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Abstract. We show that $\Sigma_{4}^{1}$-Amoeba-absoluteness implies that $\forall a \in \mathbb{R}\left(\omega_{1}^{L[a]}<\omega_{1}^{V}\right)$, and hence $\Sigma_{3}^{1}-$ measurability. This answers a question of Haim Judah (private communication).

## Introduction

We study the relationship between Amoeba forcing and projective measurability. Recall that the Amoeba partial order $\mathbb{A}$ is defined as follows.

$$
\begin{aligned}
& A \in \mathbb{A} \Longleftrightarrow A \subseteq 2^{\omega} \wedge A \text { open } \wedge \mu(A)<\frac{1}{2} \\
& A \leq B \Longleftrightarrow B \subseteq A
\end{aligned}
$$

Amoeba forcing generically adds a measure one set of random reals. Its importance in the investigation of measurability of projective sets stems from the classical result, due to Solovay, that
(*) all $\Sigma_{2}^{1}$-sets are measurable $\Longleftrightarrow \forall a \in \mathbb{R}(\mu(\operatorname{Ra}(L[a]))=1)$

[^0](see, e.g., [JS 2, 0.1. and §3]). Here $R a(M)$ denotes the set of reals random over a model $M$ of set theory.

The connection between Amoeba forcing and projective measurability was made more explicit through Judah's study of absoluteness between models $V \subseteq W$ of set theory such that $W$ is a forcing extension of $V[\mathrm{Ju}]$.

Definition (Judah [Ju, § 2]). Let $V$ be a universe of set theory. Given a forcing notion $\mathbb{P} \in V$ we say that $V$ is $\Sigma_{n}^{1}-\mathbb{P}$-absolute iff for every $\Sigma_{n}^{1}$-sentence $\phi$ with parameters in $V$ we have $V \models \phi$ iff $V^{\mathbb{P}} \models \phi$. (So this is equivalent to saying that $\mathbb{R}^{V} \prec_{\Sigma_{n}^{1}} \mathbb{R}^{V^{\mathbb{P}}}$.) Note that Shoenfield's Absoluteness Lemma [Je, Theorem 98] says that $V$ is always $\Sigma_{2}^{1}-\mathbb{P}-$ absolute. Furthermore, using $\left({ }^{*}\right)$, Judah showed [Ju, § 2]
(**) all $\Sigma_{2}^{1}$-sets are measurable in $V \Longleftrightarrow V$ is $\Sigma_{3}^{1}-\mathbb{A}$-absolute.
Whereas there is no way of getting a characterization of $\Sigma_{3}^{1}$-measurability analogous to $\left(^{*}\right),\left({ }^{* *}\right)$ suggests the investigation of the relation between $\Sigma_{3}^{1}$-measurability and $\Sigma_{4}^{1}-\mathbb{A}-$ absoluteness. The main goal of this note is to establish one implication, namely that $\Sigma_{4}^{1}-\mathbb{A}$-absoluteness implies $\Sigma_{3}^{1}-$ measurability (Theorem 5 in $\S 2$ ). Our tools for proving this theorem are a partial earlier result of Judah's, who showed Theorem 5 under the additional assumption that $\forall a \in \mathbb{R}\left(\omega_{1}^{L[a]}<\omega_{1}^{V}\right)$, and combinatorial ideas due to Cichoń and Pawlikowski [CP], which will eventually yield that Judah's additional assumption is in fact a consequence of $\Sigma_{4}^{1}-\mathbb{A}$-absoluteness ( $\S 1$ and Theorem 4 in $\S 2$ ).

