Aspects of Differential Geometry I

Synthesis Lectures on Mathematics and Statistics

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Peter Gilkey, JeongHyeong Park, and Ramón Vázquez-Lorenzo
2015

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Aspects of Differential Geometry
I

Peter Gilkey
University of Oregon, Eugene, OR

JeongHyeong Park
Sungkyunkwan University, Suwon, Korea
Institute for Advanced Study, Seoul, Korea

Ramón Vázquez-Lorenzo
University of Santiago de Compostela, Santiago de Compostela, Spain

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ABSTRACT

Differential Geometry is a wide field. We have chosen to concentrate upon certain aspects that are appropriate for an introduction to the subject; we have not attempted an encyclopedic treatment.

In Book I, we focus on preliminaries. Chapter 1 provides an introduction to multivariable calculus and treats the Inverse Function Theorem, Implicit Function Theorem, the theory of the Riemann Integral, and the Change of Variable Theorem. Chapter 2 treats smooth manifolds, the tangent and cotangent bundles, and Stokes' Theorem. Chapter 3 is an introduction to Riemannian geometry. The Levi–Civita connection is presented, geodesics introduced, the Jacobi operator is discussed, and the Gauss–Bonnet Theorem is proved. The material is appropriate for an undergraduate course in the subject.

We have given some different proofs than those that are classically given and there is some new material in these volumes. For example, the treatment of the Chern–Gauss–Bonnet Theorem for pseudo-Riemannian manifolds with boundary is new.

KEYWORDS

Change of Variable Theorem, derivative as best linear approximation, Fubini's Theorem, Gauss–Bonnet Theorem, Gauss's Theorem, geodesic, Green's Theorem, Implicit Function Theorem, improper integrals, Inverse Function Theorem, Levi–Civita connection, partitions of unity, pseudo-Riemannian geometry, Riemann integral, Riemannian geometry, Stokes' Theorem

This book is dedicated to
Alison, Arnie, Carmen, Junmin,
Junpyo, Manuel, Montse, Rosalía, and Susana.

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Preface

This two-volume series arose out of work by the three authors over a number of years both in teaching various courses and also in their research endeavors.

The present volume (Book I) is comprised of three chapters. Chapter 1 provides an introduction to multivariable calculus. It begins with two introductory sections on metric spaces and linear algebra. Various notions of differentiability are introduced and the chain rule is proved. The Inverse and Implicit Function Theorems are established. One then turns to the theory of integration. The Riemann integral is introduced and it is shown that a bounded function is integrable if and only if it is integrable almost everywhere. Compact exhaustions by Jordan measurable sets, mesa functions, and partitions of unity are used to define improper integrals. Chapter 1 concludes with a proof of the Change of Variable Theorem; upper and lower sums defined by cubes (rather than rectangles) together with partitions of unity are the fundamental tools employed.

Chapter 2 completes the discussion of multivariable calculus. The basic materials concerning smooth manifolds are introduced. It is shown any compact manifold embeds smoothly in $\mathbb{R}^m$ for some m. A brief introduction to fiber bundle theory and vector bundle theory is given and the tangent and cotangent bundles are introduced. This formalism is then combined with the results of Chapter 1 to establish the generalized Stokes' Theorem. The classical Green's Theorem, Gauss's Theorem, and Stokes' Theorem are then established. The Brauer Fixed Point Theorem, the Fundamental Theorem of Algebra, and the Combing the Hair on a Billiard Ball Theorem are presented as applications.

Chapter 3 presents an introduction to Riemannian and pseudo-Riemannian geometry. The volume form is introduced. The notion of a connection on an arbitrary vector bundle is presented and the discussion is then specialized to the Levi–Civita connection. Geodesics are treated and the classical Hopf–Rinow Theorem giving various equivalent notions of completeness is established in the Riemannian setting. The Jacobi operator is introduced and used to establish the Myers Theorem that if the Ricci tensor on a complete Riemannian manifold is uniformly positive, then the manifold is compact and has finite fundamental group. Riemann surfaces are introduced and the classical Gauss–Bonnet Theorem is established. The Chern–Gauss–Bonnet Theorem in higher dimensions is treated and analytic continuation used to establish an analogous result in the pseudo-Riemannian setting.

We have tried whenever possible to give the original references to major theorems in this area. We have provided a number of pictures to illustrate the discussion, especially in Chapters 1 and 2. Chapters 1 and 2 are suitable for an undergraduate course on "Calculus on Manifolds" and arose in that context out of a course at the University of Oregon. Chapter 3 is designed for an undergraduate course in Differential Geometry. Thus Book I is suitable as an undergraduate text although, of course, it also forms the foundation of a graduate course in Differential Geometry as well. Book II can be used as a graduate text in Differential Geometry and arose in that context out of a second-year graduate course in Differential Geometry at the University of Oregon. The material can, however, also form the basis of a second-semester course at the undergraduate level as well. While much of the material is, of course, standard, many of the proofs are a bit different from those given classically and we hope provide a new viewpoint on the subject. There are also new results in the book; our treatment of the generalized Chern–Gauss–Bonnet Theorem in the indefinite signature context arose out of our study of Euler–Lagrange equations using perturbations of complex metrics (i.e., metrics where the g_{ij} tensor is $\mathbb{C}$-valued). Similarly, our treatment of curves in $\mathbb{R}^m$ given by the solution to constant coefficient ODEs which have finite total curvature is new. There are other examples; Differential Geometry is of necessity a vibrant and growing field – it is not static! There are, of course, many topics that we have not covered – this is a work on "Aspects of Differential Geometry" and of necessity must omit more topics than can possibly be included.

For technical reasons, the material is divided into two books and each book is largely self-sufficient. To facilitate cross references between the two books, we have numbered the chapters of Book I from 1 to 3, and the chapters of Book II from 4 to 8.

Peter Gilkey, JeongHyeong Park, and Ramón Vázquez-Lorenzo
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Acknowledgments

We have provided many images of famous mathematicians in these two books; mathematics is created by real people and we think having such images makes this point more explicit. The older pictures are in the public domain. We are grateful to the Archives of the Mathematisches Forschungsinstitut Oberwolfach for permitting us to use many images from their archives (R. Brauer, H. Cartan, S. Chern, G. de Rham, S. Eilenberg, H. Hopf, E. Kähler, H. Künneth, L. Nirenberg, H. Poincaré, W. Rinow, L. Vietoris, and H. Weyl); the use of these images was granted to us for the publication of these books only and their further reproduction is prohibited without their express permission. Some of the images (E. Beltrami, E. Cartan, G. Frobenius, and F. Klein) provided to us by the MFO are from the collection of the Mathematische Gesellschaft Hamburg; again, the use of any of these images was granted to us for the publication of these books only and their further reproduction is prohibited without their express permission.

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