alpha\in\ell og^3$ if and only if there exists a sequence $\langle w_0, w_1, \cdots, w_\alpha\rangle$ of length $(\alpha+1)$ such that $w_0 = \exp^3(0) = 2^2$, and for any $j < \alpha$, $w_{j+1} = \omega_1(w_j)$. Noting that $\omega_1(\exp^3(j)) = \exp^3(j+1)$ one can then see that $w_\alpha = \exp^3(\alpha)$, and so $\alpha\in\ell og^3$. This can be formalized in $\mathrm{I}\Delta_0 + \Omega_1$ by an arithmetical formula. Note that the code of the above sequence is bounded by $\mathcal{P}(\prod_{j=0}^{j=\alpha} w_j) \leqslant \mathcal{P}\big(\exp(\sum_{j=0}^{j=\alpha}\exp^2(j))\big) \leqslant \mathcal{P}\left(\exp^3(\alpha+1)\right) \leqslant \mathcal{P}\left(\omega_1(\exp^3(\alpha))\right)$. So, in the presence of Ω_1, the existence of $\exp^3(\alpha)$ guarantees the existence of the above sequence of w_j's.

For proving Theorem 23 let us assume that we have a model

$$\mathcal{M} \models (\mathrm{I}\Delta_0 + \Omega_1) + \big(\alpha \in \ell og^2 \wedge \theta(\alpha)\big) + \mathrm{HCon}(\mathrm{I}\Delta_0 + \Omega_1),$$

for some bounded formula $\theta(x)$ and some non-standard $\alpha \in \mathcal{M}$, and then we construct a model

$$\mathcal{N} \models (\mathrm{I}\Delta_0 + \Omega_1) + \exists x \in \ell og^3 \theta(x).$$

If our language of arithmetic $\mathcal{L}_A$ contains the successor function $\mathfrak{s}$, then define the terms $\underline{j}$'s by induction: $\underline{0} = 0$, and $\underline{j+1} = \mathfrak{s}(\underline{j})$. If $\mathcal{L}_A$ does not contain $\mathfrak{s}$, then it should have the constant $\overline{1}$, and in this case we can put $\underline{j+1} = \underline{j} + \overline{1}$. The term $\underline{j}$ represents the (standard or non-standard) number j. For the sake of simplicity, assume $\mathfrak{w}$ denotes the Skolem function symbol $\mathfrak{f}_{\exists y(y=\omega_1(x))}$. Put $\mathsf{w}_0 = \underline{4}$ and inductively $\mathsf{w}_{j+1} = \mathfrak{w}(\mathsf{w}_j)$. Then w_k, in the theory $\mathrm{I}\Delta_0 + \Omega_1$, is the term which represents $\exp^3(k)$. Finally, put $\Lambda = \{\underline{0}, \ldots, \underline{\omega_1(\alpha)}, \mathsf{w}_0, \ldots, \mathsf{w}_\alpha\} = \{\underline{j} \mid j \leqslant \omega_1(\alpha)\} \cup \{\mathsf{w}_j \mid j \leqslant \alpha\}$. We can now estimate an upper bound for the code of Λ: $\ulcorner \Lambda \urcorner \leqslant \mathcal{P}\left(\prod_{j=1}^{j=\omega_1(\alpha)} 2^j\right) \leqslant \mathcal{P}\left(\exp(\omega_1(\alpha)^2)\right)$.

So Λ has a code in $\mathcal{M}$ (since $\mathcal{M} \models \alpha \in \ell og^2$), and moreover $\omega_2(\ulcorner \Lambda \urcorner)$ exists in $\mathcal{M}$, because $\omega_2(\ulcorner \Lambda \urcorner) \leqslant \mathcal{P}\left(\omega_2(\exp(\omega_1(\alpha)^2))\right) \leqslant \mathcal{P}\left(\exp(\omega_1(\omega_1(\alpha)^2))\right) \leqslant \mathcal{P}\left(\exp^2\left(4(\log \alpha)^4\right)\right) \leqslant \mathcal{P}\left(\exp^2(\alpha)\right)$.

Thus by Theorem 20 there exists a non-standard j such that $\Lambda^{\langle j \rangle}$ has a code in $\mathcal{M}$. Since by the assumption above we have $\mathcal{M} \models \mathrm{HCon}(\mathrm{I}\Delta_0 + \Omega_1)$, then there exists an $(\mathrm{I}\Delta_0 + \Omega_1)-$evaluation p on $\Lambda^{\langle j \rangle}$ (in $\mathcal{M}$). Now, by what was said after the proof of Theorem 20 one can construct the model $\mathfrak{M}(\Lambda, p) = \mathcal{N}$. By Lemma 18 we have $\mathcal{N} \models (\mathrm{I}\Delta_0 + \Omega_1)$, and also $\mathcal{N} \models \underline{\alpha}/p \in \ell og^3$ follows from the existence of w_j/p's. It remains (only) to show that $\mathfrak{M}(\Lambda, p) \models \theta(\underline{\alpha}/p)$.

For this purpose we prove the following lemmas where we assume that $\mathcal{M}$ is as above and there are some non-standard set of terms and evaluation Λ, p in $\mathcal{M}$ such that $\Lambda \supseteq \{\underline{0}, \ldots, \underline{\omega_1(\alpha)}\}$ for a non-standard $\alpha \in \mathcal{M}$, and p is an $\mathrm{I}\Delta_0-$evaluation on $\Lambda^{\langle \infty \rangle}$.

Lemma 25 *If $\mathfrak{M}(\Lambda, p) \models t/p \leqslant \underline{i}/p$ holds for a term t and $i \leqslant \omega_1(\alpha)$ in $\mathcal{M}$, then $\mathfrak{M}(\Lambda, p) \models t/p = \underline{j}/p$ for some $j \leqslant i$.*

Proof. By the assumption $\mathcal{M} \models$ "$p \models t \leqslant \underline{i}$". We prove by induction on i that there exists some $j \leqslant i$ in $\mathcal{M}$ such that $\mathcal{M} \models$ "$p \models t = \underline{j}$".

