Aspects of Differential Geometry
II

Synthesis Lectures on Mathematics and Statistics

Editor
Steven G. Krantz, ***Washington University, St. Louis***

Statistics is Easy! Second Edition
Dennis Shasha and Manda Wilson
2010

Lectures on Financial Mathematics: Discrete Asset Pricing
Greg Anderson and Alec N. Kercheval
2010

Jordan Canonical Form: Theory and Practice
Steven H. Weintraub
2009

The Geometry of Walker Manifolds
Miguel Brozos-Vázquez, Eduardo García-Río, Peter Gilkey, Stana Nikčević, and Ramón Vázquez-Lorenzo
2009

An Introduction to Multivariable Mathematics
Leon Simon
2008

Jordan Canonical Form: Application to Differential Equations
Steven H. Weintraub
2008

Statistics is Easy!
Dennis Shasha and Manda Wilson
2008

A Gyrovector Space Approach to Hyperbolic Geometry
Abraham Albert Ungar
2008

Aspects of Differential Geometry II

Peter Gilkey, JeongHyeong Park, and Ramón Vázquez-Lorenzo

ISBN: 978-3-031-01280-8 paperback
ISBN: 978-3-031-02408-5 ebook

DOI 10.1007/978-3-031-02408-5

A Publication in the Springer series
SYNTHESIS LECTURES ON MATHEMATICS AND STATISTICS

Lecture #16
Series Editor: Steven G. Krantz, *Washington University, St. Louis*
Series ISSN
Print 1938-1743 Electronic 1938-1751

Aspects of Differential Geometry II

Peter Gilkey
University of Oregon, Eugene, OR

JeongHyeong Park
Sungkyunkwan University, Suwon, Korea
Institute for Advanced Study, Seoul, Korea

Ramón Vázquez-Lorenzo
University of Santiago de Compostela, Santiago de Compostela, Spain

SYNTHESIS LECTURES ON MATHEMATICS AND STATISTICS #16

ABSTRACT

Differential Geometry is a wide field. We have chosen to concentrate upon certain aspects that are appropriate for an introduction to the subject; we have not attempted an encyclopedic treatment.

Book II deals with more advanced material than Book I and is aimed at the graduate level. Chapter 4 deals with additional topics in Riemannian geometry. Properties of real analytic curves given by a single ODE and of surfaces given by a pair of ODEs are studied, and the volume of geodesic balls is treated. An introduction to both holomorphic and Kähler geometry is given. In Chapter 5, the basic properties of de Rham cohomology are discussed, the Hodge Decomposition Theorem, Poincaré duality, and the Künneth formula are proved, and a brief introduction to the theory of characteristic classes is given. In Chapter 6, Lie groups and Lie algebras are dealt with. The exponential map, the classical groups, and geodesics in the context of a bi-invariant metric are discussed. The de Rham cohomology of compact Lie groups and the Peter–Weyl Theorem are treated. In Chapter 7, material concerning homogeneous spaces and symmetric spaces is presented. Book II concludes in Chapter 8 where the relationship between simplicial cohomology, singular cohomology, sheaf cohomology, and de Rham cohomology is established.

We have given some different proofs than those that are classically given and there is some new material in these volumes. For example, the treatment of the total curvature and length of curves given by a single ODE is new as is the discussion of the total Gaussian curvature of a surface defined by a pair of ODEs.

KEYWORDS

Chern classes, Clifford algebra, connection, de Rham cohomology, geodesic, Jacobi operator, Kähler geometry, Levi–Civita connection, Lie algebra, Lie group, Peter–Weyl Theorem, pseudo-Riemannian geometry, Riemannian geometry, sheaf cohomology, simplicial cohomology, singular cohomology, symmetric space, volume of geodesic balls

This book is dedicated to
Alison, Arnie, Carmen, Junmin,
Junpyo, Manuel, Montse, Rosalía, and Susana.

Contents

Preface

This two-volume series arose out of work by the three authors over a number of years both in teaching various courses and also in their research endeavors.

The present volume (Book II) is comprised of five chapters that continue the discussion of Book I. In Chapter 4, we examine the geometry of curves which are the solution space of a constant coefficient ordinary differential equation. We give necessary and sufficient conditions that the curves give a proper embedding and we examine when the total extrinsic curvature is finite. We examine similar questions for the total Gaussian curvature of a surface defined by a pair of ODEs and apply the Gauss–Bonnet Theorem to express the total Gaussian curvature in terms of the curves associated to the individual ODEs. We then examine the volume of a small geodesic ball in a Riemannian manifold. We show that if the scalar curvature is positive, then volume grows more slowly than it does in flat space while if the scalar curvature is negative, then volume grows more rapidly than it does in flat space. Chapter 4 concludes with a brief introduction to holomorphic and Kähler geometry.

