Stochastic Partial Differential Equations for Computer Vision with Uncertain Data

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Stochastic Partial Differential Equations for Computer Vision with Uncertain Data

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SYNTHESIS LECTURES ON VISUAL COMPUTING: COMPUTER GRAPHICS, ANIMATION, COMPUTATIONAL PHOTOGRAPHY, AND IMAGING #28

ABSTRACT

In image processing and computer vision applications such as medical or scientific image data analysis, as well as in industrial scenarios, images are used as input measurement data. It is good scientific practice that proper measurements must be equipped with error and uncertainty estimates. For many applications, not only the measured values but also their errors and uncertainties, should be—and more and more frequently are—taken into account for further processing. This error and uncertainty propagation must be done for every processing step such that the final result comes with a reliable precision estimate.

The goal of this book is to introduce the reader to the recent advances from the field of uncertainty quantification and error propagation for computer vision, image processing, and image analysis that are based on partial differential equations (PDEs). It presents a concept with which error propagation and sensitivity analysis can be formulated with a set of basic operations. The approach discussed in this book has the potential for application in all areas of quantitative computer vision, image processing, and image analysis. In particular, it might help medical imaging finally become a scientific discipline that is characterized by the classical paradigms of observation, measurement, and error awareness.

This book is comprised of eight chapters. After an introduction to the goals of the book (Chapter 1), we present a brief review of PDEs and their numerical treatment (Chapter 2), PDE-based image processing (Chapter 3), and the numerics of stochastic PDEs (Chapter 4). We then proceed to define the concept of stochastic images (Chapter 5), describe how to accomplish image processing and computer vision with stochastic images (Chapter 6), and demonstrate the use of these principles for accomplishing sensitivity analysis (Chapter 7). Chapter 8 concludes the book and highlights new research topics for the future.

KEYWORDS

image processing, computer vision, stochastic images, uncertainty quantification, stochastic partial differential equation, polynomial chaos

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Preface

Over the past two decades, the topic of uncertainty quantification within simulation science has emerged as a requirement for the advancement of many scientific and engineering endeavors. The confluence of our ability to sense and measure data at scale, advances in applied mathematics and data science, and the power of computing has enabled us to reexamine the simulation science pipeline in terms of errors and uncertainties.

This work started with our attempt to consider the world of image processing and computer vision in light of new mathematical perspectives and algorithms devised within the uncertainty quantification world. We soon realized that, in principle, it was possible to formulate error propagation and uncertainty quantification in computer vision and image processing as an integral part of basal operations. Working out the details for the respective image processing models and their numerical treatments has taken several years. This book summarizes our efforts since we initiated this field with a publication entitled *Building Blocks for Computer Vision with Stochastic Partial Differential Equations* almost a decade ago.

The target group for this book is researchers starting at an advanced graduate level who may have existing knowledge about, on the one hand, computer vision/image processing with PDEs or, on the other hand, the numerics of SPDEs. This book is meant to represent a toolbox for initiating research in this area. As a summary of our efforts, we have not written this book from the perspective of competing with other methods. Correspondingly, we do not show all the extensive connections with existing approaches for stochastic computer vision or to non-PDE-based computer vision approaches such as graph cuts, etc. Our examples have a "bottom-up" characteristic and not that of emphasizing the highly challenging and large-scale application problems that we acknowledge exist in this area. We will give only some anchor points to the stochastic image processing community related to Bayesian inference, etc. The reader should keep in mind that going deep into this is not our focus; our focus is to build momentum and interest in the area of stochastic PDEs in computer vision.

In terms of what is needed to read this book, we assume that all readers are familiar with linear algebra and calculus in several variables. For those readers without a background in numerical methods for PDEs and without a background in probability, we recommend reading through the book linearly and also following up on the background reading references given herein. Readers who are familiar with classical (deterministic) PDE-based computer vision and image processing may chose to skip Chapter 3. Those who are experienced in numerics of PDEs and/or numerics of stochastic PDEs may skip parts of Chapter 4. The central part of our approach to SPDE-based computer vision and image processing is discussed in Chapter 5. An easy entry point into the applications of stochastic PDEs to computer vision and image processing is the sensitivity analysis in Chapter 7.