• For $i = 0$ by Example 11 the assumption $\mathcal{M} \models$ "$p \models t \leqslant 0$" implies $\mathcal{M} \models$ "$p \models t = 0$", noting that p is an $\mathrm{Q}-$evaluation on $\Lambda^{\langle \infty \rangle}$, and thus all the needed Skolem terms are in p's disposal.

• For $i + 1$ we have $\mathcal{M} \models$ "$p \models t \leqslant \underline{i} \bigvee t = \mathfrak{s}(\underline{i})$" by Example 11 and the assumed satisfaction $\mathcal{M} \models$ "$p \models t \leqslant \mathfrak{s}(\underline{i})$". Then if $\mathcal{M} \models$ "$p \models t = \mathfrak{s}(\underline{i})$" we are done, and if $\mathcal{M} \models$ "$p \models t \leqslant \underline{i}$" by the induction hypothesis there must exist some $j \leqslant i$ in $\mathcal{M}$ such that $\mathcal{M} \models$ "$p \models t = \underline{j}$". ⊚

Remark 26 The proof of the above lemma does not depend on the axioms of Q (and $\mathrm{I}\Delta_0$). Indeed, in some axiomatization of Q in the literature, the sentences

$$\psi = \forall x(x \leqslant 0 \to x = 0) \text{ and } \varphi = \forall x \forall y(x \leqslant \mathfrak{s}y \to x = \mathfrak{s}y \vee x \leqslant y) \text{ (see Example 11)}$$

are accepted as axioms. In our axiomatization, the above sentences were derivable theorems. In some axiomatizations of Q our axiom A_4 is replaced with $A_4' : \forall x, y(x \leqslant y \leftrightarrow \exists z[z + x = y])$; note the difference of $x + z$ in A_4 and $z + x$ in A_4' (see e.g. [7]). In this new axiomatization the sentence φ is not derivable. However, since we have $\mathrm{I}\Delta_0 \vdash \psi \wedge \varphi$, then by the argument of Remark 13, the above Lemma 25 can be proved by using the fact that p is an $\mathrm{I}\Delta_0-$evaluation on $\Lambda^{\langle \infty \rangle}$. ⊙

Lemma 27 *For any $\mathcal{L}_A-$term $t(x_1,\dots,x_m)$ and $i_1,\dots,i_m \leqslant \omega_1(\alpha)$, if $\mathcal{M} \models x \leqslant t(i_1,\dots,i_m)$ for some x, then for an $\mathcal{L}_A-$term $t'(x_1,\dots,x_k)$ and some $j_1,\dots,j_k \leqslant \omega_1(\alpha)$ we have $\mathcal{M} \models x = t'(j_1,\dots,j_k)$.*

Proof. By induction on (the complexity of) the term t.

• For $t = 0$ and $t = x_1$ the proof is straightforward.

• For $t = \mathfrak{s}u$ the assumption $\mathcal{M} \models x \leqslant \mathfrak{s}u(i_1,\dots,i_m)$ implies that either $\mathcal{M} \models x = \mathfrak{s}u(i_1,\dots,i_m)$ or $\mathcal{M} \models x \leqslant u(i_1,\dots,i_m)$ is true, and then the conclusion follows from the induction hypothesis.

• For $t = u+v$, and the assumption $\mathcal{M} \models x \leqslant u(i_1,\dots,i_m)+v(i_1,\dots,i_m)$, we consider two cases. First if $\mathcal{M} \models x \leqslant u(i_1,\dots,i_m)$ then we are done by the induction hypothesis. Second if $\mathcal{M} \models u(i_1,\dots,i_m) \leqslant x$ then there exists a y such that $\mathcal{M} \models x = u(i_1,\dots,i_m) + y$ and moreover $\mathcal{M} \models y \leqslant v(i_1,\dots,i_m)$. Now, by the induction hypothesis there are a term $t'(x_1,\dots,x_k)$ and some elements $j_1,\dots,j_k \leqslant \omega_1(\alpha)$ such that $\mathcal{M} \models y = t'(j_1,\dots,j_k)$. Whence we finally get the conclusion $\mathcal{M} \models x = u(i_1,\dots,i_m) + t'(j_1,\dots,j_k)$.

• For $t = u\cdot v$, by an argument similar to that of the previous case, we can assume $\mathcal{M} \models u(i_1,\dots,i_m) \leqslant x \leqslant u(i_1,\dots,i_m)\cdot v(i_1,\dots,i_m)$. There are some q, r such that $\mathcal{M} \models x = u(i_1,\dots,i_m)\cdot q + r$ and $\mathcal{M} \models r \leqslant u(i_1,\dots,i_m)$. We also have $\mathcal{M} \models q \leqslant v(i_1,\dots,i_m)$. By the induction hypothesis there are terms t', t'' and $j_1,\dots,j_k \leqslant \omega_1(\alpha)$ such that $\mathcal{M} \models q = t'(j_1,\dots,j_k) \bigwedge r = t''(j_1,\dots,j_k)$. Thus we finally have $\mathcal{M} \models x = u(i_1,\dots,i_m)\cdot t'(j_1,\dots,j_k) + t''(j_1,\dots,j_k)$. ⊚

Lemma 28 *For $i,j,k \leqslant \omega_1(\alpha)$ in $\mathcal{M}$ we have*

(1) if $i \leqslant j \leqslant \omega_1(\alpha)$ then $\mathfrak{M}(\Lambda,p) \models \underline{i}/p \leqslant \underline{j}/p$;

(2) if $i + j \leqslant \omega_1(\alpha)$ then $\mathfrak{M}(\Lambda,p) \models \underline{i}/p + \underline{j}/p = \underline{i+j}/p$;

(3) if $i\cdot j \leqslant \omega_1(\alpha)$ then $\mathfrak{M}(\Lambda,p) \models \underline{i}/p \cdot \underline{j}/p = \underline{i\cdot j}/p$.