Notation. We shall mostly work with $2^{\omega}$ or $\omega^{\omega}$ instead of $\mathbb{R}$. $\mathcal{L}$ denotes the ideal of Lebesgue measure zero sets, and $\mathcal{B}$ is the ideal of meager sets. $\Sigma_{n}^{1}(\mathcal{L})$ stands for all $\Sigma_{n}^{1}$-sets are Lebesgue measurable; and $\Sigma_{n}^{1}(\mathcal{B})$ means all $\Sigma_{n}^{1}$-sets have the property of Baire. For a non-trivial $\sigma$-ideal $\mathcal{I} \subseteq P\left(2^{\omega}\right)$, let $\operatorname{add}(\mathcal{I})$ be the size of the smallest family of members in $\mathcal{I}$ whose union is not in $\mathcal{I} ; \operatorname{cov}(\mathcal{I})$ denotes the least $\kappa$ such that $2^{\omega}$ can be covered by $\kappa$ sets from $\mathcal{I}$; unif( $\mathcal{I})$ is the cardinality of he smallest subset of the reals which does not lie in $\mathcal{I}$; and $\operatorname{cof}(\mathcal{I})$ is the size of the smallest $\mathcal{F} \subseteq \mathcal{I}$ such that every member of $\mathcal{I}$ is included in a member of $\mathcal{F}$. We always have $\operatorname{add}(\mathcal{I}) \leq \operatorname{cov}(\mathcal{I}) \leq \operatorname{cof}(\mathcal{I})$ and $\operatorname{add}(\mathcal{I}) \leq \operatorname{unif}(\mathcal{I}) \leq \operatorname{cof}(\mathcal{I})$ (see, e.g., $[\mathrm{CP}]$ for details concerning these invariants in case $\mathcal{I}=\mathcal{L}$ or $\mathcal{B}$ ).

Our forcing notation is rather standard (see [Je] for any notion left undefined here). We confuse to some extent Boolean-valued models $V^{\mathbb{P}}$ and forcing extensions $V[G], G$
$\mathbb{P}$-generic over $V$. For p.o.s $\mathbb{P}, \mathbb{Q}, \mathbb{P}<_{c} \mathbb{Q}$ means that $\mathbb{P}$ can be completely embedded in $\mathbb{Q}$. For a sentence of the $\mathbb{P}$-forcing language $\phi,\|\phi\|$ is the Boolean value of $\phi . \mathbb{P}$-names for objects in the forcing extension are denoted by symbols like $\breve{r}$. Finally, $\mathbb{B}$ will stand for the random algebra, $\mathbb{C}$ for the Cohen algebra, and $\mathbb{D}$ for the Hechler p.o. (see, e.g., [BJS]).

Acknowledgments. I am very much indebted to both Haim Judah (for sharing with me his insight into projective measurability and motivating me to work in the area) and Andrzej Rosłanowski (for several stimulating discussions, concerning mainly the material in § 1).

## § 1. The combinatorial component

We start with a straightforward generalization of one version of the main result of [CP]. The proof is included for completeness' sake.

Theorem 1 (Cichoń - Pawlikowski $[\mathrm{CP}, \S 1]$ ). Assume that $\mathbb{C} \leq_{c} \mathbb{P}$, and that for any uncountable $T \subseteq \mathbb{P}$ there is an $s \in \mathbb{C}$ such that for all $\ell \in \omega$ there exists $F \subseteq T$ of size $\ell$ such that any $t$ extending $s$ is compatible with $\bigcap F \in \mathbb{P}$. Then there is a family $\left\{A_{x} ; x \in \omega^{\omega} \cap V\right\}$ of Lebesgue measure zero sets in $V^{\mathbb{C}}$ such that for all $z \in V^{\mathbb{P}},\left\{x \in \omega^{\omega} \cap V ; z \notin A_{x}\right\}$ is at most countable.

Proof. Let $\left\{\tau_{n} ; n \in \omega\right\}$ be a one-to-one enumeration of $\omega^{<\omega}$; set $\operatorname{code}(\tau)=n$ iff $\tau=\tau_{n}$ for any $\tau \in \omega^{<\omega}$. Let $\left\{C^{n}(i) ; i \in \omega\right\}$ be an enumeration of all open intervals in the unit interval $\mathbb{I}=[0,1]$ with rational endpoints of length $2^{-n}$, For $x, y \in \omega^{\omega}$ let

$$
B_{x, y}^{n}= \begin{cases}C^{n}\left(\tau_{y(n)}(\operatorname{code}(x \upharpoonright y(n+1)))\right) & \text { if } \operatorname{code}(x \upharpoonright y(n+1)) \in \operatorname{dom}\left(\tau_{y(n)}\right) \\ \emptyset & \text { if not }\end{cases}
$$

