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THE PERFECT SET THEOREM AND DEFINABLE WELLORDERINGS OF THE CONTINUUM

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Abstract. Let Γ be a collection of relations on the reals and let M be a set of reals. We call M a *perfect set basis* for Γ if every set in Γ with parameters from M which is not totally included in M contains a perfect subset with code in M. A simple elementary proof is given of the following result (assuming mild regularity conditions on Γ and M): *If M is a perfect set basis for Γ, the field of every wellordering in Γ is contained in M.* An immediate corollary is Mansfield's Theorem that the existence of a Σ^1_2 wellordering of the reals implies that every real is constructible. Other applications and extensions of the main result are also given.

§1. **Preliminaries.** Let $\omega = \{0, 1, 2, \dots\}$ be the set of natural numbers and $\mathscr{R} = \omega^\omega$ the set of all functions from ω to ω or (for simplicity) *reals*. We study subsets of the *product spaces* $\mathscr{X} = X_1 \times X_2 \times \cdots \times X_k$, where X_i is ω or $\mathscr{R}$. We call such subsets *pointsets*. Sometimes we think of them as relations and we write interchangeably $x \in A \Leftrightarrow A(x)$. A *pointclass* is a class of pointsets, usually in all product spaces. We shall be concerned primarily in this paper with the *analytical* pointclasses Σ^1_n, Π^1_n, Δ^1_n and their corresponding *projective* pointclasses $\boldsymbol{\Sigma}^1_n$, $\boldsymbol{\Pi}^1_n$, $\boldsymbol{\Delta}^1_n$. For information about them we refer to [8], [11] and [12]. For a pointclass Γ, *Determinacy* (Γ) abbreviates the statement: Every set of reals in Γ is determined. *Projective determinacy* is the hypothesis that every projective set is determined. For information about games, determinacy, etc., the reader can consult [1], [8] and [9].

We shall make considerable use of perfect sets of reals in the following. To avoid unnecessary repetition we assume that a perfect set is always nonempty. If $P \subseteq \mathscr{R}$ is perfect then a *code* of P is a real coding in any reasonable fashion the tree associated with P i.e. the set of all finite sequences from ω which are initial segments of elements of P. We shall also talk frequently about continuous functions mapping closed subsets of $\mathscr{R}$ into $\mathscr{R}$. Any such function can be completely described by a countable amount of information (e.g. its values at a reasonable countable dense subset of its domain), which in turn can be coded by a real called a *code* of the given continuous function.

Sometimes it will be convenient to work with the subspace 2^ω of $\mathscr{R}$ consisting of all binary reals. In this case, it is well known that for every perfect set $P \subseteq 2^\omega$

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there is a canonical homeomorphism $h: P \to 2^\omega$. Moreover a code of h can be recursively obtained from a code of P.

§2. **The main result.** A classical theorem of Gödel asserts that if every real is constructible (i.e. V = L holds for reals) then there is a Σ^1_2 wellordering of $\mathscr{R}$. Recently, Mansfield [6] proved the converse of this theorem. In attempting, as it is fashionable these days, to find an appropriate generalization of Mansfield's Theorem to higher levels of the analytical hierarchy from Projective Determinacy (see Corollary 2 below), we have discovered a general result which immediately implies Mansfield's Theorem and all its desired generalizations and which has a surprisingly simple proof. Our result is best explained by introducing some terminology first.

Let $M \subseteq \mathscr{R}$ be a set of reals. We say that M is *Δ^1_1-closed* if for all $\alpha_1 \cdots \alpha_n \in M$, if β is Δ^1_1 in $\alpha_1 \cdots \alpha_n$ then β belongs to M. If Γ is a pointclass and M a Δ^1_1-closed set of reals, we call M a *perfect set basis* for Γ if for all $P(\alpha, \beta_1 \cdots \beta_n)$ in Γ and all parameters $\alpha_1 \cdots \alpha_n$ from M, if $A = \{\alpha : P(\alpha, \alpha_1 \cdots \alpha_n)\}$ contains a real not in M then it contains a perfect subset with code in M. In this terminology, the Perfect Set Theorem of Solovay [13] and Mansfield [5] asserts that the set of constructible reals is a perfect set basis for Σ^1_2. Finally, let us call a pointclass Γ *reasonable* if it contains all Δ^1_1 pointsets and is closed under $\wedge$, $\vee$ and substitutions by Δ^1_1 functions (i.e. if $f: \mathscr{X} \to \mathscr{Y}$ is Δ^1_1 and $P \subseteq \mathscr{Y}$ is in Γ, then $Q(x) \Leftrightarrow P(f(x))$ is also in Γ). Obviously all the analytical pointclasses Σ^1_n, Π^1_n, Δ^1_n are reasonable. We can now state our main result.

