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A REMARK ON INFINITARY LANGUAGES

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In his paper [1] Chang provides among other things answers to questions of the following type: Given two models $\mathfrak{A}$ and $\mathfrak{B}$ of powers α and β, respectively, what is the least λ such that $\mathfrak{A} \equiv_{\lambda\kappa} \mathfrak{B}$ implies $\mathfrak{A} \equiv_{\infty\kappa} \mathfrak{B}$? His proofs are by induction on the quantifier rank of formulas and they use an idea which in the case of ordinary first-order language goes back to Ehrenfeucht and Fraïssé. But, as we show, one can easily prove that if λ is big compared with κ and with the cardinality of the universe of the structure $\mathfrak{A}$, then every $L_{\infty\kappa}$-formula is equivalent modulo the set of all $L_{\lambda\kappa}$-sentences which hold in $\mathfrak{A}$ to a $L_{\lambda\kappa}$-formula. From this, Chang's results follow immediately. The same method can be applied to similar problems concerning generalized languages.

We assume familiarity with the basic concepts of the model theory of the infinitary languages $L_{\lambda\kappa}$. We employ standard terminology and notation (see [1]). We assume throughout that we have a fixed similarity type and all models are of this type. If $\mathfrak{A}$ and $\mathfrak{B}$ are models then, by convention, α (resp. β) shall always denote the cardinality of the universe of $\mathfrak{A}$ (resp. $\mathfrak{B}$). $Th_{\lambda\kappa}(\mathfrak{A})$ is the set of all $L_{\lambda\kappa}$-sentences which hold in $\mathfrak{A}$. If φ_1 and φ_2 are $L_{\infty\kappa}$-formulas we say that φ_1 and φ_2 are $Th_{\lambda\kappa}(\mathfrak{A})$-equivalent if and only if $Th_{\lambda\kappa}(\mathfrak{A}) \models \varphi_1 \leftrightarrow \varphi_2$.—If α and κ are cardinals, we let $\alpha^{*\kappa} = \sum_{\mu<\kappa}[\alpha^\mu]^+$.

THEOREM 1. *Suppose that $\mathfrak{A}$ is a model and $\lambda \geq \alpha^{*\kappa}$. Then every $L_{\infty\kappa}$-formula is $Th_{\lambda\kappa}(\mathfrak{A})$-equivalent to a $L_{\lambda\kappa}$-formula.*

PROOF. By obvious substitution properties we need only show that the theorem holds for every $\xi < \kappa$ and every $L_{\infty\kappa}$-formula whose free variables are among $\{v_\eta \mid \eta < \xi\}$. The proof is by induction on $L_{\infty\kappa}$-formulas. The only nontrivial step is that of the infinite conjunction (resp. disjunction). Let $\varphi = \bigwedge_{i\in I}\varphi_i$. By induction hypothesis, for every $i \in I$ there exists $\psi_i \in L_{\lambda\kappa}$ such that $Th_{\lambda\kappa}(\mathfrak{A}) \models \varphi_i \leftrightarrow \psi_i$, therefore $Th_{\lambda\kappa}(\mathfrak{A}) \models \bigwedge_{i\in I}\varphi_i \leftrightarrow \bigwedge_{i\in I}\psi_i$. For $a \in {}^\xi A$ choose $i_a \in I$ such that if $\mathfrak{A} \models \neg\bigwedge_{i\in I}\psi_i[a]$ then $\mathfrak{A} \models \neg\psi_{i_a}[a]$. Then for every $i \in I$ we have $\mathfrak{A} \models \bigwedge_{a\in{}^\xi A}\psi_{i_a} \rightarrow \psi_i$. As the cardinality of ${}^\xi A$ is less than λ, $\bigwedge_{a\in{}^\xi A}\psi_{i_a} \rightarrow \psi_i$ is an $L_{\lambda\kappa}$-formula; therefore $Th_{\lambda\kappa}(\mathfrak{A}) \models \bigwedge_{a\in{}^\xi A}\psi_{i_a} \rightarrow \psi_i$ and $Th_{\lambda\kappa}(\mathfrak{A}) \models \bigwedge_{a\in{}^\xi A}\psi_{i_a} \rightarrow \bigwedge_{i\in I}\psi_i$. As the other implication obviously holds, we get that φ is $Th_{\lambda\kappa}(\mathfrak{A})$-equivalent to the $L_{\lambda\kappa}$-formula $\bigwedge_{a\in{}^\xi A}\psi_{i_a}$.

A similar proof works for every language $L_{\lambda\kappa}(Q_i \mid i \in I]$ with added unary (or $<\kappa$-ary) extensional quantifiers. Therefore the following corollary, which is an immediate consequence of Theorem 1, is also true for these languages.

