Journal of Information Processing Vol.22 No.1 40–43 (Jan. 2014)

[DOI: 10.2197/ipsjjip.22.40]

Technical Note

Yosenabe is NP-complete

Chuzo Iwamoto[1,a)]

Received: July 30, 2013, Accepted: September 13, 2013

Abstract: Yosenabe is one of Nikoli's pencil puzzles, which is played on a rectangular grid of cells. Some of the cells are colored gray, and two gray cells are considered *connected* if they are adjacent vertically or horizontally. A set of connected gray cells is called a *gray area*. Some of the gray areas are labeled by numbers, and some of the non-gray cells contain circles with numbers. The object of the puzzle is to draw arrows, vertically or horizontally, from all circles to gray areas so that (i) the arrows do not bend, and do not cross other circles or lines of other arrows, (ii) the number in a gray area is equal to the total of the numbers of the circles which enter the gray area, and (iii) gray areas with no numbers may have any sum total, but at least one circle must enter each gray area. It is shown that deciding whether a Yosenabe puzzle has a solution is NP-complete.

Keywords: NP-complete, computational complexity, one-player game, Yosenabe

1. Introduction

Yosenabe is one of Nikoli's pencil puzzles [20], which is played on a rectangular grid of cells (see **Fig. 1** (a)). Some of the cells are colored gray, and two gray cells are considered *connected* if they are adjacent vertically or horizontally. A set of connected gray cells is called a *gray area*, which is regarded as a "deep pot" ("Nabe (鍋)" in Japanese). Some of the gray areas are labeled by numbers, and the remaining gray areas have no numbers. Some of the non-gray cells contain circles with numbers, where circles are regarded as "ingredients" ("Guzai (具材)" in Japanese). The Japanese word "Yosenabe (寄せ鍋)" means a "mixed stew."

The object of the puzzle is to draw arrows, vertically or horizontally, from all circles to gray areas (see Fig. 1 (b)) so that (i) the arrows do not bend, and do not cross other circles or lines of other arrows, (ii) the number in a gray area is equal to the total of the numbers of the circles which enter the gray area, and (iii) gray areas with no numbers may have any sum total, but at least one circle (arrow tip) must enter each gray area. It should be noted that only one arrow tip can enter a gray cell.

In this paper, it is shown that deciding whether a Yosenabe puzzle has a solution is NP-complete. The puzzle is trivially in NP, since the puzzle can be solved by drawing an arrow from every circle one by one.

There has been a huge amount of literature on the computational complexities of games and puzzles. In 2009, a survey of games, puzzles, and their complexities was reported by Hearn and Demaine [9]. After the publication of this book, the following Nikoli's pencil puzzles were shown to be NP-complete: Hashiwokakero [1], Kurodoko [15], Shakashaka [6], Shikaku and Ripple Effect [19], Yajilin and Country Road [11].