Proof. We need to show for the $i, j \leqslant \omega_1(\alpha)$ that

(1) if $\mathcal{M} \models i \leqslant j$ then $\mathcal{M} \models$ "$p \models \underline{i} \leqslant \underline{j}$",

(2) if $\mathcal{M} \models i + j \leqslant \omega_1(\alpha)$ then $\mathcal{M} \models$ "$p \models \underline{i} + \underline{j} = \underline{i+j}$", and

(3) if $\mathcal{M} \models i\cdot j \leqslant \omega_1(\alpha)$ then $\mathcal{M} \models$ "$p \models \underline{i}\cdot\underline{j} = \underline{i\cdot j}$".

First we note that the statement (2) above implies already (1), since if we have $\mathcal{M} \models i \leqslant j$, then for some k we should have $\mathcal{M} \models i + k = j$, and then by (2), $\mathcal{M} \models$ "$p \models \underline{i} + \underline{k} = \underline{j}$" which implies (by A_4 of Q - see Example 11) that $\mathcal{M} \models$ "$p \models \underline{i} \leqslant \underline{j}$". By induction on j, very similarly to the proof of Lemma 25, one can prove the statements (2) and (3), noting that the evaluation p must satisfy the following axioms of Q:

$$A_5: \ \forall x(x+0=x); \qquad A_6: \forall x\forall y(x+\mathfrak{s}y = \mathfrak{s}(x+y));$$
$$A_7: \ \forall x(x\cdot 0 = 0); \qquad A_8: \forall x\forall y(x\cdot \mathfrak{s}y = x\cdot y + x).$$

⊚

Corollary 29 *Suppose for an $\mathcal{L}_A-$term $t(x_1,\dots,x_m)$ and some elements $i_1,\dots,i_m, i \leqslant \omega_1(\alpha)$, we have $\mathcal{M} \models t(i_1,\dots,i_m) = i$. Then we must also have $\mathfrak{M}(\Lambda,p) \models t(\underline{i_1}/p,\dots,\underline{i_m}/p) = \underline{i}/p$.*

Proof. By induction on t using Lemma 28. ⊚

Lemma 30 *Suppose $t(x_1,\ldots,x_m), t'(x_1,\ldots,x_m)$ are two $\mathcal{L}_A-$terms and $i_1,\ldots,i_m \leqslant \alpha^k$ are elements of $\mathcal{M}$ for some standard number $k\in\mathbb{N}$. Then, if $\mathcal{M}\models t(i_1,\ldots,i_m)=t'(i_1,\ldots,i_m)$ holds, $\mathfrak{M}(\Lambda,p)\models t(\underline{i_1}/p,\ldots,\underline{i_m}/p)=t'(\underline{i_1}/p,\ldots,\underline{i_m}/p)$ must hold too.*

Proof. By $i_1,\ldots,i_m\leqslant\alpha^k$ we have $t(i_1,\ldots,i_m)\leqslant\omega_1(\alpha)$. Put i be the common value $i=t(i_1,\ldots,i_m)=t'(i_1,\ldots,i_m)$. Then By Corollary 29 we have $\mathfrak{M}(\Lambda,p)\models t(\underline{i_1}/p,\ldots,\underline{i_m}/p)=\underline{i}/p=t'(\underline{i_1}/p,\ldots,\underline{i_m}/p)$. ⊚

Lemma 31 *Suppose $t(x_1,\ldots,x_m), t'(x_1,\ldots,x_m)$ are two $\mathcal{L}_A-$terms and $i_1,\ldots,i_m\leqslant\alpha^k$ are elements of $\mathcal{M}$ for some standard number $k\in\mathbb{N}$. If we have $\mathcal{M}\models t(i_1,\ldots,i_m)\leqslant t'(i_1,\ldots,i_m)$ then we must also have the satisfaction $\mathfrak{M}(\Lambda,p)\models t(\underline{i_1}/p,\ldots,\underline{i_m}/p)\leqslant t'(\underline{i_1}/p,\ldots,\underline{i_m}/p)$.*

Proof. Noting that $\mathrm{Q}\vdash\forall x,y\big(x\leqslant y\leftrightarrow\exists z(x+z=y)\big)$ by the assumption there exists an $\beta\in\mathcal{M}$ such that $\mathcal{M}\models t(i_1,\ldots,i_m)+\beta=t'(i_1,\ldots,i_m)$. On the other hand $\mathcal{M}\models\beta\leqslant t'(i_1,\ldots,i_m)$, so by Lemma 27 there exist a term u and some $j_1,\ldots,j_k\leqslant\omega_1(\alpha)$ such that $\mathcal{M}\models\beta=u(j_1,\ldots,j_k)$. Thus the equality $t(i_1,\ldots,i_m)+u(j_1,\ldots,j_k)=s(i_1,\ldots,i_m)$ holds in $\mathcal{M}$. Now, by Lemma 30,

$$\mathfrak{M}(\Lambda,p)\models t(\underline{i_1}/p,\ldots,\underline{i_m}/p)+u(\underline{j_1}/p,\ldots,\underline{j_k}/p)=t'(\underline{i_1}/p,\ldots,\underline{i_m}/p),$$

whence $t(\underline{i_1}/p,\ldots,\underline{i_m}/p)\leqslant t'(\underline{i_1}/p,\ldots,\underline{i_m}/p)$ is satisfied in $\mathfrak{M}(\Lambda,p)$. ⊚