Let $B_{x, y}=\bigcap_{n} \bigcup_{m>n} B_{x, y}^{m}$. Clearly $\mu\left(B_{x, y}\right)=0$. We claim that if $c$ is Cohen over $V$, $A_{x}=B_{x, c}$ for $x \in \omega^{\omega} \cap V$, then $\left\{A_{x} ; x \in \omega^{\omega} \cap V\right\}$ is the required family.

For suppose not. Then there are a $\mathbb{P}$-name $\breve{z}$, an uncountable set $T \subseteq \omega^{\omega} \cap V, T \in V$, conditions $p_{x} \in \mathbb{P}$, and $k_{x} \in \omega(x \in T)$ such that

$$
\begin{equation*}
p_{x} \Vdash_{\mathbb{P}} \forall n \geq k_{x}\left(\breve{z} \notin B_{x, \breve{c}}^{n}\right) \tag{}
\end{equation*}
$$

Choose $T^{\prime} \subseteq T$ uncountable and $k \in \omega$ such that $\forall x \in T^{\prime}\left(k_{x}=k\right)$. Fix $s \in \mathbb{C}$ according to $T^{\prime}$. Let $\ell \geq k, \ell \geq l h(s)$, and choose $F \subseteq \omega^{\omega}$ of size $2^{\ell}$ such that $\left\{p_{x} ; x \in F\right\}$ satisfies the requirements of the Theorem. Next let $n>\ell$ be such that $|\{x \mid n ; x \in F\}|=2^{\ell}$. Let $F=\left\{x_{i} ; i \in 2^{\ell}\right\}$, and choose $i_{0}, \ldots, i_{2^{\ell}-1}$ such that $C^{\ell}\left(i_{0}\right) \cup \ldots \cup C^{\ell}\left(i_{2^{\ell}-1}\right)=\mathbb{I}$. Let $m \in \omega$ be such that $\tau_{m}\left(\operatorname{code}\left(x_{0} \upharpoonright n\right)\right)=i_{0}, \ldots, \tau_{m}\left(\operatorname{code}\left(x_{2^{\ell}-1} \upharpoonright n\right)\right)=i_{2^{\ell}-1}$. Let $t \leq s$ be such that $t(\ell)=m, t(\ell+1)=n$. Then $\bigcup_{i \in 2^{\ell}} C^{\ell}\left(\tau_{t(\ell)}\left(\operatorname{code}\left(x_{i} \upharpoonright t(l+1)\right)\right)\right)=\mathbb{I}$, i.e.

$$
t \cap \bigcap\left\{p_{x} ; x \in F\right\} \Vdash_{\mathbb{P}} \breve{z} \in \bigcup_{i \in 2^{\ell}} C^{\ell}\left(\tau_{\breve{c}(\ell)}\left(\operatorname{code}\left(x_{i} \mid \breve{c}(\ell+1)\right)\right)\right)=\bigcup_{i \in 2^{\ell}} B_{x_{i}, \breve{c}}^{\ell},
$$

contradicting $(*)$.

As each open set in $2^{\omega}$ can be written as a countable disjoint union of sets of the form $[\sigma]=\left\{f \in 2^{\omega} ; \sigma \subseteq f\right\}$, where $\sigma \in 2^{<\omega}$, we can think of a condition $A$ in the Amoeba algebra $\mathbb{A}$ as a function $\phi: \omega \rightarrow \bigcup_{i \in \omega} P\left(2^{i}\right)$ with $\phi(i) \in P\left(2^{i}\right)$ such that $\sigma \in \phi(i)$ iff $\sigma \in 2^{i}$ and $\sigma$ lies in the countable disjoint decomposition of $A$. We can furthermore assume that $\phi$ has the property:

$$
\begin{equation*}
\forall \sigma \in 2^{i} \backslash \phi(i)\left(\mu(\cup\{[\tau] ; \tau \supseteq \sigma \wedge \exists j>i(\tau \in \phi(j))\})<2^{-i}\right) \tag{*}
\end{equation*}
$$

