# THE THEORY OF TRACIAL VON NEUMANN ALGEBRAS DOES NOT HAVE A MODEL COMPANION 

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#### Abstract

In this note, we show that the theory of tracial von Neumann algebras does not have a model companion. This will follow from the fact that the theory of any locally universal, $\mathrm{McDuff} \mathrm{II}_{1}$ factor does not have quantifier elimination. We also show how a positive solution to the Connes Embedding Problem implies that there can be no modelcomplete theory of $\mathrm{II}_{1}$ factors.


## 1. Introduction

The model theoretic study of operator algebras is at a relatively young stage in its development (although many interesting results have already been proven, see [7, [8, [9) and thus there are many foundational questions that need to be answered. In this note, we study the question that appears in the title: does the theory of tracial von Neumann algebras have a model companion? (Recall that a theory is said to be model-complete if every embedding between models of the theory is elementary and a model-complete theory $T^{\prime}$ is a model companion of a theory $T$ if every model of $T$ embeds into a model of $T^{\prime}$ and vice-versa.) We show that the answer to this question is: no! Indeed, we prove that a locally universal, McDuff $\mathrm{II}_{1}$ factor cannot have quantifier elimination. (See below for the definitions of locally universal and McDuff.) Since a model companion of the theory of tracial von Neumann algebras will have to be a model completion as well as the theory of a locally universal, McDuff $\mathrm{II}_{1}$ factor, the result follows.

We then pose a weaker question: can there exist a model-complete theory of $\mathrm{I}_{1}$ factors? Here, we show that a positive solution to the Connes Embedding Problem implies that the answer is once again: no!

Another motivation for this work came from considering independence relations in $\mathrm{II}_{1}$ factors. Although all $\mathrm{I}_{1}$ factors are unstable (see [7), it is still possible that there are other reasonably well-behaved independence relations to consider. Indeed, the independence relation stemming from conditional expectation is a natural candidate. In the end of this note, we show how the failure of quantifier elimination seems to pose serious hurdles in showing that conditional expectation yields a strict independence relation in the sense of [1.

[^0]We thank Dima Shlyakhtenko for patiently explaining Brown's work when we posed the question to him of the existence of non-extendable embeddings of pairs $\mathcal{M} \subset \mathcal{N}$ into $\mathcal{R}^{\omega}$. (See the proof of Theorem 2.1 below.)

Throughout, $\mathcal{L}$ denotes the signature for tracial von Neumann algebras and $\mathcal{R}$ denotes the hyperfinite $\mathrm{I}_{1}$ factor. We recall that $\mathcal{R}$ embeds into any $\mathrm{II}_{1}$ factor. We will say that a von Neumann algebra is $\mathcal{R}^{\omega}$-embeddable if it embeds into $\mathcal{R}^{\mathcal{U}}$ for some $\mathcal{U} \in \beta \mathbb{N} \backslash \mathbb{N}$. If $M$ is $\mathcal{R}^{\omega}$ embeddable, then $M$ embeds into $\mathcal{R}^{\mathcal{U}}$ for all $\mathcal{U} \in \beta \mathbb{N} \backslash \mathbb{N}$; see Corollary 4.15 of [8]. For this reason, we fix $\mathcal{U} \in \beta \mathbb{N} \backslash \mathbb{N}$ throughout this note.

## 2. Model Companions

In the proof of our first theorem, we use the crossed product construction for von Neumann algebras; a good reference is [4, Chapter 4].

Theorem 2.1. $\operatorname{Th}(\mathcal{R})$ does not have quantifier elimination.
Proof. It is enough to find separable, $\mathcal{R}^{\omega}$-embeddable tracial von Neumann algebras $M \subset N$ and an embedding $\pi: M \rightarrow \mathcal{R}^{\mathcal{U}}$ that does not extend to an embedding $N \rightarrow \mathcal{R}^{\mathcal{U}}$. Indeed, if this is so, let $N_{1}$ be a separable model of $\operatorname{Th}(\mathcal{R})$ containing $N$. Then $\pi$ does not extend to an embedding $N_{1} \rightarrow \mathcal{R}^{\mathcal{U}}$; since $\mathcal{R}^{\mathcal{U}}$ is $\aleph_{1}$-saturated, this shows that $\operatorname{Th}(\mathcal{R})$ does not have QE.

