Aspects of Differential Geometry IV

Synthesis Lectures on Mathematics and Statistics

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Aspects of Differential Geometry IV

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ABSTRACT

Book IV continues the discussion begun in the first three volumes. Although it is aimed at first-year graduate students, it is also intended to serve as a basic reference for people working in affine differential geometry. It also should be accessible to undergraduates interested in affine differential geometry. We are primarily concerned with the study of affine surfaces which are locally homogeneous. We discuss affine gradient Ricci solitons, affine Killing vector fields, and geodesic completeness. Opozda has classified the affine surface geometries which are locally homogeneous; we follow her classification. Up to isomorphism, there are two simply connected Lie groups of dimension 2. The translation group $\mathbb{R}^2$ is Abelian and the $ax + b$ group is non-Abelian. The first chapter presents foundational material. The second chapter deals with Type $\mathcal{A}$ surfaces. These are the left-invariant affine geometries on $\mathbb{R}^2$. Associating to each Type $\mathcal{A}$ surface the space of solutions to the quasi-Einstein equation corresponding to the eigenvalue $\mu = -1$ turns out to be a very powerful technique and plays a central role in our study as it links an analytic invariant with the underlying geometry of the surface. The third chapter deals with Type $\mathcal{B}$ surfaces; these are the left-invariant affine geometries on the $ax + b$ group. These geometries form a very rich family which is only partially understood. The only remaining homogeneous geometry is that of the sphere S^2. The fourth chapter presents relations between the geometry of an affine surface and the geometry of the cotangent bundle equipped with the neutral signature metric of the modified Riemannian extension.

KEYWORDS

affine gradient Ricci solitons, affine Killing vector fields, geodesic completeness, locally homogeneous affine surfaces, locally symmetric affine surfaces, projectively flat, quasi-Einstein equation

This book is dedicated to
Alison, Carmen, Celia, Fernanda, Hugo, Junmin, Junpyo, Luis,
Manuel, Mateo, Montse, Rosalía, Sara, and Susana.

Contents

Preface

This four-volume series arose out of work by the authors over a number of years both in teaching various courses and in their research endeavors. For technical reasons, the material is divided into four books and each book is largely self-sufficient. To facilitate cross references between the books, we have numbered the chapters of Book I from 1–3, the chapters of Book II from 4–8, the chapters of Book III from 9–11, and the chapters of the present Book IV on affine surfaces from 12–15. A final book in the series dealing with elliptic operator theory and its applications to Differential Geometry is proposed.

Up to isomorphism, there are two simply connected Lie groups of dimension 2. The translation group $\mathbb{R}^2$ is Abelian and acts on $\mathbb{R}^2$ by translation; the group structure is given by $(a,b)+(a',b')=(a+a',b+b')$. The $ax+b$ group $\mathbb{R}^+\times\mathbb{R}$ acts on $\mathbb{R}^+\times\mathbb{R}$ by

$$(x^1,x^2)\to(ax^1,ax^2+b)\quad\text{for}\quad a>0\quad\text{and}\quad b\in\mathbb{R}\,;$$

the group structure is given by composition and is non-Abelian;

$$(a,b)*(a',b')=(aa',ab'+b)\,.$$

An affine surface $\mathcal{M}$ is a pair (M,∇) where M is a smooth surface and ∇ is a torsion-free connection on the tangent bundle of M. One says $\mathcal{M}=(M,\nabla)$ is locally homogeneous if given any two points of M, there is the germ of a diffeomorphism mapping one point to the other point which preserves the connection ∇. Opozda [53] showed that any locally homogeneous affine surface geometry is *modeled* on one of the following three geometries:

- **Type $\mathcal{A}$.** $\mathcal{M}=(\mathbb{R}^2,\nabla)$ where ∇ has constant Christoffel symbols $\Gamma_{ij}{}^k=\Gamma_{ji}{}^k$. This geometry is homogeneous; the Type $\mathcal{A}$ connections are the left-invariant connections on the Lie group $\mathbb{R}^2$. An affine surface is modeled on such a geometry if and only if there exists a coordinate atlas so that the Christoffel symbols ${}^\alpha\Gamma_{ij}{}^k=\Gamma_{ij}{}^k$ are constant in each chart of the atlas.
- **Type $\mathcal{B}$.** $\mathcal{M}=(\mathbb{R}^+\times\mathbb{R},\nabla)$ where ∇ has Christoffel symbols $\Gamma_{ij}{}^k=(x^1)^{-1}A_{ij}{}^k$ where $A_{ij}{}^k=A_{ji}{}^k$ is constant. This geometry is homogeneous; the action of the $ax+b$ group sending $(x^1,x^2)\to(ax^1,ax^2+b)$ acts transitively on $\mathbb{R}^+\times\mathbb{R}$. If we identify the $ax+b$ group with $\mathbb{R}^+\times\mathbb{R}$, then the Type $\mathcal{B}$ connections are the left-invariant connections. An affine surface is modeled on such a geometry if there is a coordinate atlas so the Christoffel symbols ${}^\alpha\Gamma_{ij}{}^k=(x^1_\alpha)^{-1}A_{ij}{}^k$ in each chart of the atlas.
- **Type $\mathcal{C}$.** $\mathcal{M}=(M,\nabla)$ where ∇ is the Levi–Civita connection of a metric of constant non-zero sectional curvature.