Lemma 32 *Suppose $t(x_1,\ldots,x_m), t'(x_1,\ldots,x_m)$ are two $\mathcal{L}_A-$terms and $i_1,\ldots,i_m\leqslant\alpha^k$ are elements of $\mathcal{M}$ for some standard number $k\in\mathbb{N}$. If it is true that $\mathcal{M}\models t(i_1,\ldots,i_m)\neq t'(i_1,\ldots,i_m)$ then $\mathfrak{M}(\Lambda,p)\models t(\underline{i_1}/p,\ldots,\underline{i_m}/p)\neq t'(\underline{i_1}/p,\ldots,\underline{i_m}/p)$ must be true too. And if $\mathcal{M}\models t(i_1,\ldots,i_m)\not\leqslant t'(i_1,\ldots,i_m)$ then $\mathfrak{M}(\Lambda,p)\models t(\underline{i_1}/p,\ldots,\underline{i_m}/p)\not\leqslant t'(\underline{i_1}/p,\ldots,\underline{i_m}/p)$.*

Proof. It follows from Lemma 31 (and Remark 13) noting that p is an $\mathrm{I}\Delta_0-$evaluation on $\Lambda^{\langle\infty\rangle}$ and

$\mathrm{I}\Delta_0\vdash\forall x,y\big(x\neq y\longleftrightarrow\mathfrak{s}y\leqslant x\vee\mathfrak{s}x\leqslant y\big)$, and

$\mathrm{I}\Delta_0\vdash\forall x,y\big(x\not\leqslant y\longleftrightarrow\mathfrak{s}y\leqslant x\big)$. ⊚

Theorem 33 *Suppose $\psi(x_1,\ldots,x_m)$ is an open RNNF $\mathcal{L}_A-$formula and $i_1,\ldots,i_m\leqslant\alpha^k$ are elements of $\mathcal{M}$ for some standard number $k\in\mathbb{N}$. If we have $\mathcal{M}\models\psi(i_1,\ldots,i_m)$ then we also have $\mathfrak{M}(\Lambda,p)\models\psi(\underline{i_1}/p,\ldots,\underline{i_m}/p)$.*

Proof. Lemmas 30 and 31 prove the theorem for atomic formulas, and Lemma 32 proves it for negated atomic formulas. For the disjunctive and conjunctive compositions of those formulas one can prove the theorem by a simple induction. ⊚

Theorem 34 *Suppose that $\varphi(x_1,\ldots,x_m)$ is a bounded $\mathcal{L}_A-$formula and that $i_1,\ldots,i_m\leqslant\alpha^k$ are elements of $\mathcal{M}$ for some standard number $k\in\mathbb{N}$. If $\mathcal{M}\models\varphi(i_1,\ldots,i_m)$ then $\mathfrak{M}(\Lambda,p)\models\varphi(\underline{i_1}/p,\ldots,\underline{i_m}/p)$.*

Proof. Every bounded formula can be written as an (equivalent) RNNF formula. By Lemma 27 the range of bounded quantifiers of a formula whose all parameters belong to the set

$$\{t(i_1,\ldots,i_m)\mid i_1,\ldots,i_m\leqslant\alpha\ \&\ t\text{ is an }\mathcal{L}_A-\text{term}\}$$

is indeed that set again. Now the conclusion follows from Theorem 33.

► *An alternative proof:* To make this important theorem more clear, we sketch another proof, which is not really too different but has more model-theoretic flavor. Consider the above set again

$$\langle [0,\alpha]\rangle_{\mathcal{M}} = \{t(i_1,\ldots,i_m) \mid i_1,\ldots,i_m \leqslant \alpha \ \&\ t \text{ is an } \mathcal{L}_A - \text{term}\}$$

which is a subset of $\mathcal{M}$ closed under the successor, addition, and multiplication, and thus forms a submodel of $\mathcal{M}$ (generated by $[0,\alpha] = \{x \in \mathcal{M} \mid x \leqslant \alpha\}$). This submodel is an initial segment of $\mathcal{M}$ by Lemma 27. Hence, whenever $\mathcal{M} \models \varphi$, for a bounded formula φ with parameters in $[0,\alpha]$, then $\langle [0,\alpha]\rangle_{\mathcal{M}} \models \varphi$.

Now, similarly, the set

$$\langle [\underline{0}/p, \underline{\alpha}/p]\rangle_{\mathcal{N}} = \{t(\underline{i_1}/p,\ldots,\underline{i_m}/p) \mid i_1,\ldots,i_m \leqslant \alpha \ \&\ t \text{ is an } \mathcal{L}_A - \text{term}\}$$

is an initial segment and a submodel of $\mathcal{N} = \mathfrak{M}(\Lambda, p)$. Thus if $\langle [\underline{0}/p, \underline{\alpha}/p]\rangle_{\mathcal{N}} \models \varphi$, where φ is a bounded formula with parameters in $[\underline{0}/p, \underline{\alpha}/p]$, then $\mathfrak{M}(\Lambda, p) \models \varphi$. Finally, we note that the mapping $t(i_1,\ldots,i_m) \mapsto t(\underline{i_1}/p,\ldots,\underline{i_m}/p)$ defines a bijection between $\langle [0,\alpha]\rangle_{\mathcal{M}}$ and $\langle [\underline{0}/p, \underline{\alpha}/p]\rangle_{\mathcal{N}}$ which is also an isomorphism by Lemmas 30, 31 and 32. So the proof of the theorem goes as follows:

If $\mathcal{M} \models \varphi(i_1,\ldots,i_m)$ then $\langle [0,\alpha]\rangle_{\mathcal{M}} \models \varphi(i_1,\ldots,i_m)$, so $\langle [\underline{0}/p, \underline{\alpha}/p]\rangle_{\mathcal{N}} \models \varphi(\underline{i_1}/p,\ldots,\underline{i_m}/p)$ hence $\mathfrak{M}(\Lambda, p) \models \varphi(\underline{i_1}/p,\ldots,\underline{i_m}/p)$. ⊚

Corollary 35 *By the above assumptions,* $\mathfrak{M}(\Lambda, p) \models \theta(\underline{\alpha}/p)$. ⊙

Let us summarize what was argued in the last few pages.