THEOREM. *Let Γ be a reasonable pointclass and let M be a perfect set basis for Γ. If $\leq$ is a wellordering of a set of reals and $\leq \in \Gamma$, then the field of $\leq$ (i.e. the set $\{\alpha : \alpha \leq \alpha\}$) is contained in M.*

PROOF. Without loss of generality we can assume that the field of $\leq$ is contained in 2^ω so that we can work below with this space instead of $\mathscr{R}$. So let $\leq$ be a wellordering in Γ with field $A \subseteq 2^\omega$ and strict part $<$, where $\alpha < \beta \Leftrightarrow \alpha \leq \beta$ & $\alpha \neq \beta$. Assuming $A - M \neq \varnothing$, we shall easily obtain a contradiction after proving the following

LEMMA. *If $P \subseteq 2^\omega$ is a perfect set with code in M and $f: P \to A$ is* 1-1 *and continuous with code in M, then there is $Q \subseteq P$ perfect with code in M and $g: Q \to A$ also* 1-1 *and continuous with code in M, such that for all $\alpha \in Q$, $g(\alpha) < f(\alpha)$.*

PROOF OF LEMMA. Let α_0 be the least (in the sense of $\leq$) member of $A - M$. For each real $\alpha \in 2^\omega$ put $\bar{\alpha}(n) = 1 - \alpha(n)$. Let $h: P \to 2^\omega$ be the canonical homeomorphism and consider $h^{-1}(\alpha_0)$, $h^{-1}(\bar{\alpha}_0)$. These reals are distinct and not in M so that the same is true for $f(h^{-1}(\alpha_0))$, $f(h^{-1}(\bar{\alpha}_0))$. Since they both belong to A one of them must be bigger (in the sense of $<$) than α_0. Say $\alpha_0 < f(h^{-1}(\bar{\alpha}_0))$. Consider then the set $B = \{\alpha : \alpha \in P \;\&\; \overline{h(\alpha)} < f(\alpha)\}$. It is in Γ with parameters from M and contains $h^{-1}(\bar{\alpha}_0) \notin M$, so it contains a perfect subset Q with code in M. To finish the proof of the lemma put $g(\alpha) = \overline{h(\alpha)}$, for $\alpha \in Q$.

Let now $P_0 \subseteq 2^\omega$ be a perfect subset of A with code in M and let $f_0 =$ identity on P_0. By repeated use of the lemma we can construct a decreasing sequence of perfect sets $P_0 \supseteq P_1 \supseteq P_2 \supseteq \cdots$ and a sequence of functions $f_n: P_n \to A$ such that for all $\alpha \in P_{n+1}$, $f_{n+1}(\alpha) < f_n(\alpha)$. Since $\bigcap_n P_n \neq \emptyset$ this gives us immediately a contradiction. $\perp$

§3. Some corollaries and applications. As an immediate consequence of the preceding theorem we can now obtain.

COROLLARY 1 (MANSFIELD [6]). *The field of a Σ_2^1 wellordering of reals is contained in L. In particular, if there is a Σ_2^1 wellordering of $\mathscr{R}$, then $V = L$ holds for reals.*

More generally, one can see that if $\leq$ is a wellordering which is κ-Souslin with tree T, then the field of $\leq$ is contained in every inaccessible admissible set M containing T. (An admissible set M is inaccessible if for all $x \in M$ there is an admissible set N such that $x \in N \in M$. For the rest of the terminology and some facts about κ-Souslin pointsets see [2], [3], [7] and [8].)

Another generalization of Corollary 1 can be obtained by looking at the higher levels of the analytical hierarchy. For the definition of the set $\mathscr{C}_{2n+2}$ ($n \geq 0$) which is the analog of the set of constructible reals at level $2n + 2$ (so that $\mathscr{C}_2 =$ the set of constructible reals), the reader can consult [3] and [4].

