Numerical Integration of Space Fractional Partial Differential Equations

Vol 2 - Applications from Classical Integer PDEs

Synthesis Lectures on Mathematics and Statistics

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Vol 2 - Applications from Classical Integer PDEs

Younes Salehi
Razi University

William E. Schiesser
Lehigh University

SYNTHESIS LECTURES ON MATHEMATICS AND STATISTICS #20

ABSTRACT

Partial differential equations (PDEs) are one of the most used widely forms of mathematics in science and engineering. PDEs can have partial derivatives with respect to (1) an initial value variable, typically time, and (2) boundary value variables, typically spatial variables. Therefore, two fractional PDEs can be considered, (1) fractional in time (TFPDEs), and (2) fractional in space (SFPDEs). The two volumes are directed to the development and use of SFPDEs, with the discussion divided as:

- Vol 1: Introduction to Algorithms and Computer Coding in R
- Vol 2: Applications from Classical Integer PDEs.

Various definitions of space fractional derivatives have been proposed. We focus on the *Caputo* derivative, with occasional reference to the *Riemann-Liouville* derivative.

In the second volume, the emphasis is on applications of SFPDEs developed mainly through the extension of classical integer PDEs to SFPDEs. The example applications are:

- Fractional diffusion equation with Dirichlet, Neumann and Robin boundary conditions
- Fisher-Kolmogorov SFPDE
- Burgers SFPDE
- Fokker-Planck SFPDE
- Burgers-Huxley SFPDE
- Fitzhugh-Nagumo SFPDE

These SFPDEs were selected because they are integer first order in time and integer second order in space. The variation in the spatial derivative from order two (parabolic) to order one (first order hyperbolic) demonstrates the effect of the spatial fractional order α with $1 \leq \alpha \leq 2$. All of the example SFPDEs are one dimensional in Cartesian coordinates. Extensions to higher dimensions and other coordinate systems, in principle, follow from the examples in this second volume.

The examples start with a statement of the integer PDEs that are then extended to SF-PDEs. The format of each chapter is the same as in the first volume.

The R routines can be downloaded and executed on a modest computer (R is readily available from the Internet).

KEYWORDS

partial differential equations, value variables; space fractional partial differential equations, fractional calculus

To

Mahnaz and Dolores

for their encouragement and patience.

Contents

Preface

Partial differential equations (PDEs) are one of the most used widely forms of mathematics in science and engineering. PDEs can have partial derivatives with respect to (1) an initial value variable, typically time, and (2) boundary value variables, typically spatial variables. Therefore, two fractional PDEs (FPDEs) can be considered, (1) fractional in time (TFPDEs), and (2) fractional in space (SFPDEs). The books[1] are directed to the development and use of SFPDEs.

FPDEs have features and solutions that go beyond the established integer PDEs (IPDEs), for example, the classical field equations including the Euler, Navier-Stokes, Maxwell and Einstein equations. FPDEs therefore offer the possibility of solutions that have features that better approximate physical/chemical/biological phenomena than IPDEs.

Fractional calculus dates back to the beginning of calculus (e.g., to Leibniz, Riemann and Liouville), but recently there has been extensive reporting of applications, typically as expressed by TFPDE/SFPDEs. In particular, SFPDEs are receiving broad attention in the research literature, especially when applied to the computer-based modeling of heterogeneous media. For example, SFPDEs are being applied to living tissue (with potential applications in biomedical engineering, biology and medicine).

Various definitions of space fractional derivatives have been proposed. Therefore, as a first step in the use of SFPDEs, a definition of the derivative must be selected. In the books, we focus on the *Caputo* derivative, with occasional reference to the *Riemann-Liouville* derivative.

The Caputo derivative has at least two important advantages:

1. For the special case of an integer derivative, the usual properties of integer calculus follow. For example, the Caputo derivative of a constant is zero.

2. The definition of a Caputo derivative is based the integral of an integer derivative. Therefore, the established algorithms for approximating integer derivatives can be used. For the numerical methods that follow, the integer derivatives are approximated with splines.

The Caputo derivative is defined as a convolution integral. Thus, rather than being *local* (with a value at a particular point in space), the Caputo derivative is *non-local*, (it is based on an integration in space), which is one of the reasons that it has properties not shared by integer derivative).

[1]The two volume set has the titles:
Numerical Integration of Space Fractional Partial Differential Equations
Vol 1: Introduction to Algorithms and Computer Coding in R
Vol 2: Applications from Classical Integer PDEs.

A parameter of the Caputo derivative that is of primary interest is the order of the derivative, which is fractional, with integer order as a special case. The various example applications that follow generally permit the variation of the fractional order in computer-based analysis.

The papers cited as a source of the SFPDE models generally consist of a statement of the equations followed by reported numerical solutions. Generally, little or no information is given about how the solutions were computed (the algorithms) and in all cases, the computer code that was used to calculate the solutions is not provided.

In other words, what is missing is: (1) a detailed discussion of the numerical methods used to produce the reported solutions and (2) the computer routines used to calculate the reported solutions. For the reader to complete these two steps to verify the reported solutions with reasonable effort is essentially impossible.

A principal objective of the books is therefore to provide the reader with a set of documented R routines that are discussed in detail, and can be downloaded and executed without having to first master the details of the relevant numerical analysis and then code a set of routines.

The example applications are intended as introductory and open ended. They are based mainly on classical (legacy) IPDEs. The focus in each chapter is on:

1. A statement of the SFPDE system, including initial conditions (ICs), boundary conditions (BCs) and parameters.

2. The algorithms for the calculation of numerical solutions, with particular emphasis on splines.

3. A set of R routines for the calculation of numerical solutions, including a detailed explanation of each section of the code.

4. Discussion of the numerical solution.

5. Summary and conclusions about extensions of the computer-based analysis.

In summary, the presentation is not as formal mathematics, e.g., theorems and proofs. Rather, the presentation is by examples of SFPDE applications, including the details for computing numerical solutions, particularly with documented source code. The authors would welcome comments, especially pertaining to this format and experiences with the use of the R routines. Comments and questions can be directed to wes1@lehigh.edu.

Younes Salehi and William E. Schiesser
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