COROLLARY. *Suppose that $\mathfrak{A}$ is a model and $\lambda \geq \alpha^{*\kappa}$. Then*

(1) *$Th_{\lambda\kappa}(\mathfrak{A})$ is a complete $L_{\mu\kappa}$-theory for every μ such that $\lambda \leq \mu \leq \infty$.*

(2) *If $\mathfrak{A} \equiv_{\lambda\kappa} \mathfrak{B}$, then $\mathfrak{A} \equiv_{\infty\kappa} \mathfrak{B}$.*

(3) *If $\mathfrak{A} \equiv_{\lambda\kappa} \mathfrak{B}$ and $\mathfrak{B} \prec_{\lambda\kappa} \mathfrak{C}$, then $\mathfrak{B} \prec_{\infty\kappa} \mathfrak{C}$.*

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We shall refer to $P_{\lambda\kappa}$, $E_{\lambda\kappa}$, $U_{\lambda\kappa}$ as the positive formulas, existential formulas, universal formulas, respectively, of the language $L_{\lambda\kappa}$.

THEOREM 2. *Suppose that* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are models and* $\lambda \geq \alpha^{*\kappa} + \beta^{*\kappa}$. *Then to every* $P_{\infty\kappa}$*-formula* φ, *there exists a* $P_{\lambda\kappa}$*-formula* ψ *such that* φ *and* ψ *are both* $Th_{\lambda\kappa}(\mathfrak{A})$*-equivalent and* $Th_{\lambda\kappa}(\mathfrak{B})$*-equivalent.*

Theorem 2 remains true if we replace $P_{\lambda\kappa}$ by $U_{\lambda\kappa}$ (or $E_{\lambda\kappa}$) and $P_{\infty\kappa}$ by $U_{\infty\kappa}$ (or $E_{\infty\kappa}$).

PROOF. The proof is similar to that of Theorem 1. Suppose that the free variables of φ are among $\{v_\eta \mid \eta < \xi\}$ where φ is the conjunction $\bigwedge_{i\in I}\varphi_i$, and, for every $i \in I$, ψ_i is a $P_{\lambda\kappa}$-formula such that $Th_{\lambda\kappa}(\mathfrak{A}) \models \varphi_i \leftrightarrow \psi_i$ and $Th_{\lambda\kappa}(\mathfrak{B}) \models \varphi_i \leftrightarrow \psi_i$. Then, for $a \in {}^{\xi}A$ (resp. $b \in {}^{\xi}B$) choose i_a (resp. i_b) such that if $\mathfrak{A} \models \neg\bigwedge_{i\in I}\psi_i[a]$ (resp. $\mathfrak{B} \models \neg\bigwedge_{i\in I}\psi_i[b]$) then $\mathfrak{A} \models \neg\psi_{i_a}[a]$ (resp. $\mathfrak{B} \models \neg\psi_{i_b}[b]$). As in the proof of Theorem 1 we show that: $Th_{\lambda\kappa}(\mathfrak{A}) \models \varphi \leftrightarrow \bigwedge_{a\in{}^{\xi}A}\psi_{i_a} \wedge \bigwedge_{b\in{}^{\xi}B}\psi_{i_b}$ and $Th_{\lambda\kappa}(\mathfrak{B}) \models \varphi \leftrightarrow \bigwedge_{a\in{}^{\xi}A}\psi_{i_a} \wedge \bigwedge_{b\in{}^{\xi}B}\psi_{i_b}$. But $\bigwedge_{a\in{}^{\xi}A}\psi_{i_a} \wedge \bigwedge_{b\in{}^{\xi}B}\psi_{i_b}$ is a $P_{\lambda\kappa}$-formula.

Note that Theorem 2 may be also proved for every language $L_{\lambda\kappa}(Q_i \mid i \in I)$ if the corresponding classes of positive, existential and universal formulas are properly defined.

COROLLARY. *Suppose that* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are models and* $\lambda \geq \alpha^{*\kappa} + \beta^{*\kappa}$. *Then*

(1) *If* $\mathfrak{A}P_{\lambda\kappa}\mathfrak{B}$, *then* $\mathfrak{A}P_{\infty\kappa}\mathfrak{B}$.

(2) *If* $\mathfrak{A}E_{\lambda\kappa}\mathfrak{B}$, *then* $\mathfrak{A}E_{\infty\kappa}\mathfrak{B}$.

(3) *If* $\mathfrak{A}U_{\lambda\kappa}\mathfrak{B}$, *then* $\mathfrak{A}U_{\infty\kappa}\mathfrak{B}$.

REFERENCES

[1] C. C. CHANG, *Some remarks on the model theory of infinitary languages*, ***The syntax and semantics of infinity languages***, edited by Jon Barwise, Springer-Verlag, Berlin, 1968, pp. 36–63.

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