

# Synthesis of Quantum Circuits vs. Synthesis of Classical Reversible Circuits

# Synthesis Lectures on Digital Circuits and Systems

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The *Synthesis Lectures on Digital Circuits and Systems* series is comprised of 50- to 100-page books targeted for audience members with a wide-ranging background. The Lectures include topics that are of interest to students, professionals, and researchers in the area of design and analysis of digital circuits and systems. Each Lecture is self-contained and focuses on the background information required to understand the subject matter and practical case studies that illustrate applications. The format of a Lecture is structured such that each will be devoted to a specific topic in digital circuits and systems rather than a larger overview of several topics such as that found in a comprehensive handbook. The Lectures cover both well-established areas as well as newly developed or emerging material in digital circuits and systems design and analysis.

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# Synthesis of Quantum Circuits

vs.

# Synthesis of Classical Reversible Circuits

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## ABSTRACT

At first sight, quantum computing is completely different from classical computing. Nevertheless, a link is provided by reversible computation.

Whereas an arbitrary quantum circuit, acting on  $w$  qubits, is described by an  $n \times n$  unitary matrix with  $n = 2^w$ , a reversible classical circuit, acting on  $w$  bits, is described by a  $2^w \times 2^w$  permutation matrix. The permutation matrices are studied in group theory of finite groups (in particular the symmetric group  $\mathbf{S}_n$ ); the unitary matrices are discussed in group theory of continuous groups (a.k.a. Lie groups, in particular the unitary group  $\mathbf{U}(n)$ ).

Both the synthesis of a reversible logic circuit and the synthesis of a quantum logic circuit take advantage of the decomposition of a matrix: the former of a permutation matrix, the latter of a unitary matrix. In both cases the decomposition is into three matrices. In both cases the decomposition is not unique.

## KEYWORDS

quantum computing, reversible computing, unitary matrix, permutation matrix, group theory, matrix decomposition, circuit synthesis

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