Any work of this size and scope has benefitted from the involvement of many people both indirectly and directly. We wish to thank our collaborators that inspired us with discussions about computer vision and uncertainty quantification in various models and application areas. We also thank the various faculty, students, and colleagues at the SCI Institute (University of Utah), Fraunhofer MEVIS (Bremen, Germany), and Jacobs University (Bremen, Germany) with whom we sharpened our ideas. In addition, we would like to thank the various Federal Funding Agencies that have supported our research efforts over the years. The papers we reference that we coauthored detail those acknowledgments. Lastly, we would like to thank our spouses, without whose patience and encouragement, we would probably not have made it this far.

Tobias Preusser, Robert M. Kirby, and Torben Pätz
June 2017

Notation

Symbols

$(\cdot,\cdot)_{2,D}$ L^2 scalar product on D 12

Δ Laplacian 31

$\cdot$ Euclidean scalar product 18

∇ Nabla operator, gradient 8

$|\cdot|$ Euclidean norm 31

$||\cdot||_{2,D}$ L^2 norm on D 12

$||\cdot||_{\infty,D}$ supremum/maximum norm on D 12

$*$ convolution operator 32

$<\cdot,\cdot>$ scalar product on $L^2(\Omega)$ 50

$:$ scalar product on matrices 15

2^Ω power set of Ω 47

A

$\mathcal{A}$ σ-algebra 47

a.s. almost surely 50

C

C^0 space of continuous functions 12

$C_{\alpha\beta\gamma}$ stochastic lookup table 57

C^k space of k-times continuously differentiable functions 12

Cov covariance 49

D

D image domain 28

∂_m partial derivative with respect to space coordinate m 44

δ_ε regularized Dirac distribution . . . 41

δ_{ij} Kronecker delta 20

div divergence operator 7

∂_s partial derivative w.r.t. sequence time 43

∂_t partial derivative w.r.t. time/scale 31

∂_{ss} 2nd partial derivative w.r.t. sequence time 45

∂_t partial derivative w.r.t. time/scale . . 7

∂_{xx} 2nd partial derivative w.r.t. x 62

E

$\mathbb{E}$ expected value 49

G

G_σ Gaussian kernel of variance σ 32

H

$H_0^1(D)$ Sobolev space of weakly differentiable functions with zero trace on the boundary 19

$H^1(D)$ Sobolev space of weakly differentiable functions $W^{1,2}$ 13

$\mathcal{H}^n$	n-dimensional Hausdorff measure	35
H_ε	regularized Heaviside function	41

I

$\mathcal{I}$	pixel/voxel index set	28

J

J_σ	smoothed structure tensor	45

L

$L^2(\Omega)$	Lebesgue space on $(\Omega, \mathcal{A}, \Pi)$	50
L^p	Lebesque spaces	12

M

M^n	n^{th} stochastic moment	49

O

$\mathcal{O}$	Landau symbol	15
Ω	sample space	47

P

P_i	shape function	30
Π	probability measure	47
Π_X	probabilty measure induced by random variable X	49

Q

Q	time space cylinder	29

R

$\mathbb{R}^+$	positive real numbers	8
$\mathbb{R}^+_0$	nonnegative real numbers	38

S

supp	support	30

U

$\mathcal{U}$	uniform distribution	103
u	image function	28
u_α	stochastic mode α of image u	70
u^i	digital image function, value of pixel/voxel x_i	28
u^i_α	stochastic mode α of pixel/voxel x_i	70

V

Var	variance	49

W

$\mathbf{w}$	optical flow field, deformation field	44
$W^{k,p}$	Sobolev spaces of k-times weakly differentiable functions	13

X

x_i	pixel/voxel	28