THE THEORY OF TRACIAL VON NEUMANN ALGEBRAS DOES NOT HAVE A MODEL COMPANION

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ABSTRACT. In this note, we show that the theory of tracial von Neumann algebras does not have a model companion. This will follow from the fact that the theory of any locally universal, McDuff II₁ factor does not have quantifier elimination. We also show how a positive solution to the Connes Embedding Problem implies that there can be no modelcomplete theory of II₁ factors.

1. INTRODUCTION

The model theoretic study of operator algebras is at a relatively young stage in its development (although many interesting results have already been proven, see [7],[8], [9]) and thus there are many foundational questions that need to be answered. In this note, we study the question that appears in the title: does the theory of tracial von Neumann algebras have a model companion? (Recall that a theory is said to be *model-complete* if every embedding between models of the theory is elementary and a model-complete theory T' is a *model companion* of a theory T if every model of T embeds into a model of T' and vice-versa.) We show that the answer to this question is: no! Indeed, we prove that a locally universal, McDuff II₁ factor cannot have quantifier elimination. (See below for the definitions of *locally universal* and *McDuff*.) Since a model companion of the theory of tracial von Neumann algebras will have to be a model completion as well as the theory of a locally universal, McDuff II₁ factor, the result follows.

We then pose a weaker question: can there exist a model-complete theory of II₁ factors? Here, we show that a positive solution to the *Connes Embedding Problem* implies that the answer is once again: no!

Another motivation for this work came from considering independence relations in II₁ factors. Although all II₁ factors are unstable (see [7]), it is still possible that there are other reasonably well-behaved independence relations to consider. Indeed, the independence relation stemming from conditional expectation is a natural candidate. In the end of this note, we show how the failure of quantifier elimination seems to pose serious hurdles in showing that conditional expectation yields a strict independence relation in the sense of [1].

Goldbring's work was partially supported by NSF grant DMS-1007144.

We thank Dima Shlyakhtenko for patiently explaining Brown's work when we posed the question to him of the existence of non-extendable embeddings of pairs $\mathcal{M} \subset \mathcal{N}$ into \mathcal{R}^{ω} . (See the proof of Theorem 2.1 below.)

Throughout, \mathcal{L} denotes the signature for tracial von Neumann algebras and \mathcal{R} denotes the hyperfinite II₁ factor. We recall that \mathcal{R} embeds into any II₁ factor. We will say that a von Neumann algebra is \mathcal{R}^{ω} -embeddable if it embeds into $\mathcal{R}^{\mathcal{U}}$ for some $\mathcal{U} \in \beta \mathbb{N} \setminus \mathbb{N}$. If M is \mathcal{R}^{ω} embeddable, then Membeds into $\mathcal{R}^{\mathcal{U}}$ for all $\mathcal{U} \in \beta \mathbb{N} \setminus \mathbb{N}$; see Corollary 4.15 of [8]. For this reason, we fix $\mathcal{U} \in \beta \mathbb{N} \setminus \mathbb{N}$ throughout this note.

2. Model Companions

In the proof of our first theorem, we use the crossed product construction for von Neumann algebras; a good reference is [4, Chapter 4].

Theorem 2.1. $\operatorname{Th}(\mathcal{R})$ does not have quantifier elimination.

Proof. It is enough to find separable, \mathcal{R}^{ω} -embeddable tracial von Neumann algebras $M \subset N$ and an embedding $\pi : M \to \mathcal{R}^{\mathcal{U}}$ that does not extend to an embedding $N \to \mathcal{R}^{\mathcal{U}}$. Indeed, if this is so, let N_1 be a separable model of $\operatorname{Th}(\mathcal{R})$ containing N. Then π does not extend to an embedding $N_1 \to \mathcal{R}^{\mathcal{U}}$; since $\mathcal{R}^{\mathcal{U}}$ is \aleph_1 -saturated, this shows that $\operatorname{Th}(\mathcal{R})$ does not have QE.