In order to achieve the goal of the above paragraph, we claim that it is enough to find a countable discrete group $\Gamma$ such that $L(\Gamma)$ is $\mathcal{R}^{\omega}$-embeddable, an embedding $\pi: L(\Gamma) \rightarrow \mathcal{R}^{\mathcal{U}}$, and $\alpha \in \operatorname{Aut}(L(\Gamma))$ such that there exists no unitary $u \in \mathcal{R}^{\mathcal{U}}$ satisfying $(\pi \circ \alpha)(x)=u \pi(x) u^{*}$ for all $x \in L(\Gamma)$. (We should remark that we are using the usual trace on $L(\Gamma)$ and that $\operatorname{Aut}(L(\Gamma))$ refers to the group of $*$-automorphisms preserving this trace.) First, we abuse notation and also use $\alpha$ to denote the homomorphism $\mathbb{Z} \rightarrow \operatorname{Aut}(L(\Gamma))$ which sends the generator of $\mathbb{Z}$ to the aforementioned $\alpha$. Set $\mathcal{M}=L(\Gamma)$ and $\mathcal{N}=\mathcal{M} \rtimes_{\alpha} \mathbb{Z}$. Then $N$ is a tracial von Neumann algebra. Moreover, we have that $\mathcal{N}$ is $\mathcal{R}^{\omega}$-embeddable if and only if $\mathcal{M}$ is-in fact, this is true for any crossed product algebra $\mathcal{M} \rtimes_{\alpha} G$ where $G$ is amenable [2, Prop. 3.4(2)]. Now suppose, towards a contradiction, that $\pi$ were to extend to an embed$\operatorname{ding} \widetilde{\pi}: \mathcal{N} \rightarrow \mathcal{R}^{\mathcal{U}}$. If $u \in L(\mathbb{Z}) \subset \mathcal{M} \rtimes_{\alpha} \mathbb{Z}$ is the generator of $\mathbb{Z}$, then setting $\tilde{u}=\tilde{\pi}(u) \in \mathcal{R}^{\mathcal{U}}$, we would have that $\tilde{u} \pi(x) \tilde{u}^{*}=\pi\left(u x u^{*}\right)=\pi(\alpha(x))$ for all $x \in \mathcal{M}$, contradicting the fact that $\pi \circ \alpha$ is not unitarily conjugate to the embedding $\pi$ in $\mathcal{R}^{\mathcal{U}}$.

An explicit construction of $\Gamma, \pi$ and $\alpha$ as above has already appeared in the work of N. P. Brown [6]. Indeed, by Corollary 6.11 of [6], we may choose $\Gamma=\operatorname{SL}(3, \mathbb{Z}) * \mathbb{Z}$ and $\alpha=\operatorname{id} * \theta$ for any nontrivial $\theta \in \operatorname{Aut}(L(\mathbb{Z}))$.

We say that a separable $\mathrm{II}_{1}$ factor $\mathcal{S}$ is locally universal if every separable $\mathrm{II}_{1}$ factor embeds into $\mathcal{S}^{\mathcal{U}}$. (By [8, Corollary 4.15], this notion is independent of $\mathcal{U}$.) In [9], it is shown that a locally universal $\mathrm{II}_{1}$ factor exists. The Connes Embedding Problems (CEP) asks whether $\mathcal{R}$ is locally universal.

We say that a separable $\mathrm{II}_{1}$ factor $M$ is $M c D u f f$ if $M \otimes \mathcal{R} \cong M$. For example, $\mathcal{R}$ is McDuff as is $M \otimes \mathcal{R}$ for any separable $\mathrm{II}_{1}$ factor $M$. By examining Brown's argument in [6], we see that the only properties of $\mathcal{R}$ that are used (other than it being finite) is that $L(\Gamma)$ (for $\Gamma$ as in the previous proof) is $\mathcal{R}^{\omega}$-embeddable and that $\mathcal{R}$ is McDuff . We thus have:

Corollary 2.2. If $\mathcal{S}$ is a locally universal, $M c D u f f ~ I I_{1}$ factor, then $\operatorname{Th}(\mathcal{S})$ does not have $Q E$.

Let $T_{0}$ be the theory of tracial von Neumann algebras in the signature $\mathcal{L}$. $T_{0}$ is a universal theory; see [8]. Let $T$ be the theory of $\mathrm{II}_{1}$ factors, a $\forall \exists$-theory by [8]. Moreover, since every tracial von Neumann algebra is contained in a $\mathrm{II}_{1}$ factor, we see that $T_{0}=T_{\forall}$. Thus, an existentially closed model of $T_{0}$ is a model of $T$.

By [9, Proposition 3.9], there is a set $\Sigma$ of $\forall \exists$-sentences in the language of tracial von Neumann algebras such that $M$ is McDuff if and only if $M \models \Sigma$. Since every $\mathrm{II}_{1}$ factor is contained in a $\mathrm{McDuff} \mathrm{II}_{1}$ factor ( as $M \subseteq M \otimes \mathcal{R}$ ), it follows that an existentially closed $\mathrm{II}_{1}$ factor is McDuff .

We can now prove our main result:
Theorem 2.3. $T_{0}$ does not have a model companion.
Proof. Suppose that $T$ is a model companion for $T_{0}$. Since $T_{0}$ is univerally axiomatizable and has the amalgamation property (see [4, Chapter 4]), we have that $T$ has QE.