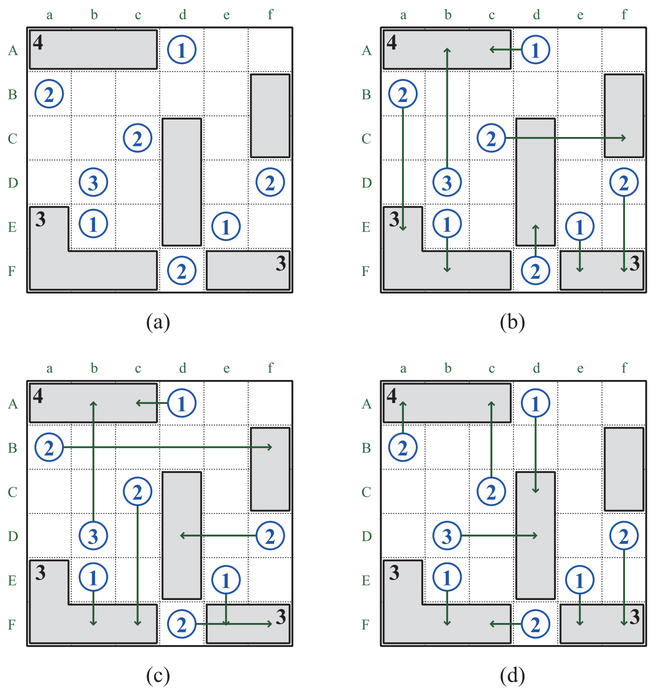


Fig. 1 (a) A Yosenabe puzzle. (b) A solution of (a). All circles are moved, vertically or horizontally, so that they enter the gray areas. The number in a gray area is equal to the total of the numbers of the circles which enter the gray area. At least one circle enters every gray area with no number. (c) is not a solution, since pairs of arrows intersect at cells (B, b) and (F, e). (d) is not a solution, since there is a gray area (on cells (B, f) and (C, f)) which does not receive any arrow.

Furthermore, it was also shown that Block Sum [8], Kaboozle [2], Magnet Puzzle [16], Pandemic [17], Shisen-Sho [14], String Puzzle [13], single-player UNO [5] and Zen Puzzle Garden [10] are NP-complete. As for higher complexity classes, Chat Noir [12], Rolling Block Maze [3], and two-player UNO [5] were shown to be PSPACE-complete.

1 Hiroshima University, Graduate School of Engineering, Higashi-Hiroshima, Hiroshima 739–8527, Japan

a) chuzo@hiroshima-u.ac.jp

2. NP-completeness of Yosenabe

We present a polynomial-time transformation from an arbitrary instance C of PLANAR 3SAT to a Yosenabe puzzle Y such that C is satisfiable if and only if Y has a solution.

2.1 PLANAR 3SAT Problem

The definition of PLANAR 3SAT is mostly from [LO1] of Ref. [7]. Let $U = \{x_1, x_2, \ldots, x_n\}$ be a set of Boolean *variables*. Boolean variables take on values 0 (false) and 1 (true). If x is a variable in U, then x and $\overline{x}$ are *literals* over U. The value of $\overline{x}$ is 1 (true) if and only if x is 0 (false). A *clause* over U is a set of literals over U, such as $\{\overline{x_1}, x_3, x_4\}$. It represents the disjunction of those literals and is *satisfied* by a truth assignment if and only if at least one of its members is true under that assignment.

An instance of PLANAR 3SAT is a collection $C = \{c_1, c_2, \ldots, c_m\}$ of clauses over U such that (i) $|c_j| \leq 3$ for each $c_j \in C$ and (ii) the bipartite graph $G = (V, E)$, where $V = U \cup C$ and E contains exactly those pairs $\{x, c\}$ such that either literal x or $\overline{x}$ belongs to the clause c, is planar.

The PLANAR 3SAT problem asks whether there exists some truth assignment for U that simultaneously satisfies all the clauses in C. This problem is known to be NP-complete. For example, $U = \{x_1, x_2, x_3, x_4\}$, $C = \{c_1, c_2, c_3, c_4\}$, and $c_1 = \{x_1, x_2, \overline{x_3}\}$, $c_2 = \{\overline{x_1}, \overline{x_2}, x_4\}$, $c_3 = \{\overline{x_1}, x_3, \overline{x_4}\}$, $c_4 = \{\overline{x_2}, \overline{x_3}, \overline{x_4}\}$ provide an instance of PLANAR 3SAT. In this instance, the answer is "yes," since there is a truth assignment $(x_1, x_2, x_3, x_4) = (1, 0, 1, 1)$ satisfying all clauses. It is known that PLANAR 3SAT is NP-complete even if each variable occurs exactly once positively and exactly twice negatively in C [4].

2.2 Transformation from an Instance of PLANAR 3SAT to a Yosenabe Puzzle

Each variable $x_i \in \{x_1, x_2, \ldots, x_n\}$ is transformed into the variable gadget (as illustrated in **Fig. 2** (a)), which consists of two gray areas having number 2 (labeled with x_i and $\overline{x_i}$), one circle with number 2 (labeled with r), and two pairs of circles with number 1 (labeled with s, t and u, v).