COROLLARY 2. *Assume Determinacy ($\boldsymbol{\Delta}_{2n}^1$), $n \geq 0$. The field of every Σ_{2n+2}^1 wellordering is contained in $\mathscr{C}_{2n+2}$. In particular, $\mathscr{C}_{2n+2}$ is the largest set or reals carrying a Σ_{2n+2}^1 wellordering.*

PROOF. By the results in [3] and [4], $\mathscr{C}_{2n+2}$ is a perfect set basis for Σ_{2n+2}^1, granting Determinacy ($\boldsymbol{\Delta}_{2n}^1$). $\perp$

It is well known that if the class of projective sets *satisfies property P* (i.e. every uncountable projective set contains a perfect subset) then every projective wellordering has countable field. A more detailed analysis of the usual constructions of uncountable sets with no perfect subsets, shows in fact that if the class of $\boldsymbol{\Delta}_{n+1}^1$ sets satisfies property P then every $\boldsymbol{\Delta}_n^1$ wellordering of reals has countable field. The following result improves on this estimate.

COROLLARY 3. *Let $n \geq 1$ and assume the class of $\boldsymbol{\Delta}_n^1$ sets of reals satisfies property P. Then every $\boldsymbol{\Delta}_n^1$ wellordering of reals has countable field.*

PROOF. Assume that our hypothesis is satisfied and let $\leq$ be a wellordering of a set of reals such that $\leq \, \in \Delta_n^1(\alpha_0)$, where α_0 is the parameter in a $\boldsymbol{\Delta}_n^1$ definition of $\leq$. Using the Skolem-Löwenheim Theorem we can find easily a countable set of reals M which is a perfect set basis for $\Gamma = \Delta_n^1(\alpha_0)$. Then the field of $\leq$ is included in M so it is also countable. $\perp$

We do not know if the hypothesis of Corollary 3 can be weakened further by replacing $\boldsymbol{\Delta}_n^1$ by $\boldsymbol{\Pi}_{n-1}^1$.

As we mentioned before, from the proofs of the theorems in [3] and [4] it follows that, assuming Determinacy($\boldsymbol{\Delta}_{2n}^1$), every thin (i.e. containing no perfect subset) Σ_{2n+2}^1 set is contained in $\mathscr{C}_{2n+2}$. (For $n = 0$ this is of course the Perfect Set Theorem of Solovay and Mansfield.) Our last application provides a common generalization of this fact and Corollary 2 above. It will be convenient to introduce some terminology first.

Let $\prec$ be a wellfounded relation on reals (i.e. relation for which there is no infinite descending chain $\cdots \alpha_3 \prec \alpha_2 \prec \alpha_1 \prec \alpha_0$). Let $F = \{\alpha : \exists \beta\, (\alpha \prec \beta$ or $\beta \prec \alpha)\}$ be the field of $\prec$. We say that $\prec$ satisfies the *thin antichain condition* if the field of $\prec$ contains no perfect set of pairwise incomparable under $\prec$ elements or equivalently if for every perfect set $P \subseteq F$ there are $\alpha, \beta \in P$ such that $\alpha \prec \beta$.

THEOREM. *Assume Determinacy*(Δ^1_{2n}), $n \geq 0$. *The field of every* Σ^1_{2n+2} *wellfounded relation which satisfies the thin antichain condition is contained in* $\mathscr{C}_{2n+2}$.

PROOF. Let us first mention two basic properties of $\mathscr{C}_{2n+2}$ (and its relativization $\mathscr{C}_{2n+2}(\beta)$) that will be used below (and can be proved from Determinacy (Δ^1_{2n})).

(i) $\mathscr{C}_{2n+2}$ is a perfect set basis for Σ^1_{2n+2}.

(ii) $\alpha \in \mathscr{C}_{2n+2}(\beta)$ & $\beta \in \mathscr{C}_{2n+2}(\gamma) \Rightarrow \alpha \in \mathscr{C}_{2n+2}(\gamma)$.

Let $\prec$ be a Σ^1_{2n+2} wellfounded relation which satisfies the thin antichain condition and let F be its field. Assume that $F - \mathscr{C}_{2n+2} \neq \emptyset$ towards a contradiction.