(Then $\phi$ is unique.) We define a p.o. $\mathbb{A}^{\prime}$ as follows.

$$
\begin{gathered}
(u, \phi) \in \mathbb{A}^{\prime} \Longleftrightarrow\left\{\begin{array}{l}
\text { 1) } \operatorname{dom}(\phi)=\omega \wedge \forall i \in \omega\left(\phi(i) \in P\left(2^{i}\right)\right) \wedge \phi \text { satisfies }(*) \\
\text { 2) } u \subseteq \phi(u \text { is an initial segment of } \phi) \\
\text { 3) } \mu(\cup\{[\sigma] ; \exists i \in \omega(\sigma \in \phi(i))\})<\frac{1}{2}
\end{array}\right. \\
(u, \phi) \leq(v, \psi) \Longleftrightarrow u \supseteq v \wedge \forall i \forall \sigma \in \psi(i) \exists j \leq i \exists \tau \in \phi(j)(\sigma \supseteq \tau)
\end{gathered}
$$

Lemma 1. $\mathbb{A}$ and $\mathbb{A}^{\prime}$ are equivalent.
Proof. We define $\Phi: \mathbb{A} \rightarrow \mathbb{A}^{\prime}$ as follows. $\Phi(\phi)=(u, \phi)$, where $u \subseteq \phi$ is such that $\operatorname{dom}(u)$ is maximal with the following property: for any extension $\psi \supseteq \phi$ in $\mathbb{A}$, $\psi \upharpoonright \operatorname{dom}(u)=\phi \upharpoonright \operatorname{dom}(u)$. We claim that $\Phi$ is a dense embedding.

Clearly $\psi \leq \phi$ implies $\Phi(\psi) \leq \Phi(\phi)$, and $\psi \perp \phi$ implies $\Phi(\psi) \perp \Phi(\phi)$. To check density, choose $(u, \phi) \in \mathbb{A}^{\prime}$. Let $i:=\operatorname{dom}(u)-1$; and set $S_{\phi}:=\left\{\sigma \in 2^{i}\right.$; for no $j \leq i$ does there exist $\tau \in u(j)$ such that $\sigma \supseteq \tau\}$. For $\sigma \in S_{\phi}$ we have $m_{\sigma}:=\mu([\sigma] \backslash \cup\{[\tau] ; \tau \supseteq \sigma \wedge \exists i \geq$ $\operatorname{dom}(u)(\tau \in \phi(i))\})>0$. Let $a:=\min \left\{m_{\sigma} ; \sigma \in S_{\phi}\right\} ;$ and note that $\sum_{\sigma \in S_{\phi}} m_{\sigma}>\frac{1}{2}$.

Now define $\psi$ satisfying $\left({ }^{*}\right)$ such that

1) $\forall i \in \operatorname{dom}(u)(\psi(i)=\phi(i))$
2) $\forall i \geq \operatorname{dom}(u) \forall \tau_{1} \in \phi(i) \exists j \leq i \exists \tau_{2} \in \psi(j)\left(\tau_{2} \subseteq \tau_{1}\right)$
3) $\frac{1}{2}>\mu(\cup\{[\tau] ; \exists i \in \omega(\tau \in \psi(i))\})>\frac{1}{2}-\frac{a}{2}$
4) for each $\sigma \in S_{\phi}, \mu([\sigma] \backslash \cup\{[\tau] ; \tau \supseteq \sigma \wedge \exists i \geq n(\tau \in \psi(i))\}) \geq \frac{a}{2}$