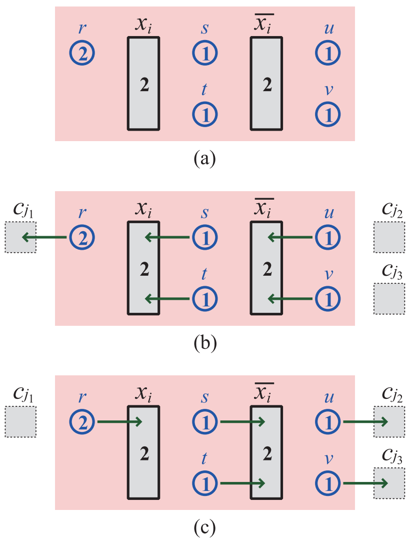


Fig. 2 (a) Variable gadget transformed from x_i. (b) If x_i is filled by s, t, then $\overline{x_i}$ must be filled by u, v. In this case, gray area c_{j_1} can be filled by r, but c_{j_2}, c_{j_3} cannot be filled by u, v. (c) If $\overline{x_i}$ is filled by s, t, then x_i must be filled by r. In this case, c_{j_2}, c_{j_3} can be filled by u, v, but c_{j_1} cannot be filled by r.

Suppose that there are gray areas c_{j_1} and c_{j_2}, c_{j_3} to the left and right sides of the variable gadget (see Fig. 2 (b)). If gray area x_i is filled by circles s, t, then gray area $\overline{x_i}$ must be filled by circles u, v. In this case, gray area c_{j_1} can be filled by circle r, but c_{j_2}, c_{j_3} cannot be filled by circles u, v. This configuration corresponds to $x_i = 1$.

On the other hand, if gray area $\overline{x_i}$ is filled by circles s, t (see Fig. 2 (c)), then gray area x_i must be filled by circle r. In this case, c_{j_2}, c_{j_3} can be filled by circles u, v, but c_{j_1} cannot be filled by r. This corresponds to $\overline{x_i} = 1$.

Figure 3 is the right-and-left turn gadget, which consists of two gray areas having number k (labeled with A and B) and three circles with number k (labeled with p, q, and r) , where $k \in \{1, 2\}$. If gray area c_j is filled by circle p, then gray areas A and B must be filled by circles q and r, respectively. Conversely, if x_i is filled by circle r, then gray areas B and A must be filled by circles q and p, respectively.

Figure 4 is a Yosenabe puzzle Y transformed from $C = \{c_1, c_2, c_3, c_4\}$ and $U = \{x_1, x_2, x_3, x_4\}$, where $c_1 = \{x_1, x_2, \overline{x_3}\}$, $c_2 = \{\overline{x_1}, \overline{x_2}, x_4\}$, $c_3 = \{\overline{x_1}, x_3, \overline{x_4}\}$, and $c_4 = \{\overline{x_2}, \overline{x_3}, \overline{x_4}\}$. If either literal x_i or $\overline{x_i}$ belongs to the clause $c_j \in C$, then the variable gadget for $x_i \in U$ is connected to gray area c_j via right-and-left turn gadgets.

Let $G = (V, E)$ be a graph, where $V = U \cup C$ and E contains exactly those pairs $\{x_i, c_j\}$ such that either literal x_i or $\overline{x_i}$ belongs to the clause c_j. Now one can see that E corresponds to *white regions* of Fig. 4, and $V = U \cup C$ corresponds to *red regions* (labeled with $x_i, \overline{x_i}$) and *gray rectangle areas* (labeled with c_j), respectively. *Regions labeled with color* $l \in \{3, 4, 5\}$ in Fig. 4 correspond to G's faces.

Regions labeled with color $l \in \{3, 4, 5\}$ are used as "walls" (see also **Fig. 5**), which are composed of pairs of gray areas (squares) with number l and circles having number l. For example, the center region labeled with color 3 in Fig. 4 is composed of seven gray areas (squares) with number 3 and seven circles having number 3 (see Fig. 5).

Each gray square of Fig. 5 must be filled by a *single* circle (and not by two or more circles), since it is composed of a single cell. The center region of color 3 (see Fig. 5) can be filled up with seven arrows connecting circles to gray squares in the region.