Proof. (Of Theorem 23.) By the assumption of the theorem, the theory $(\mathrm{I}\Delta_0 + \Omega_1) + \exists x \in \ell og^2 \theta(x) + \mathrm{HCon}(\mathrm{I}\Delta_0 + \Omega_1)$ is consistent. So there is a model

$$\mathcal{M} \models (\mathrm{I}\Delta_0 + \Omega_1) + \big(\alpha \in \ell og^2 \wedge \theta(\alpha)\big) + \mathrm{HCon}(\mathrm{I}\Delta_0 + \Omega_1),$$

where $\alpha \in \mathcal{M}$. We wish to show the consistency of $(\mathrm{I}\Delta_0 + \Omega_1) + \exists x \in \ell og^3 \theta(x)$ by constructing another model

$$\mathcal{N} \models (\mathrm{I}\Delta_0 + \Omega_1) + \exists x \in \ell og^3 \theta(x).$$

If α is standard (i.e., $\alpha \in \mathbb{N}$) then one can take $\mathcal{N} = \mathcal{M}$. But if $\alpha \in \mathcal{M}$ is non-standard, then we proceed as follows: Take Λ to be the following set of terms: $\Lambda = \{\underline{j} \mid j \leqslant \omega_1(\alpha)\} \cup \{\mathsf{w}_j \mid j \leqslant \alpha\}$ in which the terms $\underline{j}$'s and w_j's are defined inductively as $\underline{0} = 0$, $\underline{j+1} = \mathfrak{s}\underline{j}$; and $\mathsf{w}_0 = \underline{4}$, $\mathsf{w}_{j+1} = \mathfrak{w}(\mathsf{w}_j)$. Here $\mathfrak{s}$ is the successor function, and $\mathfrak{w}$ denotes the Skolem function symbol $\mathfrak{f}_{\exists y(y=\omega_1(x))}$. Now $\omega_2(\ulcorner\Lambda\urcorner)$ is of order (far less than) 2^{2^α} which exists by the assumption $\mathcal{M} \models \alpha \in \ell og^2$. Then by Theorem 20 for a non-standard j the set of terms $\Lambda^{\langle j\rangle}$ has a code in $\mathcal{M}$. Thus the assumption $\mathcal{M} \models \mathrm{HCon}(\mathrm{I}\Delta_0 + \Omega_1)$ implies that there must exists an $(\mathrm{I}\Delta_0 + \Omega_1)-$evaluation p on $\Lambda^{\langle j\rangle}$. Then one can form the model $\mathcal{N} = \mathfrak{M}(\Lambda, p)$. Now $\mathcal{N} \models \mathrm{I}\Delta_0 + \Omega_1$ by Lemma 18, and also $\mathcal{N} \models \underline{\alpha}/p \in \ell og^3$ by the definition of w_α. Finally, $\mathcal{N} \models \theta(\underline{\alpha}/p)$ by Corollary 35 (of Theorem 34). Whence $\mathcal{N}$ is a model of the theory $(\mathrm{I}\Delta_0 + \Omega_1) + \exists x \in \ell og^3 \theta(x)$; and this finishes the proof of its consistency. ⊚

4 Herbrand Consistency of $\mathrm{I}\Delta_0$

Our definition of Herbrand consistency is not best suited for $\mathrm{I}\Delta_0$: there are $\omega_1(\ulcorner\Lambda\urcorner)-$many evaluations on a given set of terms Λ. Though this may not seem a big problem in the first glance (one can change or modify the definition accordingly) but special care is needed for generalizing the results to the case of $\mathrm{I}\Delta_0$. In the first subsection we pinpoint the critical usages of Ω_1 and in the second subsection we tailor the definitions and theorems in a way that we can prove our main theorem for $\mathrm{I}\Delta_0$ finally.

4.1 Essentiality of Ω_1

We made an essential use of Ω_1 in the following parts of our arguments:

1- The totality of the ω_1 function was needed for the upper bound of the code of an evaluation on a given set of terms Λ. Namely, the code of any evaluation on Λ is of order $\omega_1(\ulcorner\Lambda\urcorner)$, see Lemma 16. And indeed there is no escape from this bound since, as it was explained after Lemma 16, there are $\exp(2|\Lambda|^2)$ evaluations on Λ, and if $|\Lambda| \approx \log\ulcorner\Lambda\urcorner$ then there could exist as many as $\omega_1(\ulcorner\Lambda\urcorner)^2$ evaluations on Λ. So, if Ω_1 is not available, then there could be a large and non-standard set of terms Γ in a model $\mathcal{M}$ such that $\mathcal{M}$ cannot see all the evaluations on Γ. One of those evaluations could be a $T-$evaluation, that an end-extension of $\mathcal{M}$, say $\mathcal{K}$, can see. Then Γ is a Herbrand proof of contradiction in $\mathcal{M}$ because in $\mathcal{M}$'s view there is no $T-$evaluation on Γ. But there could be indeed a very large $T-$evaluation on Γ which $\mathcal{M}$ could not see, but $\mathcal{K}$ can. Thus the definition of HCon is deficient for $\mathrm{I}\Delta_0$ (where Ω_1 is not there) and one cannot consider all the set of terms; those for which the ω_1 of their codes exist, should be considered instead.