Chapter 5 treats de Rham cohomology. The basic properties are introduced and it is shown that de Rham cohomology satisfies the Eilenberg–Steenrod axioms; these are properties that all homology and cohomology theories have in common. We shall postpone until Chapter 8 a discussion of the Mayer–Vietoris sequence and the homotopy property as these depend upon some results in homological algebra that will be treated there. We determine the de Rham cohomology of the sphere and of real projective space. We introduce Clifford algebras and present the Hodge Decomposition Theorem. This is used to establish the Künneth formula and Poincaré duality. We treat the first Chern class in some detail and use it to determine the ring structure of the de Rham cohomology of complex projective space. A brief introduction to the higher Chern classes and the Pontrjagin classes is given.

Chapter 6 contains an introduction to the theory of Lie groups and Lie algebras. We restrict for the most part to matrix groups so that the exponential and log functions can be given explicitly in terms of convergent power series. We show in this setting that a closed subgroup of a matrix group is a Lie subgroup; this result is used to treat the geometry of the classic matrix groups (special linear group, orthogonal group in arbitrary signature, unitary group in arbitrary signature, symplectic group, etc.). If M is a compact Lie group with a bi-invariant metric, we show the one-parameter subgroups of the exponential map are geodesics. We express the de Rham cohomology of a compact connected Lie group in terms of the left-invariant differential forms and prove the Hopf structure theorem that shows the de Rham cohomology of a compact connected Lie group is a finitely generated exterior algebra on odd-dimensional generators. We use these results to determine the ring structure of the de Rham cohomology of the unitary group.

We conclude Chapter 6 by discussing the orthogonality relations and the Peter–Weyl Theorem which decomposes L^2 as the direct sum of irreducible representations for a compact connected Lie group.

Chapter 7 presents an introduction to the theory of homogeneous spaces and of symmetric spaces. We examine coset spaces and the group of isometries of a pseudo-Riemannian manifold. We introduce material related to the Lie derivative and Killing vector fields. We outline the geometry of homogeneous spaces, of local symmetric spaces, and of global symmetric spaces. Chapter 8 concludes Book II with a discussion of other cohomology theories. We introduce the necessary homological machinery to show that de Rham cohomology is a homotopy functor and has the Mayer–Vietoris long exact sequence; these two results also have a significant geometric input. We relate simplicial cohomology, singular cohomology, and sheaf cohomology to de Rham cohomology.

We have tried whenever possible to give the original references to major theorems in this area. We have provided a number of pictures to illustrate the discussion. Chapters 1 and 2 of Book I are suitable for an undergraduate course on "Calculus on Manifolds" and arose in that context out of a course at the University of Oregon. Chapter 3 is designed for an undergraduate course in Differential Geometry. Therefore, Book I is suitable as an undergraduate text although, of course, it also forms the foundation of a graduate course in Differential Geometry as well. Book II can be used as a graduate text in Differential Geometry and arose in that context out of second year graduate courses in Differential Geometry at the University of Oregon and at Sungkyunkwan University. The material can, however, also form the basis of a second semester course at the undergraduate level as well. While much of the material is, of course, standard, many of the proofs are a bit different from those given classically and we hope provide a new viewpoint on the subject. Our treatment of curves in $\mathbb{R}^m$ given by the solution to constant coefficient ODEs which have finite total curvature is new as is the corresponding treatment of the total Gaussian curvature of a surface given by a pair of ODEs. There are other examples; Differential Geometry is of necessity a vibrant and growing field; it is not static! There are, of course, many topics that we have not covered. This is a work on "Aspects of Differential Geometry" and of necessity must omit more topics than can be possibly included.

For technical reasons, the material is divided into two books and each book is largely self-sufficient. To facilitate cross references between the two books, we have numbered the chapters of Book I from 1 to 3, and the chapters of Book II from 4 to 8.

Peter Gilkey, JeongHyeong Park, and Ramón Vázquez-Lorenzo
May 2015

Acknowledgments

We have provided many images of famous mathematicians in these two books; mathematics is created by real people and we think having such images makes this point more explicit. The older pictures are in the public domain. We are grateful to the Archives of the Mathematisches Forschungsinstitut Oberwolfach for permitting us to use many images from their archives (R. Brauer, H. Cartan, S. Chern, G. de Rham, S. Eilenberg, H. Hopf, E. Kähler, H. Künneth, L. Nirenberg, H. Poincaré, W. Rinow, L. Vietoris, and H. Weyl); the use of these images was granted to us for the publication of these books only and their further reproduction is prohibited without their express permission. Some of the images (E. Beltrami, E. Cartan, G. Frobenius, and F. Klein) provided to us by the MFO are from the collection of the Mathematische Gesellschaft Hamburg; again, the use of any of these images was granted to us for the publication of these books only and their further reproduction is prohibited without their express permission.

The research of the authors was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2014053413) and by Project MTM2013-41335-P with FEDER funds (Spain). The authors are very grateful to Esteban Calviño-Louzao and Eduardo García-Río for constructive suggestions and assistance in proofreading. The assistance of Ekaterina Puffini of the Krill Institute of Technology has been invaluable. Wikipedia has been a useful guide to tracking down the original references and was a source of many of the older images that we have used that are in the public domain.

Peter Gilkey, JeongHyeong Park, and Ramón Vázquez-Lorenzo
May 2015