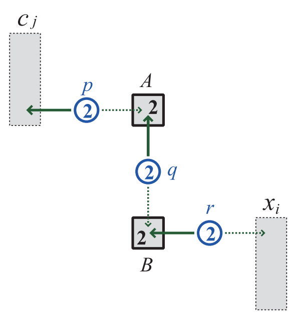


Fig. 3 Right-and-left turn gadget when $k = 2$. If c_j is filled by p, then A and B must be filled by q and r, respectively. If x_i is filled by r, then B and A must be filled by q and p, respectively.

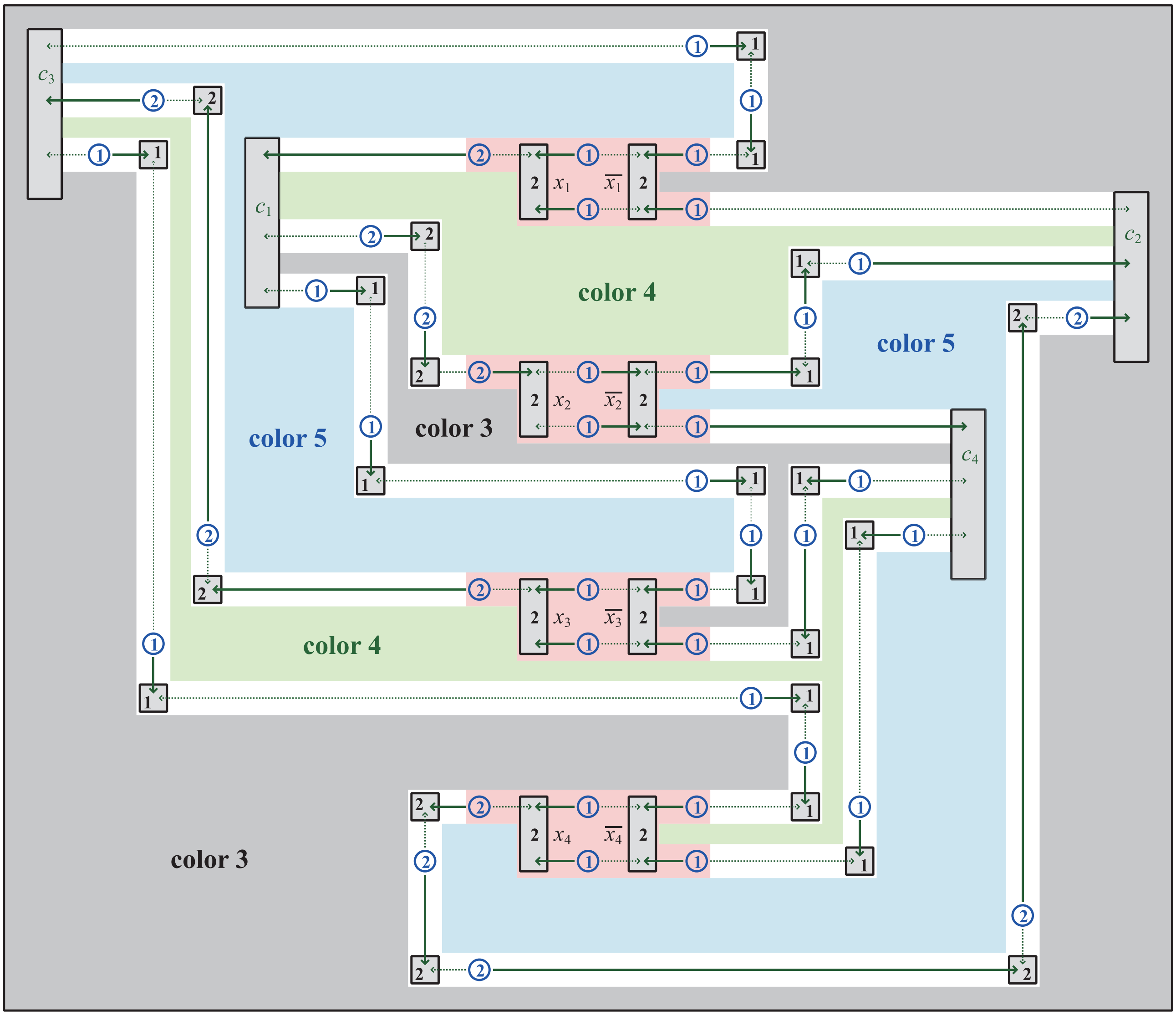


Fig. 4 Yosenabe puzzle Y transformed from $C = \{c_1, c_2, c_3, c_4\}$, where $c_1 = \{x_1, x_2, \overline{x_3}\}$, $c_2 = \{\overline{x_1}, \overline{x_2}, x_4\}$, $c_3 = \{\overline{x_1}, x_3, \overline{x_4}\}$, and $c_4 = \{\overline{x_2}, \overline{x_3}, \overline{x_4}\}$. From the solution indicated by solid arrows, one can see that the assignment $(x_1, x_2, x_3, x_4) = (1, 0, 1, 1)$ satisfies all clauses of C.

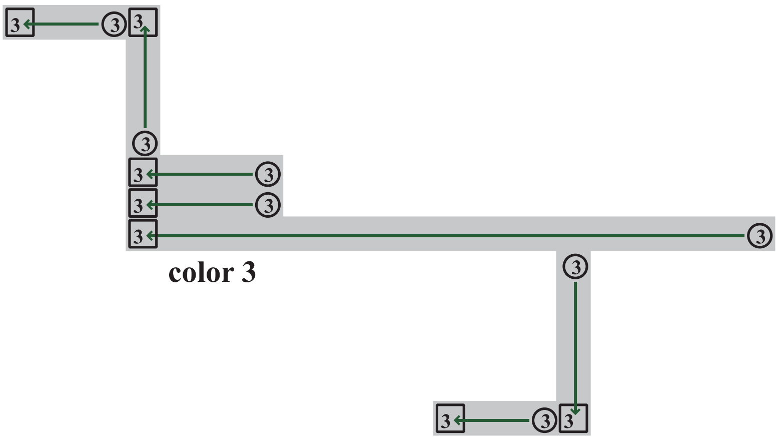


Fig. 5 The center region labeled with color 3 of Fig. 4 is composed of seven gray areas (squares) with number 3 and seven circles having number 3. This region will be filled up with seven arrows connecting circles to gray squares in the region.

By using the quadratic-time four-coloring algorithm [18], we can assign four colors $\{3, 4, 5, 6\}$ to every face of G so that no two adjacent faces have the same color. (In Fig. 4, color 6 is not used.) Since the maximum degree of G is three, such a coloring satisfies that no two faces sharing a single vertex have the same color. Therefore, any pair of