Notice first that we may assume that $\prec$ has the following extra property:

$$(*) \qquad \text{For all } \alpha, \beta, \ \alpha \prec \beta \Rightarrow \alpha \in \mathscr{C}_{2n+2}(\beta).$$

To see this, let $\alpha \prec \beta \Rightarrow \exists \gamma\, B(\alpha, \beta, \gamma)$, where $B \in \Pi^1_{2n+1}$. Uniformize (using the results of [10]) B by $B^* \in \Pi^1_{2n+1}$, so that $\alpha \prec \beta \Leftrightarrow \exists \gamma\, B^*(\alpha, \beta, \gamma) \Leftrightarrow \exists!\, \gamma\, B^*(\alpha, \beta, \gamma)$. Following an idea of Mansfield, call $\beta \in F$ *bad* if $I_\beta = \{(\alpha, \gamma)\colon B^*(\alpha, \beta, \gamma)\}$ contains a perfect set. If there are no bad β's, clearly for every $\beta \in F$, $I_\beta \subseteq \mathscr{C}_{2n+2}(\beta)$, so $\alpha \prec \beta \Rightarrow \alpha \in \mathscr{C}_{2n+2}(\beta)$ and we are done. Otherwise, since the set of bad β's is Σ^1_{2n+2} we can find (by the basis theorem of [10]) a bad $\beta' \in \Delta^1_{2n+2}$. If β' is not $\prec$-minimal bad, there is a bad $\beta'' \prec \beta'$ also in Δ^1_{2n+2} etc. Proceeding this way we can find a $\prec$-minimal bad real β_0 which is also in Δ^1_{2n+2}. Assuming, as we may without loss of generality, that the constant 0 real α_0 is not in F let

$$\alpha \prec' \beta \Leftrightarrow (\alpha \prec \beta \,\&\, \beta \prec \beta_0) \quad \text{or} \quad (\alpha = \alpha_0 \,\&\, \beta \prec \beta_0).$$

Clearly $\prec' \in \Sigma^1_{2n+2}$, $\prec'$ satisfies the thin antichain condition and $\alpha \prec' \beta \Rightarrow \alpha \in \mathscr{C}_{2n+2}(\beta)$ by the minimality of β_0. Moreover the field F' of $\prec'$ contains $\{\beta : \beta \prec \beta_0\}$, so it contains a perfect subset P with code in Δ^1_{2n+2}, therefore $F' - \mathscr{C}_{2n+2} \neq \emptyset$. (Since otherwise $P \subseteq \mathscr{C}_{2n+2}$, therefore $\mathscr{R} \subseteq \mathscr{C}_{2n+2}$ violating $F - \mathscr{C}_{2n+2} \neq \emptyset$.) So if $\prec$ itself does not satisfy $(*)$ we can replace it by $\prec'$. Therefore it is safe to assume that $\prec$ satisfies $(*)$ to start with.

Let us call a partial function from reals to reals a Σ^1_n *function* if its graph is Σ^1_n. As in the proof of the theorem in §2 we can immediately obtain a contradiction after proving the following

LEMMA. *Let P be a perfect set with code in $\mathscr{C}_{2n+2}$ and let $f\colon P \to F$ be a partial Σ^1_{2n+2} in parameters from $\mathscr{C}_{2n+2}$ 1-1 function. Then we can find a perfect set $Q \subseteq P$ with code in $\mathscr{C}_{2n+2}$ and $g\colon Q \to F$ a partial Σ^1_{2n+2} with parameters from $\mathscr{C}_{2n+2}$ 1-1 function such that for all $\alpha \in Q$, $g(\alpha) \prec f(\alpha)$.*

PROOF OF LEMMA. Let $f[P] = A \subseteq F$. Clearly A is Σ^1_{2n+2} with parameters from $\mathscr{C}_{2n+2}$ and contains a real not in $\mathscr{C}_{2n+2}$ (namely $f(\alpha)$, where $\alpha \in P - \mathscr{C}_{2n+2}$), so it contains a perfect subset R with code in $\mathscr{C}_{2n+2}$ and consequently it contains a perfect set $S \subseteq R$ such that $S \cap \mathscr{C}_{2n+2} = \emptyset$. ($S$ can be taken to be $i^{-1}\{\alpha \in 2^\omega : \forall n\, (\alpha(2n) = \beta_0(n))\}$, where $i\colon R \to 2^\omega$ is the canonical homeomorphism and $\beta_0 \notin \mathscr{C}_{2n+2}$.) Find then $\bar{\alpha}, \beta \in S$ such that $\bar{\alpha} \prec \beta$. By the basis theorem we can find $\bar{\beta} \in \Delta^1_{2n+2}(\bar{\alpha}, \gamma)$, where γ is a parameter in $\mathscr{C}_{2n+2}$, such that $\bar{\alpha} \prec \bar{\beta} \in R$. Then by standard prewellordering arguments (see [8]) we can find a partial Σ^1_{2n+2} with parameters from $\mathscr{C}_{2n+2}$ function h such that $h(\bar{\alpha}) = \bar{\beta}$. (We are using here the fact that there are partial Σ^1_{2n+2} functions $\{d_i\}_{i \in \omega}$ from $\mathscr{R}$ into $\mathscr{R}$ such that if $\gamma \in$