Fix a separable model $\mathcal{S}$ of $T$. Then $\mathcal{S}$ is a locally universal $\mathrm{II}_{1}$ factor. Indeed, given an arbitrary separable $\mathrm{II}_{1}$ factor $M$, we have a separable model $\mathcal{S}_{1} \models T$ containing $M$. Since $\mathcal{S}^{\mathcal{U}}$ is $\aleph_{1}$-saturated, we have that $\mathcal{S}_{1}$ embeds into $\mathcal{S}^{\mathcal{U}}$, yielding an embedding of $M$ into $\mathcal{S}^{\mathcal{U}}$. Meanwhile, since $T$ is the theory of existentially closed models of $T_{0}$, we see that $\mathcal{S}$ is McDuff. Thus, by Corollary $2.2, T$ does not have QE, a contradiction.

## 3. Model Complete $\mathrm{II}_{1}$ Factors

While we have proven that the theory of tracial von Neumann algebras does not have a model companion, at this point it is still possible that there is a model complete theory of $\mathrm{II}_{1}$ factors. In this section, we show that a positive solution to the CEP implies that there is no model-complete theory of $\mathrm{II}_{1}$ factors.

We begin by observing the following:
Lemma 3.1. Every embedding $\mathcal{R} \rightarrow \mathcal{R}^{\omega}$ is elementary.
Proof. This follows from the fact that every embedding $\mathcal{R} \rightarrow \mathcal{R}^{\omega}$ is unitarily equivalent to the diagonal embedding; see [10].

Remark. The previous lemma shows that $\mathcal{R}$ is the unique prime model of its theory. Indeed, to show that $\mathcal{R}$ is a prime model of its theory, by Downward

Löwenheim-Skolem (DLS), it is enough to show that whenever $M \equiv \mathcal{R}$ is separable, then $\mathcal{R}$ elementarily embeds into $M$. Well, since $\mathcal{R}^{\mathcal{U}}$ is $\aleph_{1}$-saturated, we have that $M$ elementarily embeds into $\mathcal{R}^{\mathcal{U}}$. Composing an embedding $\mathcal{R} \rightarrow M$ with the elementary embedding $M \rightarrow \mathcal{R}^{\mathcal{U}}$ and applying Lemma 3.1, we see that the embedding $\mathcal{R} \rightarrow M$ is elementary.

Proposition 3.2. Suppose that $M$ is an $\mathcal{R}^{\omega}$-embeddable $I_{1}$ factor such that $\operatorname{Th}(M)$ is model-complete. Then $M \equiv \mathcal{R}$.

Proof. Without loss of generality, we may assume that $M$ is separable. Fix embeddings $\mathcal{R} \rightarrow M$ and $M \rightarrow \mathcal{R}^{\mathcal{U}}$. By Lemma 3.1, the composition

$$
\mathcal{R} \rightarrow M \rightarrow \mathcal{R}^{\mathcal{U}}
$$

is elementary. By DLS, we can take a separable elementary substructure $\mathcal{R}_{1}$ of $\mathcal{R}^{\mathcal{U}}$ such that $M$ embeds in $\mathcal{R}_{1}$; observe that the composition $\mathcal{R} \rightarrow M \rightarrow$ $\mathcal{R}_{1}$ is elementary. By DLS again, take a separable elementary substructure $M_{1}$ of $M^{\mathcal{U}}$ such that $\mathcal{R}_{1}$ embeds in $M_{1}$. We now repeat this process with $M_{1}$ : embed $M_{1}$ in $\mathcal{R}^{\mathcal{U}}$, take separable elementary substructure $\mathcal{R}_{2}$ of $\mathcal{R}^{\mathcal{U}}$ such that $M_{1}$ embeds in $\mathcal{R}_{2}$ and then embed $\mathcal{R}_{2}$ in a separable elementary substructure $M_{2}$ of $M^{\mathcal{U}}$. Iterate this construction countably many times, obtaining

$$
\mathcal{R} \rightarrow M \rightarrow \mathcal{R}_{1} \rightarrow M_{1} \rightarrow \mathcal{R}_{2} \rightarrow M_{2} \rightarrow \cdots,
$$

where each $\mathcal{R}_{n}$ is a separable elementary substructure of $\mathcal{R}^{\mathcal{U}}$ and each $M_{i}$ is a separable elementary substructure of $M^{\mathcal{U}}$. Set $\mathcal{R}_{\omega}=\bigcup_{n} \mathcal{R}_{n}=\bigcup_{n} M_{n}$. Then $\mathcal{R}$ is an elementary substructure of $\mathcal{R}_{\omega}$ since $\mathcal{R} \rightarrow \mathcal{R}_{1}$ is elementary and $\mathcal{R}_{n} \rightarrow \mathcal{R}_{n+1}$ is elementary for each $n \geq 1$. Meanwhile, observe that $M_{n} \equiv M$ for each $n$, so by model-completeness of $\operatorname{Th}(M)$, we have that the $M_{n}$ 's form an elementary chain, whence $M$ is an elementary substructure of $\mathcal{R}_{\omega}$. Consequently, $\mathcal{R} \equiv M$.