2- The second critical use of Ω_1 was in the definition of w_j's for shrinking the (double-)logarithmic witness $\mathcal{M} \models \alpha \in \ell og^2$ to $\mathcal{N} \models \mathsf{w}_\alpha/p \in \ell og^3$. There we constructed the sequence $\langle \mathsf{w}_0, \ldots, \mathsf{w}_\alpha \rangle$ of terms such that $\mathsf{w}_0 = \underline{4}$ and $\mathsf{w}_{j+1} = \mathfrak{w}(\mathsf{w}_j)$ where $\mathfrak{w}$ is the Skolem function symbol $\mathfrak{f}_{\exists y[y=\omega_1(x)]}$. And this was in our disposal because $\Omega_1 = \forall x \exists y[y = \omega_1(x)]$ was one of the axioms (of $\mathrm{I}\Delta_0 + \Omega_1$) and thus every $(\mathrm{I}\Delta_0 + \Omega_1)-$evluation must have satisfied $\mathfrak{w}(t) = \omega_1(t)$.

Note that we also required Λ to contain $\{\underline{j} \mid j \leqslant \omega_1(\alpha)\}$, but for this we did not need the existence of $\omega_1(\alpha)$; it was guaranteed by the assumption $\mathcal{M} \models \alpha \in \ell og^2$.

4.2 Tailoring for $\mathrm{I}\Delta_0$

Here we introduce the necessary modifications on the above two points.

4.2.1 The Definition of HCon*

The first point can be dealt with by tailoring the definition of HCon for $\mathrm{I}\Delta_0$:

Definition 36 A theory T is called Herbrand Consistent*, denoted symbolically as HCon$^*(T)$, when for all set of terms Λ, if $\omega_1(\ulcorner\Lambda\urcorner)$ exists then there is an $T-$evaluation on Λ. ⎔

This, obviously, can again be formalized in the language of arithmetic. The new definition cannot harm our arguments too much, because we needed HCon only for some special set of terms. And it was $\Lambda^{\langle j \rangle}$ for a non-standard j where $\Lambda = \{\underline{j} \mid j \leqslant \omega_1(\alpha)\} \cup \{\mathsf{w}_j \mid j \leqslant \alpha\}$. For constructing the model $\mathfrak{M}(\Lambda, p)$ we already needed the existence of $\omega_2(\ulcorner\Lambda\urcorner)$ (see the beginning of the proof of Theorem 23 before Lemma 25). Thus if we require the existence of $\omega_1(\ulcorner\Lambda\urcorner)$ in the definition of HCon*, then we will need the existence of $\omega_2(\ulcorner\Lambda\urcorner)$ later in the proof! Thus the first deficiency can be overcome.

4.2.2 The Cuts $\mathcal{I}$ and $\mathcal{J}$

In the absence of Ω_1 we cannot define the above sequence $\langle \mathsf{w}_0, \ldots, \mathsf{w}_\alpha \rangle$ satisfying $\mathsf{w}_{j+1} = \omega_1(\mathsf{w}_j)$. The most we can do inside $\mathrm{I}\Delta_0$ is to define a sequence like $\langle v_0, \ldots, v_\beta \rangle$ where $v_0 = m$ and $v_{j+1} = (v_j)^n$ for some fixed $m, n \in \mathbb{N}$. Then $v_\beta = a^{n2^\beta} \leqslant \mathcal{P}(\exp^2(\beta))$. Thus we cannot get anything larger than $\exp^2$, and so for shortening a witness we should start from ℓog and remain in the realm of ℓog^2. Indeed by the arguments

of the beginning of the proof of Theorem 23 before Lemma 25 we did not need the existence of $\exp^2(\alpha)$ for the existence of $\omega_2(\ulcorner\Lambda\urcorner)$. We needed only $\exp^2\big(4(\log\alpha)^4\big)$. Thus it seems natural to consider the cut $\mathcal{I} = \{x \mid \exists y[y = \exp^2\big(4(\log\alpha)^4\big)]\}$ and its logarithm $\mathcal{J} = \{x \mid \exists y[y = \exp^2\big(4\alpha^4\big)]\}$. We first note that Adamowicz's theorem (Theorem 22) holds for $\mathrm{I}\Delta_0$ and any $n \in \mathbb{N}$; i.e., there exists a bounded formula whose ℓog^n-witness cannot *consistently* be shortened to ℓog^{n+1}. Indeed this theorem holds for any cut I and its logarithm which is definition the cut $J = \{x \mid \exists y[y = \exp(x) \wedge y \in I]\}$. The only relation between ℓog^n and ℓog^{n+1} needed in the proof of Theorem 21 is that $2^x \in \ell og^n \iff x \in \ell og^{n+1}$; see [3]. And the proof works for any cut I and J which satisfy $\forall x(2^x \in I \iff x \in J)$. The cuts $\mathcal{I}$ and $\mathcal{J}$ defined above satisfy this as well $(\exp(x) \in \mathcal{I} \iff x \in \mathcal{J})$. So, we repeat Theorem 21 as:

Theorem 37 ([3]) *There exists a bounded formula $\eta(\overline{x})$ such that the theory $\mathrm{I}\Delta_0 + \exists\overline{x} \in \mathcal{I}\eta(\overline{x})$ is consistent, but $\mathrm{I}\Delta_0 + \exists\overline{x} \in \mathcal{J}\eta(\overline{x})$ is not consistent.* ⎔

4.2.3 The Main Theorem for $\mathrm{I}\Delta_0$

Let us note that the following theorem together with Theorem 37 prove that $\mathrm{I}\Delta_0 \nvdash \mathrm{HCon}^*(\mathrm{I}\Delta_0)$.