This is clearly possible. By construction we have $\Phi(\psi)=(u, \psi) \leq(u, \phi)$.
Next define $\mathbb{A}^{\prime \prime} \subseteq \mathbb{A}^{\prime}$ by

$$
(u, \phi) \in \mathbb{A}^{\prime \prime} \Longleftrightarrow\left\{\begin{array}{l}
\text { for some } n \in \omega \text { we have } \mu(\cup\{[\sigma] ; \exists i \in \operatorname{dom}(u)(\sigma \in u(i))\})>\frac{1}{2}-\frac{1}{2^{n}} \\
\mu(\cup\{[\sigma] ; \exists i \in \operatorname{dom}(u)-1(\sigma \in u(i))\}) \leq \frac{1}{2}-\frac{1}{2^{n}} \\
\text { and } \mu(\cup\{[\sigma] ; \exists i \geq \operatorname{dom}(u)(\sigma \in \phi(i))\})<\frac{1}{2^{n+7}}
\end{array}\right.
$$

Clearly $\mathbb{A}^{\prime \prime}$ is dense in $\mathbb{A}^{\prime}$. Finally we want to define $h: \mathbb{A}^{\prime \prime} \rightarrow \mathbb{C}$ giving rise to a complete embedding of $\mathbb{C}$ into $\mathbb{A}$. To this end, let $f: \omega \rightarrow \omega$ be such that $\forall n \exists{ }^{\infty} i(f(i)=n)$. For $(u, \phi) \in \mathbb{A}^{\prime \prime}$ and $n \in \omega$ such that $\frac{1}{2}-\frac{1}{2^{n+1}} \geq \mu(\cup\{[\sigma] ; \exists i \in \operatorname{dom}(u)(\sigma \in u(i))\})>\frac{1}{2}-\frac{1}{2^{n}}$ and each $j \leq n$ choose $i_{j}$ minimal such that $\mu\left(\cup\left\{[\sigma] ; \exists i \in i_{j}(\sigma \in u(i))\right\}\right)>\frac{1}{2}-\frac{1}{2^{j}}$, and let $h((u, \phi))=\left\langle f\left(i_{0}\right)\right\rangle^{\wedge} \ldots \wedge\left\langle f\left(i_{n}\right)\right\rangle$. We leave it to the reader to verify that $h: \mathbb{A}^{\prime \prime} \rightarrow \mathbb{C}$ is indeed a projection (in the forcing theoretic sense). Furthermore, given $T \subseteq \mathbb{A}^{\prime \prime}$ uncountable we can find $T^{\prime} \subseteq T$ uncountable and $u$ such that all elements of $T^{\prime}$ are of the form $(u, \phi)$ for some $\phi$. Then there is an $s \in \mathbb{C}$ such that $\forall(u, \phi) \in T^{\prime}(h((u, \phi))=s)$. Next, given $\ell \in \omega$, we can find $F \subseteq T^{\prime}$ of size $\ell$ such that $\cap F \in \mathbb{A}^{\prime \prime}$. Clearly $h(\cap F)=s$ and so any extension of $s$ in $\mathbb{C}$ will be compatible with $\cap F$. Hence we have proved that $\mathbb{A}^{\prime \prime}$ satisfies the requirements of Theorem 1. Using Lemma 1 we get

Theorem 2. There is a family $\left\{A_{x} ; x \in \omega^{\omega} \cap V\right\}$ of Lebesgue measure zero sets in $V^{\mathbb{A}}$ such that for all $z \in V^{\mathbb{A}},\left\{x \in \omega^{\omega} \cap V ; z \notin A_{x}\right\}$ is at most countable.

Corollary 1. Let $V \subseteq W$ be models of $Z F C$ such that $\omega_{1}^{V}=\omega_{1}^{W}$. Then there is no real random in $W^{\mathbb{A}}$ over $V^{\mathbb{A}}$.

Proof. Let $\left\{A_{x} ; x \in \omega^{\omega} \cap W\right\}$ be as in Theorem 2 and note that $\forall z \in \omega^{\omega} \cap W^{\mathbb{A}} \exists x \in$ $\omega^{\omega} \cap V\left(z \in A_{x}\right)$. Hence any real in $W^{\mathbb{A}}$ lies in a measure zero set coded in $V^{\mathbb{A}}$.