regions having the same color are separated by a region having a different color. For example, in Fig. 4, there are two regions of color 3. The center region of color 3 is completely surrounded by colors 4 and 5, and is separated from the outer region of color 3.

Hence, every circle in a region of color $l \in \{3, 4, 5, 6\}$ has an arrow to a gray area (square) in the same region (see Fig. 5) (and not to a gray area (square) in a different region).

From this construction, the instance C of PLANAR 3SAT is satisfiable if and only if Yosenabe puzzle Y has a solution. From the solution indicated by solid arrows in Fig. 4, one can see that the assignment $(x_1, x_2, x_3, x_4) = (1, 0, 1, 1)$ satisfies all clauses of C.

References

[1] Andersson, D.: Hashiwokakero is NP-complete, *Inf. Process. Lett.*, Vol.109, pp.1145–1146 (2009).

[2] Asano, T., Demaine, E.D., Demaine, M.L. and Uehara, R.: NP-completeness of Generalized Kaboozle, *J. Inf. Process.*, Vol.20, No.3, pp.713–718 (2012).

[3] Buchin, K. and Buchin, B.: Rolling Block Mazes are PSPACE-complete, *J. Inf. Process.*, Vol.20, No.3, pp.719–722 (2012).

[4] Cerioli, M.R., Faria, L., Ferreira, T.O., Martinhon, C.A.J., Protti, F. and Reed, B.: Partition into Cliques for Cubic Graphs: Planar Case, Complexity and Approximation, *Discrete Appl. Math.*, Vol.156, No.12, pp.2270–2278 (2008).