Theorem 38 *For any bounded formula $\theta(x)$, if the theory $\mathrm{I}\Delta_0 + \exists x \in \mathcal{I}\theta(x) + \mathrm{HCon}^*(\mathrm{I}\Delta_0)$ is consistent then so is the theory $\mathrm{I}\Delta_0 + \exists x \in \mathcal{J}\theta(x)$.*

Proof. Suppose the theory $\mathrm{I}\Delta_0 + \exists x \in \mathcal{I}\theta(x) + \mathrm{HCon}^*(\mathrm{I}\Delta_0)$ is consistent. So there exists a model

$$\mathcal{M} \models \mathrm{I}\Delta_0 + \big(\alpha \in \mathcal{I} \wedge \theta(\alpha)\big) + \mathrm{HCon}^*(\mathrm{I}\Delta_0),$$

where $\alpha \in \mathcal{M}$. We will show the consistency of $\mathrm{I}\Delta_0 + \exists x \in \mathcal{J}\theta(x)$ by constructing another model

$$\mathcal{N} \models \mathrm{I}\Delta_0 + \exists x \in \mathcal{J}\theta(x).$$

If α is standard (i.e., $\alpha \in \mathbb{N}$) then one can take $\mathcal{N} = \mathcal{M}$. But if $\alpha \in \mathcal{M}$ is non-standard, then we proceed as follows:

Let $\Upsilon = \{0, 0+0, 0^2, \mathfrak{c}, \mathfrak{c}^2, \mathfrak{c}^2+0, \mathfrak{sc}, \mathfrak{qc}, (\mathfrak{sc})^2, (\mathfrak{sc})^2+0\}$ where $\mathfrak{q}$ is the Skolem function symbol for the formula $\exists y(y \leqslant x^2 \wedge y = x^2)$ and $\mathfrak{c}$ is the Skolem constant symbol for the sentence (see Example 12)

$$\exists x\big(\exists w(w \leqslant x^2 \wedge w = x^2) \wedge \forall v(v \nleqslant (\mathfrak{s}x)^2 \wedge v \neq (\mathfrak{s}x)^2)\big).$$

We can use the argument of Example 12, since for the bounded formula $\psi(x) = \exists y \leqslant x^2(y = x \cdot x)$, the sentence ind_ψ is an axiom of the theory $\mathrm{I}\Delta_0$. Take $\Lambda = \Upsilon \cup \{\underline{j} \mid j \leqslant \omega_1(\alpha)\} \cup \{\mathsf{z}_j \mid j \leqslant 4\alpha^4\}$ in which the terms $\underline{j}$'s and z_j's are defined inductively as $\underline{0} = 0$, $\underline{j+1} = \mathfrak{s}\underline{j}$; and $\mathsf{z}_0 = \underline{2}$, $\mathsf{z}_{j+1} = \mathfrak{q}(\mathsf{z}_j)$. Now $\omega_2(\ulcorner\Lambda\urcorner)$ is of order $\exp^2\big(4(\log\alpha)^4\big)$ which exists by the assumption $\mathcal{M} \models \alpha \in \mathcal{I}$. Then by Theorem 20 for a non-standard j the set of terms $\Lambda^{\langle j\rangle}$ has a code in $\mathcal{M}$. Thus the assumption $\mathcal{M} \models \mathrm{HCon}^*(\mathrm{I}\Delta_0)$ implies that there must exists an $\mathrm{I}\Delta_0-$evaluation p on $\Lambda^{\langle j\rangle}$. Then one can form the model $\mathcal{N} = \mathfrak{M}(\Lambda, p)$. Now $\mathcal{N} \models \mathrm{I}\Delta_0$ by Lemma 18, and also $\mathcal{N} \models \underline{\alpha}/p \in \mathcal{J}$ by the definition of $\mathsf{z}_{4\alpha^4}$ (which represents $\exp^2(4\alpha^4)$). Note that $p \models \mathsf{z}_{j+1} = \mathsf{z}_j \cdot \mathsf{z}_j$ by the argument of Example 12, and also the code of the sequence $\langle \mathsf{z}_0, \ldots, \mathsf{z}_{4\alpha^4}\rangle$ is of order $\exp\big((4\alpha^4)^2\big) \leqslant \exp^2(4(\log\alpha)^4)$ which exists since $\alpha \in \mathcal{I}$. Finally, $\mathcal{N} \models \theta(\underline{\alpha}/p)$ by Corollary 35 (of Theorem 34). Whence $\mathcal{N}$ is a model of the theory $\mathrm{I}\Delta_0 + \exists x \in \mathcal{J}\theta(x)$; what proves its consistency. ⎔

5 Conclusions

An important property of Herbrand consistency of the theories $\mathrm{I}\Delta_0 + \Omega_1$ and $\mathrm{I}\Delta_0$ has been proved. That property immediately implies Gödel's second incompleteness theorem for the notion of Herbrand consistency in those theories. However, this version of Gödel's theorem has come a long way. The original presumption of Paris & Wilkie [10] asked for a proof of $\mathrm{I}\Delta_0 \nvdash \mathrm{CFCon}(\mathrm{I}\Delta_0)$, without specifying any variant of Cut-Free Consistency CFCon: "Presumably $\mathrm{I}\Delta_0 \nvdash \mathrm{CFCon}(\mathrm{I}\Delta_0)$ although we do not know this at present". Willard [14] solved this problem for the Tableau Consistency variant. Pudlák [11] asked a more specific question: "we know only that $T \nvdash \mathrm{HCon}(T)$ for T containing at least $\mathrm{I}\Delta_0 + \mathrm{Exp}$, for weaker theories it is an open problem". In [13] this problem was studied for the theories $\mathrm{I}\Delta_0 + \Omega_1$ and $\mathrm{I}\Delta_0$ (and a theory in between these two, namely $\mathrm{I}\Delta_0$ plus the totality of the $x \mapsto x^{\log^2 x}$ function). The proof of $\mathrm{I}\Delta_0 + \Omega_1 \nvdash \mathrm{HCon}(\mathrm{I}\Delta_0 + \Omega_1)$ given here was presented for the first time in Chapter 5 of [13]. But the unprovability of $\mathrm{HCon}(\mathrm{I}\Delta_0)$ in $\mathrm{I}\Delta_0$ was not as easy as it would have seemed. In Chapter 3 of [13] this unprovability was proved for a re-axiomatization of $\mathrm{I}\Delta_0$.