Remark 3.3. Proposition 3.2 provides immediate examples of non-model complete theories of $\mathrm{II}_{1}$ factors. Indeed, for $m \geq 2$, the von Neumann group algebra of the free group on $m$ generators, $L\left(\mathbb{F}_{m}\right)$, is $\mathcal{R}^{\omega}$-embeddable but not elementarily equivalent to $\mathcal{R}$ (see 3.2.2 in [9]), whence $\operatorname{Th}\left(L\left(\mathbb{F}_{m}\right)\right.$ ) is not model-complete. It is an outstanding problem in operator algebras whether or not $L\left(\mathbb{F}_{m}\right) \cong L\left(\mathbb{F}_{n}\right)$ for all $m, n \geq 2$. A weaker, but still seemingly difficult, question is whether or $\operatorname{not} L\left(\mathbb{F}_{m}\right) \equiv L\left(\mathbb{F}_{n}\right)$ for all $m, n \geq 2$. (An equivalent formulation of this question is whether or not there is $\mathcal{U} \in \beta \mathbb{N} \backslash \mathbb{N}$ such that $\left.L\left(\mathbb{F}_{m}\right)^{\mathcal{U}} \cong L\left(\mathbb{F}_{n}\right)^{\mathcal{U}}\right)$ ?) Suppose this latter question has an affirmative answer. Then we see that the theory of free group von Neumann algebras is not model-complete, mirroring the corresponding fact that the theory of free groups is not model-complete. However, the natural embeddings $\mathbb{F}_{m} \rightarrow \mathbb{F}_{n}$, for $m<n$, are elementary. Assuming $L\left(\mathbb{F}_{m}\right) \equiv L\left(\mathbb{F}_{n}\right)$, are the natural embeddings $L\left(\mathbb{F}_{m}\right) \rightarrow L\left(\mathbb{F}_{n}\right)$, for $m<n$, elementary?

Corollary 3.4. Assume that the CEP has a positive solution. Then there is no model-complete theory of $I I_{1}$ factors.

Proof. Suppose that $T$ is a model-complete theory of $\mathrm{II}_{1}$ factors. By the positive solution to the CEP and Proposition 3.2, $T=\operatorname{Th}(\mathcal{R})$. Meanwhile, a positive solution to the CEP implies that $T_{\forall}=T_{0}$, whence $T$ is a model companion for $T_{0}$, contradicting Theorem [2.3.

## 4. Concluding Remarks

Theorem 2.1 presents a major hurdle in trying to understand the model theory of $\mathrm{II}_{1}$ factors. In particular, it places a major roadblock in trying to understand potential independence relations in theories of $\mathrm{I}_{1}$ factors. Indeed, although any $\mathrm{II}_{1}$ factor is unstable (see [7]), one might wonder whether the natural notion of independence stemming from noncommutative probability theory might show that some $\mathrm{II}_{1}$ factor is (real) rosy (see [1] for the definition of rosy theory). More precisely, fix some "large" $\mathrm{II}_{1}$ factor $M$ and consider the relation $\downarrow$ on "small" subsets of $M$ given by $A \downarrow_{C} B$ if and only if, for all $a \in\langle A C\rangle, E_{\langle C\rangle}(a)=E_{\langle B C\rangle}(a)$. Here, $\langle *\rangle$ denotes the von Neumann subalgebra generated by $*$ and $E_{\langle *\rangle}$ is the conditional expectation (or orthogonal projection) map $E_{\langle *\rangle}: L^{2} M \rightarrow L^{2}\langle *\rangle$. In trying to verify some of the natural axioms for an independence relation (see [1]), one runs into trouble when trying to verify the extension axiom: If $B \subseteq C \subseteq D$ and $A \downarrow_{B} C$, can we find $A^{\prime}$ realizing the same type as $A$ over $C$ such that $A^{\prime} \downarrow_{B} D$ ? If $M=\mathcal{R}^{\mathcal{U}}$ and "small" means "countable," then it seems quite likely that one could find an $A^{\prime}$ with the same quantifier-free type as $A$ over $C$ that is independent from $D$ over $B$ as quantifier-free types are determined by moments. Without quantifier-elimination, it seems quite difficult to prove the extension property for this purported notion of independence. (The question of whether or not the independence relation arising from conditional expectation yields a strict independence relation was also discussed in [5].)

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