[5] Demaine, E.D., Demaine, M.L., Uehara, R., Uno, T. and Uno, Y.: UNO is Hard, Even for a Single Player, *Proc. Fun with Algorithms* (*Lect. Notes Comput. Sc.*), Vol.6099, pp.133–144 (2010).

[6] Demaine, E.D., Okamoto, Y., Uehara, R. and Uno, Y.: Computational Complexity and an Integer Programming Model of Shakashaka, *Proc. 25th Canadian Conference on Computational Geometry* (2013) (online), available from ⟨http://cccg.ca/⟩ (accessed 2013-09-24).

[7] Garey, M.R. and Johnson, D.S.: *Computers and Intractability: A Guide to the Theory of NP-completeness*, W.H. Freeman, New York, NY, USA (1979).

[8] Haraguchi, K. and Ono, H.: BlockSum is NP-complete, *IEICE Trans. Inf. Syst.*, Vol.D96-D, No.3, pp.481–488 (2013).

[9] Hearn, R.A. and Demaine, E.D.: *Games, Puzzles, and Computation*, A K Peters Ltd., MA, USA (2009).

[10] Houston, R., White, J. and Amos, M.: Zen Puzzle Garden is NP-complete, *Inf. Process. Lett.*, Vol.112, pp.106–108 (2012).

[11] Ishibashi, A., Sato, Y. and Iwata, S.: NP-completeness of Two Pencil Puzzles: Yajilin and Country Road, *Utilitas Mathematica*, Vol.88, pp.237–246 (2012).

[12] Iwamoto, C., Mukai, Y., Sumida, Y. and Morita, K.: Generalized Chat Noir is PSPACE-complete, *IEICE Trans. Inf. Syst.*, Vol.E96-D, No.3, pp.502–505 (2013).

[13] Iwamoto, C., Sasaki, K. and Morita, K.: A Polynomial-Time Reduction from the 3SAT Problem to the Generalized String Puzzle Problem, *Algorithms*, Vol.5, No.2, pp.261–272 (2012).

[14] Iwamoto, C., Wada, Y. and Morita, K.: Generalized Shisen-Sho is NP-complete, *IEICE Trans. Inf. Syst.*, Vol.E95-D, No.11, pp.2712–2715 (2012).

[15] Kölker, J.: Kurodoko is NP-complete, *J. Inf. Process.*, Vol.20, No.3, pp.694–706 (2012).

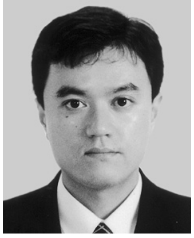
[16] Kölker, J.: The Magnet Puzzle is NP-complete, *J. Inf. Process.*, Vol.20, No.3, pp.707–708 (2012).

[17] Nakai, K. and Takenaga, Y.: NP-completeness of Pandemic, *J. Inf. Process.*, Vol.20, No.3, pp.723–726 (2012).

[18] Robertson, N., Sanders, D.P., Seymour, P. and Thomas, R.: Efficiently Four-Coloring Planar Graphs, *Proc. 28th Annual ACM Symposium on Theory of Computing*, pp.571–575, ACM Press (1996).

[19] Takenaga, Y., Aoyagi, S., Iwata, S. and Kasai, T.: Shikaku and Ripple Effect are NP-complete, 44th Southeastern International Conference on Combinatorics, Graph Theory, and Computing (2013) (online), available from ⟨http://math.fau.edu/cgtc/cgtc44/Abstracts.html⟩ (accessed 2013-09-24).

[20] WEB Nikoli – Enjoy Pencil Puzzles!, Puzzles, Yosenabe (online), available from ⟨http://www.nikoli.co.jp/en/puzzles/yosenabe.html⟩ (accessed 2013-09-24).



Chuzo Iwamoto received his B.Eng. degree from Yamaguchi University in 1990, and M.Eng. and Dr. Eng. degrees from Kyushu University in 1992 and 1995, respectively. From 1995 to 1997, he was a lecturer at Kyushu Institute of Design. Since 1997, he has been an associate professor in the Graduate School of Engineering, Hiroshima University. His present interests include computational complexity and automata theory.