In order to achieve the goal of the above paragraph, we claim that it is enough to find a countable discrete group Γ such that $L(\Gamma)$ is \mathcal{R}^{ω} -embeddable, an embedding $\pi: L(\Gamma) \to \mathcal{R}^{\mathcal{U}}$, and $\alpha \in \operatorname{Aut}(L(\Gamma))$ such that there exists no unitary $u \in \mathcal{R}^{\mathcal{U}}$ satisfying $(\pi \circ \alpha)(x) = u\pi(x)u^*$ for all $x \in L(\Gamma)$. (We should remark that we are using the usual trace on $L(\Gamma)$ and that $Aut(L(\Gamma))$ refers to the group of *-automorphisms preserving this trace.) First, we abuse notation and also use α to denote the homomorphism $\mathbb{Z} \to \operatorname{Aut}(L(\Gamma))$ which sends the generator of \mathbb{Z} to the aforementioned α . Set $\mathcal{M} = L(\Gamma)$ and $\mathcal{N} = \mathcal{M} \rtimes_{\alpha} \mathbb{Z}$. Then N is a tracial von Neumann algebra. Moreover, we have that \mathcal{N} is \mathcal{R}^{ω} -embeddable if and only if \mathcal{M} is—in fact, this is true for any crossed product algebra $\mathcal{M} \rtimes_{\alpha} G$ where G is amenable [2, Prop. 3.4(2)]. Now suppose, towards a contradiction, that π were to extend to an embedding $\widetilde{\pi} : \mathcal{N} \to \mathcal{R}^{\mathcal{U}}$. If $u \in L(\mathbb{Z}) \subset \mathcal{M} \rtimes_{\alpha} \mathbb{Z}$ is the generator of \mathbb{Z} , then setting $\tilde{u} = \tilde{\pi}(u) \in \mathcal{R}^{\mathcal{U}}$, we would have that $\tilde{u}\pi(x)\tilde{u}^* = \pi(uxu^*) = \pi(\alpha(x))$ for all $x \in \mathcal{M}$, contradicting the fact that $\pi \circ \alpha$ is not unitarily conjugate to the embedding π in $\mathcal{R}^{\mathcal{U}}$.

An explicit construction of Γ , π and α as above has already appeared in the work of N. P. Brown [6]. Indeed, by Corollary 6.11 of [6], we may choose $\Gamma = SL(3,\mathbb{Z}) * \mathbb{Z}$ and $\alpha = id *\theta$ for any nontrivial $\theta \in Aut(L(\mathbb{Z}))$.

We say that a separable II₁ factor S is *locally universal* if every separable II₁ factor embeds into $S^{\mathcal{U}}$. (By [8, Corollary 4.15], this notion is independent of \mathcal{U} .) In [9], it is shown that a locally universal II₁ factor exists. The *Connes Embedding Problems* (CEP) asks whether \mathcal{R} is locally universal.

We say that a separable II₁ factor M is McDuff if $M \otimes \mathcal{R} \cong M$. For example, \mathcal{R} is McDuff as is $M \otimes \mathcal{R}$ for any separable II₁ factor M. By examining Brown's argument in [6], we see that the only properties of \mathcal{R} that are used (other than it being finite) is that $L(\Gamma)$ (for Γ as in the previous proof) is \mathcal{R}^{ω} -embeddable and that \mathcal{R} is McDuff. We thus have:

Corollary 2.2. If S is a locally universal, McDuff II₁ factor, then Th(S) does not have QE.

Let T_0 be the theory of tracial von Neumann algebras in the signature \mathcal{L} . T_0 is a universal theory; see [8]. Let T be the theory of II₁ factors, a $\forall \exists$ -theory by [8]. Moreover, since every tracial von Neumann algebra is contained in a II₁ factor, we see that $T_0 = T_{\forall}$. Thus, an existentially closed model of T_0 is a model of T.

By [9, Proposition 3.9], there is a set Σ of $\forall \exists$ -sentences in the language of tracial von Neumann algebras such that M is McDuff if and only if $M \models \Sigma$. Since every II₁ factor is contained in a McDuff II₁ factor (as $M \subseteq M \otimes \mathcal{R}$), it follows that an existentially closed II₁ factor is McDuff.