Using a similar argument as in [CP, § 3] we can prove
Corollary 2. After adding one Amoeba $\operatorname{real}, \operatorname{cov}(\mathcal{L})=\operatorname{add}(\mathcal{L})=\omega_{1}$ and unif $(\mathcal{L})=$ $\operatorname{cof}(\mathcal{L})=2^{\omega}$.

We note that in [BJS, § 2] results much stronger than Theorem 2 and the Corollaries were proved for the Hechler p.o. $\mathbb{D}$; e.g. it was shown that after adding a Hechler real, $\operatorname{add}(\mathcal{B})=\operatorname{unif}(\mathcal{B})=\omega_{1}$ and $\operatorname{cof}(\mathcal{B})=\operatorname{cov}(\mathcal{B})=2^{\omega}$ [BJS, 2.5.]. Accordingly we ask:

Question [BJS, 2.7.]. Is unif $(\mathcal{B})=\omega_{1}$ and $\operatorname{cov}(\mathcal{B})=2^{\omega}$ after adding an Amoeba real?

Before ending this section I wish to include a few comments, some of which are due to Andrzej Rosłanowski.

Definition (implicit in [Tr 2]). A p.o. $\mathbb{P}$ is said to have $\left(\omega_{1}, \omega\right)$-caliber iff for any uncountable $T \subseteq \mathbb{P}$ of size $\omega_{1}$ there is a countable $F \subseteq T$ such that $\cap F \in \mathbb{P}$.

This is equivalent to: any set of ordinals $A$ in $V^{\mathbb{P}}$ of size $\geq \omega_{1}$ has a countable subset $B$ in $V[\operatorname{Tr} 2]$. It is easy to see that if $\mathbb{C} \leq_{c} \mathbb{P}$ and $\mathbb{P}$ has $\left(\omega_{1}, \omega\right)$-caliber, then the assumptions of Theorem 1 are satisfied. Furthermore the Amoeba algebra $\mathbb{A}$ has $\left(\omega_{1}, \omega\right)$-caliber (the proof for this is similar to the corresponding proof for the random algebra $\mathbb{B}$, given in $[\mathrm{Tr}$ 2]). This gives an alternative argument to prove Theorem 2. - Our reason for giving the (slightly more difficult) above argument involving $\mathbb{A}^{\prime}$ and $\mathbb{A}^{\prime \prime}$ is that along the same lines results corresponding to Theorem 2 and the Corollary can be proved for p.o.s not having $\left(\omega_{1}, \omega\right)$-caliber. We include two examples for such p.o.s:

- the eventually different reals p.o. $\mathbb{E}$, due to Miller [Mi]:

$$
\begin{gathered}
(s, G) \in \mathbb{E} \Longleftrightarrow s \in \omega^{<\omega} \wedge G \in\left[\omega^{\omega}\right]^{<\omega} \\
(s, G) \leq(t, H) \Longleftrightarrow s \supseteq t \wedge G \supseteq H \wedge \forall g \in H \forall i(\operatorname{dom}(t) \leq i<\operatorname{dom}(s) \rightarrow s(i) \neq g(i))
\end{gathered}
$$

— the localization p.o. $\mathbb{L}$ (see, e.g., $[\operatorname{Tr} 3, \S 2]$ ):
$(\sigma, G) \in \mathbb{L} \Longleftrightarrow \sigma \in\left([\omega]^{<\omega}\right)^{<\omega} \wedge \forall i \in \operatorname{dom}(\sigma)(|\sigma(i)|=i+1) \wedge G \in\left[\omega^{\omega}\right] \leq \operatorname{dom}(\sigma)+1$ $(\sigma, G) \leq(\tau, H) \Longleftrightarrow \sigma \supseteq \tau \wedge G \supseteq H \wedge \forall g \in H \forall i(\operatorname{dom}(\tau) \leq i<\operatorname{dom}(\sigma) \rightarrow g(i) \in \sigma(i))$