This present volume is organized around this observation. There is a non-trivial intersection between the Type $\mathcal{A}$ and the Type $\mathcal{B}$ geometries. There is no geometry which is both Type $\mathcal{A}$ and $\mathcal{C}$. And the only Type $\mathcal{C}$ geometry which is not also Type $\mathcal{B}$ is modeled on the round sphere S^2 in $\mathbb{R}^3$. Chapter 12 of the book deals with preliminary material. We introduce the basics of affine geometry, discuss the affine quasi-Einstein equation, and establish its basic properties. We discuss affine gradient Ricci solitons and other preliminary matters. For surface geometries, the Ricci tensor

$$\rho(x, y) := \mathrm{Tr}\{z \to R(z, x)y\}$$

carries the geometry; an affine connection on a surface is flat if and only if $\rho = 0$.

Chapter 13 is devoted to a discussion of the geometry of Type $\mathcal{A}$ surfaces. Any Type $\mathcal{A}$ surface is strongly projectively flat. The solution space to the quasi-Einstein equation for the critical eigenvalue $\mu = -1$ will play a central role in our discussion as it is a complete invariant of strongly projectively flat surfaces. By identifying the Christoffel symbols $\{\Gamma_{ij}{}^k\}$ with a point of $\mathbb{R}^6$, we parameterize such surfaces. The Ricci tensor of any Type $\mathcal{A}$ surface is symmetric and any such surface is strongly projectively flat. The set of flat Type $\mathcal{A}$ surfaces where Γ does not vanish identically is a smooth 4-dimensional manifold which may be identified with a $\mathbb{Z}_2$ quotient of $S^1 \times S^2 \times \mathbb{R}$. The set of Type $\mathcal{A}$ surfaces where the Ricci tensor has rank 1 and is positive or negative semi-definite is a 5-dimensional manifold which may be identified with $S^1 \times S^1 \times \mathbb{R}^3$. It is natural to identify Type $\mathcal{A}$ geometries which differ by a change of coordinates or, equivalently, by the action of the general linear group $\mathrm{GL}(2, \mathbb{R})$. The resulting moduli spaces of flat Type $\mathcal{A}$ surfaces, of Type $\mathcal{A}$ surfaces where the Ricci tensor has rank 1, and of Type $\mathcal{A}$ surfaces where the Ricci tensor is non-degenerate and has signature (p, q) is determined quite explicitly. The surfaces which are geodesically complete are described up to linear equivalence. We discuss affine Killing vector fields and affine gradient Ricci solitons for such geometries.

In Chapter 14, we present an analogous discussion for the Type $\mathcal{B}$ surfaces. These surfaces are, in general, not strongly projectively flat, and thus the solution space to the quasi-Einstein equation is of less utility. The structure group here is the $ax + b$ group rather than the general linear group where the action this time is $(x^1, x^2) \to (x^1, bx^1 + ax^2)$. Let $\ker_{\mathcal{B}}\{\rho\} - \Gamma_0$ be the the space of flat connections other than the trivial connection where all the Christoffel symbols vanish and let $\ker_{\mathcal{B}}\{\rho_s\} - \ker_{\mathcal{B}}\{\rho\}$ be the space of all connections where the Ricci tensor is purely alternating but does not vanish identically. In contrast to the situation for Type $\mathcal{A}$ geometries, these two spaces are not smooth. The set $\ker_{\mathcal{B}}\{\rho\} - \Gamma_0$ (resp. $\ker_{\mathcal{B}}\{\rho_s\} - \ker_{\mathcal{B}}\{\rho\}$) is an immersed 3-dimensional (resp. 2-dimensional) manifold with transversal intersections. We also discuss affine Killing vector fields and affine gradient Ricci solitons in this context. We determine the locally symmetric Type $\mathcal{B}$ surfaces.

In Chapter 15, we present some applications of affine surface theory. If $\mathcal{M} = (M, \nabla)$ is an affine surface, the modified Riemannian extension gives rise to a neutral signature metric $g_{\nabla,\Phi,T,S,X}$ on the cotangent bundle of M where X is a tangent vector field on M, where Φ is a symmetric 2-tensor on M, and where T and S are endomorphisms of the tangent bundle of M.

There is an intimate relation between the geometry of the affine surface $\mathcal{M}$ and the geometry of $\mathcal{N} := (T^*M, g_{\nabla,\Phi,T,S,X})$. We relate solutions to the affine quasi-Einstein equation on $\mathcal{M}$ and the Riemannian quasi-Einstein equation. We also construct Bach flat signature $(2, 2)$ metrics using the Riemannian extension and construct vanishing scalar invariant (VSI) manifolds.

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April 2019

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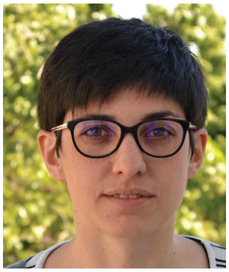

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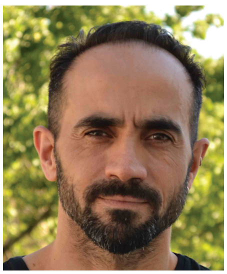

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