We can now prove our main result:

Theorem 2.3. T_0 does not have a model companion.

Proof. Suppose that T is a model companion for T_0 . Since T_0 is universally axiomatizable and has the amalgamation property (see [4, Chapter 4]), we have that T has QE.

Fix a separable model S of T. Then S is a locally universal II₁ factor. Indeed, given an arbitrary separable II₁ factor M, we have a separable model $S_1 \models T$ containing M. Since $S^{\mathcal{U}}$ is \aleph_1 -saturated, we have that S_1 embeds into $S^{\mathcal{U}}$, yielding an embedding of M into $S^{\mathcal{U}}$. Meanwhile, since T is the theory of existentially closed models of T_0 , we see that S is McDuff. Thus, by Corollary 2.2, T does not have QE, a contradiction.

3. Model Complete II_1 Factors

While we have proven that the theory of tracial von Neumann algebras does not have a model companion, at this point it is still possible that there is a model complete theory of II₁ factors. In this section, we show that a positive solution to the CEP implies that there is no model-complete theory of II₁ factors.

We begin by observing the following:

Lemma 3.1. Every embedding $\mathcal{R} \to \mathcal{R}^{\omega}$ is elementary.

Proof. This follows from the fact that every embedding $\mathcal{R} \to \mathcal{R}^{\omega}$ is unitarily equivalent to the diagonal embedding; see [10].

Remark. The previous lemma shows that \mathcal{R} is the unique prime model of its theory. Indeed, to show that \mathcal{R} is a prime model of its theory, by Downward

Löwenheim-Skolem (DLS), it is enough to show that whenever $M \equiv \mathcal{R}$ is separable, then \mathcal{R} elementarily embeds into M. Well, since $\mathcal{R}^{\mathcal{U}}$ is \aleph_1 -saturated, we have that M elementarily embeds into $\mathcal{R}^{\mathcal{U}}$. Composing an embedding $\mathcal{R} \to M$ with the elementary embedding $M \to \mathcal{R}^{\mathcal{U}}$ and applying Lemma 3.1, we see that the embedding $\mathcal{R} \to M$ is elementary.

Proposition 3.2. Suppose that M is an \mathcal{R}^{ω} -embeddable II_1 factor such that Th(M) is model-complete. Then $M \equiv \mathcal{R}$.

Proof. Without loss of generality, we may assume that M is separable. Fix embeddings $\mathcal{R} \to M$ and $M \to \mathcal{R}^{\mathcal{U}}$. By Lemma 3.1, the composition

$$\mathcal{R} \to M \to \mathcal{R}^{\mathcal{U}}$$

is elementary. By DLS, we can take a separable elementary substructure \mathcal{R}_1 of $\mathcal{R}^{\mathcal{U}}$ such that M embeds in \mathcal{R}_1 ; observe that the composition $\mathcal{R} \to M \to \mathcal{R}_1$ is elementary. By DLS again, take a separable elementary substructure M_1 of $M^{\mathcal{U}}$ such that \mathcal{R}_1 embeds in M_1 . We now repeat this process with M_1 : embed M_1 in $\mathcal{R}^{\mathcal{U}}$, take separable elementary substructure \mathcal{R}_2 of $\mathcal{R}^{\mathcal{U}}$ such that M_1 embeds in \mathcal{R}_2 and then embed \mathcal{R}_2 in a separable elementary substructure M_2 of $M^{\mathcal{U}}$. Iterate this construction countably many times, obtaining

$$\mathcal{R} \to M \to \mathcal{R}_1 \to M_1 \to \mathcal{R}_2 \to M_2 \to \cdots,$$

where each \mathcal{R}_n is a separable elementary substructure of $\mathcal{R}^{\mathcal{U}}$ and each M_i is a separable elementary substructure of $M^{\mathcal{U}}$. Set $\mathcal{R}_{\omega} = \bigcup_n \mathcal{R}_n = \bigcup_n M_n$. Then \mathcal{R} is an elementary substructure of \mathcal{R}_{ω} since $\mathcal{R} \to \mathcal{R}_1$ is elementary and $\mathcal{R}_n \to \mathcal{R}_{n+1}$ is elementary for each $n \geq 1$. Meanwhile, observe that $M_n \equiv M$ for each n, so by model-completeness of Th(M), we have that the M_n 's form an elementary chain, whence M is an elementary substructure of \mathcal{R}_{ω} . Consequently, $\mathcal{R} \equiv M$.