Our reason for using the induction formula ind_ψ, where $\psi(x)$ is the bounded formula $\exists y \leqslant x^2 (y = x^2)$, was having an Skolem function symbol for squaring $\mathfrak{q}(x) = x^2$. This way the Gödel code of $\mathfrak{q}(x)$ is $M \cdot \ulcorner x \urcorner$ for a fixed $M \in \mathbb{N}$, and thus the code of $\mathfrak{q}^n(x)$ is $M^n \cdot \ulcorner x \urcorner$ which is of order $\exp(n)$. So, we could code a term representing the number $x^{\exp(n)}$ $(= \mathfrak{q}^n(x))$ by a number of order $\exp(n)$. But if we coded the number $x^{\exp(n)}$ directly, that would be the code of $x \cdot x \cdot \ldots \cdot x$ (with $2^n -$ times x) which is of order $(\ulcorner x \urcorner)^{2^n}$ or $\exp^2(n)$. In that case, the code of the sequence $\langle \mathsf{z}_0, \ldots, \mathsf{z}_{4\alpha^4} \rangle$ would be of order $\exp^2((4\alpha^4)^2)$, but we used the order $\exp\big((4\alpha^4)^2\big)$ in the proof of Theorem 38 (since we had at most $\exp^2(4(\log \alpha)^4)$ in our disposal - which is far less than $\exp^2((4\alpha^4)^2)$). That way, we avoided accepting the totality of the squaring function $\Omega_0 : \forall x \exists y (y = x \cdot x)$ as an (additional) axiom.

This point deserves another look: define the terms $\{\mathsf{z}_i\}$, $\{\mathsf{u}_i\}$, and $\{\mathsf{v}_i\}$ inductively as $\mathsf{z}_0 = 2$, $\mathsf{z}_{j+1} = \mathfrak{q}(\mathsf{z}_j)$; $\mathsf{u}_0 = 2$, $\mathsf{u}_{j+1} = (\mathsf{z}_j)^2$; and $\mathsf{v}_0 = 2$, $\mathsf{v}_{j+1} = (\mathsf{v}_j)^2$. Then the codes of the terms z_n's and v_n's are of order $\mathcal{P}(2^n)$, but the code of v_n's are of order $\mathcal{P}(2^{2^n})$. On the other hand, the terms z_i, u_i and v_i have the same value (2^{2^i}) in any model of $\mathrm{I}\Delta_0$. In fact, for $i \leqslant \omega_1(\alpha)$ we have $\mathsf{z}_i \in \Lambda$ and also $\mathsf{u}_i \in \Lambda^{\langle 1 \rangle}$; but v_i's are too big to fit in small sets of terms.

Our treatment of Gödel's second incompleteness theorem for Herbrand consistency in weak arithmetics, can be summarized in the following improvements to the classical treatments (cf. the first paragraph of Appendix E in [15]):

(1) For Skolemizing a formula we did not transform it to a prenex normal form. This allowed a more efficient Skolemization and Herbrandization of formulas.

(2) Propositional satisfiability was achieved by evaluations, which are partial (Herbrand) models; see also [2, 3, 4, 8, 12, 13].

(3) For logarithmic shortening of bounded witnesses in $\mathrm{I}\Delta_0$, we could not go from ℓog to ℓog^2 directly. Instead we used the condition $\omega_1^2(x)^4 \in \ell og$ (equivalently $x \in \mathcal{I}$) to get to $4x^4 \in \ell og^2$ (equivalently $x \in \mathcal{J}$). For that we used the improved version of Adamowicz's theorem [3] (Theorem 37).

(4) And finally, we used the trick of ind_ψ to get an Skolem function symbol for the squaring function. Ideally, one would not use any induction axiom for proving a formula like $\Omega_0 : \forall x \exists y (y = x^2)$. This is an $\mathrm{Q}-$derivable sentence, and adding it as an axiom seems much more natural than proving it by an inductive argument. But, fortunately, there was a way of avoiding the acceptance of Ω_0 as an axiom, and that was proving its Π_1-equivalent $\forall x \exists y \leqslant x^2 (y = x^2)$ by induction on its bounded part $\exists y \leqslant x^2 (y = x^2)$ (see Example 12 and the proof of Theorem 38). That induction axiom could give us a free Skolem function symbol for the squaring operation, provided that we did not prenex normalize the induction axiom, and

instead Skolemize it more effectively − see point (1) above. Prenex normalizing and then Skolemizing the induction axioms can be so cumbersome that many would prefer avoiding them, but accepting new axioms instead! Trying to prenex normalize the induction axiom ind_ψ for $\psi = \exists y \leqslant x^2(y = x^2)$ in Example 12 can give a hint for its difficulty.

In the end, we conjecture that by using our coding techniques and definitions of Herbrand consistency, the results of L. A. Kołodziejczyk [8] can be generalized for showing the following unprovability:

Conjecture 39 $\bigcup_n(\mathrm{I}\Delta_0 + \Omega_n) \nvdash \mathrm{HCon}^*(\mathrm{I}\Delta_0)$.

Question 40 Can a Book proof (in the words of Paul Erdös) be given for Gödel's second incompleteness theorem $T \nvdash \mathcal{H}\mathfrak{Con}(T)$ for any theory $T \supseteq \mathrm{Q}$ and a canonical definition of Herbrand consistency $\mathcal{H}\mathfrak{Con}$?

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