$\Delta^1_{2n+2}(\delta)$ then there is some $i \in \omega$ so that $\gamma = d_i(\delta)$.) Consider the set $M = \{\alpha: h(\alpha)$ is defined & $\alpha \prec h(\alpha)$ & $h(\alpha) \in R\}$; M is Σ^1_{2n+2} with parameters from $\mathscr{C}_{2n+2}$ and contains $\bar{\alpha} \notin \mathscr{C}_{2n+2}$. so it contains a perfect set T with code in $\mathscr{C}_{2n+2}$. Pick $\alpha'' \in T - \mathscr{C}_{2n+2}$ and put $\beta' = h(\alpha'')$. Since $\exists\alpha(h(\alpha) = \beta'$ & $\alpha \in T)$, we can find $\alpha' \in T$, $\alpha' \in \Delta^1_{2n+2}(\beta',\delta)$, where $\delta \in \mathscr{C}_{2n+2}$, such that $h(\alpha') = \beta'$ and furthermore we can find a partial Σ^1_{2n+2} with parameters from $\mathscr{C}_{2n+2}$ function g' such that $g'(\beta') = \alpha'$. Consider now the set

$$N = \{\beta: \beta \in R \,\&\, g'(\beta) \text{ is defined } \&\, g'(\beta) \in T \,\&\, h(g'(\beta)) = \beta\};$$

N contains $\beta' \notin \mathscr{C}_{2n+2}$ (since otherwise by $(*)$ $\alpha'' \in \mathscr{C}_{2n+2}$, because $\alpha'' \prec h(\alpha'') = \beta'$) so it contains a perfect set $Q' \subseteq R$ with code in $\mathscr{C}_{2n+2}$. Clearly g' is 1-1 on Q' and $g'(\beta) \prec \beta$ for all $\beta \in Q'$. Let now Q be a perfect subset of $f^{-1}[Q']$ with code in $\mathscr{C}_{2n+2}$ and let $g: Q \to F$ be given by $g(\alpha) = g'(f(\alpha))$. $\perp$

REFERENCES

[1] J. E. Fenstad, *The axiom of determinateness*, ***Proceedings of the Second Scandinavian Logic Symposium*** (J. E. Fenstad, Editor), North-Holland, Amsterdam, pp. 41–61.

[2] A. S. Kechris, *On projective ordinals*, this Journal, vol. 39 (1974), pp. 269–282.

[3] ———, *The theory of countable analytical sets*, ***Transactions of the American Mathematical Society***, vol. 202(1975), pp. 259–297.

[4] A. S. Kechris and Y. N. Moschovakis, *Two theorems about projective sets*, ***Israel Journal of Mathematics***, vol. 12(1972), pp. 391–399.

[5] R. Mansfield, *Perfect subsets of definable sets of real numbers*, ***Pacific Journal of Mathematics***, vol. 35(1970), pp. 451–457.

[6] ———, *The nonexistence of Σ^1_2 wellorderings of the Baire space*, ***Fundamenta Mathematicae***, vol. 86(1975), pp. 279–282.

[7] D. A. Martin, *Projective sets and cardinal numbers: some questions related to the continuum problem*, this Journal (to appear).

[8] Y. N. Moschovakis, ***Descriptive set theory***, North-Holland, Amsterdam (to appear).

[9] ———, *Determinacy and prewellorderings of the continuum*, ***Mathematical logic and foundations of set theory*** (Y. Bar-Hillel, Editor), North-Holland, Amsterdam, 1970, pp. 24–62.

[10] ———, *Uniformization in a playful universe*, ***Bulletin of the American Mathematical Society***, vol. 77(1970), pp. 731–736.

[11] H. Rogers, Jr., ***Theory of recursive functions and effective computability***, McGraw-Hill, New York, 1967.

[12] J. R. Shoenfield, ***Mathematical logic***, Addison-Wesley, Reading, Massachusetts, 1967.

[13] R. M. Solovay, *On the cardinality of Σ^1_2 sets of reals*, ***Foundations of mathematics, Symposium Commemorating the 60th Birthday of K. Gödel***, Springer-Verlag, Berlin and New York, 1966, pp. 58–73.

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