Remark 3.3. Proposition 3.2 provides immediate examples of non-model complete theories of II₁ factors. Indeed, for $m \ge 2$, the von Neumann group algebra of the free group on m generators, $L(\mathbb{F}_m)$, is \mathcal{R}^{ω} -embeddable but not elementarily equivalent to \mathcal{R} (see 3.2.2 in [9]), whence $\operatorname{Th}(L(\mathbb{F}_m))$ is not model-complete. It is an outstanding problem in operator algebras whether or not $L(\mathbb{F}_m) \cong L(\mathbb{F}_n)$ for all $m, n \ge 2$. A weaker, but still seemingly difficult, question is whether or not $L(\mathbb{F}_m) \equiv L(\mathbb{F}_n)$ for all $m, n \ge 2$. (An equivalent formulation of this question is whether or not there is $\mathcal{U} \in \beta \mathbb{N} \setminus \mathbb{N}$ such that $L(\mathbb{F}_m)^{\mathcal{U}} \cong L(\mathbb{F}_n)^{\mathcal{U}}$?) Suppose this latter question has an affirmative answer. Then we see that the theory of free group von Neumann algebras is not model-complete, mirroring the corresponding fact that the theory of free groups is not model-complete. However, the natural embeddings $\mathbb{F}_m \to \mathbb{F}_n$, for m < n, are elementary. Assuming $L(\mathbb{F}_m) \equiv L(\mathbb{F}_n)$, are the natural embeddings $L(\mathbb{F}_m) \to L(\mathbb{F}_n)$, for m < n, elementary?

Corollary 3.4. Assume that the CEP has a positive solution. Then there is no model-complete theory of II_1 factors.

Proof. Suppose that T is a model-complete theory of II₁ factors. By the positive solution to the CEP and Proposition 3.2, $T = \text{Th}(\mathcal{R})$. Meanwhile, a positive solution to the CEP implies that $T_{\forall} = T_0$, whence T is a model companion for T_0 , contradicting Theorem 2.3.

4. Concluding Remarks

Theorem 2.1 presents a major hurdle in trying to understand the model theory of II_1 factors. In particular, it places a major roadblock in trying to understand potential independence relations in theories of II_1 factors. Indeed, although any II_1 factor is unstable (see [7]), one might wonder whether the natural notion of independence stemming from noncommutative probability theory might show that some II_1 factor is (real) rosy (see [1] for the definition of rosy theory). More precisely, fix some "large" II_1 factor M and consider the relation \downarrow on "small" subsets of M given by $A \downarrow_C B$ if and only if, for all $a \in \langle AC \rangle$, $E_{\langle C \rangle}(a) = E_{\langle BC \rangle}(a)$. Here, $\langle * \rangle$ denotes the von Neumann subalgebra generated by * and $E_{\langle * \rangle}$ is the conditional expectation (or orthogonal projection) map $E_{\langle * \rangle} : L^2 M^{'} \to L^2 \langle * \rangle$. In trying to verify some of the natural axioms for an independence relation (see [1]), one runs into trouble when trying to verify the extension axiom: If $B \subseteq C \subseteq D$ and $A igsquarpoint_B C$, can we find A' realizing the same type as A over C such that $A' \perp_B D$? If $M = \mathcal{R}^{\mathcal{U}}$ and "small" means "countable," then it seems quite likely that one could find an A' with the same quantifier-free type as A over C that is independent from D over B as quantifier-free types are determined by moments. Without quantifier-elimination, it seems quite difficult to prove the extension property for this purported notion of independence. (The question of whether or not the independence relation arising from conditional expectation yields a strict independence relation was also discussed in [5].)

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