Let $\left\{f_{\alpha} ; \alpha<\omega_{1}\right\} \subseteq \omega^{\omega}$ be a family of pairwise eventually different reals (i.e. $\alpha \neq \beta \rightarrow$ $\left.\exists n \forall k \geq n\left(f_{\alpha}(k) \neq f_{\beta}(k)\right)\right)$. Then $\left\{\left(\left\rangle,\left\{f_{\alpha}\right\}\right) ; \alpha<\omega_{1}\right\}\right.$ is an uncountable set of conditions in $\mathbb{E}$ (and $\mathbb{L}$ ) such that no countable subset has nontrivial intersection, thus witnessing that $\mathbb{E}$ and $\mathbb{L}$ do not have $\left(\omega_{1}, \omega\right)$-caliber. We leave it to the reader to verify that both still satisfy the assumptions of Theorem 1, however (note that both have a definition similar to, but easier than, $\left.\mathbb{A}^{\prime \prime}\right)$.
(The localization p.o. $\mathbb{L}$ arose from Bartoszyński's characterization of the cardinal $\operatorname{add}(\mathcal{L})[\mathrm{Ba}]$, and is closely related to the Amoeba algebra $\mathbb{A}$. Truss $[\operatorname{Tr} 3, \S 4]$ showed that
$\mathbb{A}<_{c} \mathbb{L}$. By the above discussion the converse cannot hold.)

## § 2. The projective part

We first introduce a notion closely related to absoluteness, and discuss the relationship between the two notions.

Definition (Judah [Ju, § 2]). Let $V$ be a universe of set theory. Given a forcing notion $\mathbb{P} \in V$ we say that $V$ is $\Sigma_{n}^{1}-\mathbb{P}$-correct iff for every $\Sigma_{n}^{1}$-formula $\phi(x)$ with parameters in $V$ and for every $\mathbb{P}$-name $\tau$ for a real we have $V[\tau] \models \phi(\tau)$ iff $V^{\mathbb{P}} \models \phi(\tau)$.

Lemma 2. Suppose $\mathbb{P}<_{c} \mathbb{Q}$. Then:
(i) $\Sigma_{n}^{1}-\mathbb{Q}$-correctness implies $\Sigma_{n}^{1}-\mathbb{P}$-correctness.
(ii) $\Sigma_{n+1}^{1}-\mathbb{Q}$-absoluteness $+\Sigma_{n}^{1}-\mathbb{Q}$-correctness implies $\Sigma_{n+1}^{1}-\mathbb{P}$-absoluteness.

Proof. We prove both (i) and (ii) by induction on $n$.
(i) $n=2$ follows from Shoenfield's Absoluteness Lemma. Suppose it is true for $n \geq 2$ and assume $V$ is $\Sigma_{n+1}^{1}-\mathbb{Q}$-correct. Let $\phi(x)$ be a $\Sigma_{n+1}^{1}-$ formula, $\phi(x)=\exists y \psi(y, x)$ where $\psi$ is $\Pi_{n}^{1}$. Suppose first that $V[\tau] \models \phi(\tau)$. Then $V[\tau] \models \exists x \psi(x, \tau)$. So there is a $\mathbb{P}$-name $\sigma$ such that $V[\tau]=V[\sigma, \tau] \models \psi(\sigma, \tau)$. By induction $V^{\mathbb{P}} \models \psi(\sigma, \tau)$; thus $V^{\mathbb{P}} \models \phi(\tau)$.

Assume now that $V^{\mathbb{P}} \models \phi(\tau)$. Hence $V^{\mathbb{P}} \models \exists x \psi(x, \tau)$; and we can again find a $\mathbb{P}_{-}$ name $\sigma$ such that $V^{\mathbb{P}} \models \psi(\sigma, \tau)$. By induction $V[\sigma, \tau] \models \psi(\sigma, \tau)$. So $\Sigma_{n}^{1}-\mathbb{Q}$-correctness implies $V^{\mathbb{Q}} \models \psi(\sigma, \tau)$; thus $V^{\mathbb{Q}} \models \phi(\tau)$. Hence by $\Sigma_{n+1}^{1}-\mathbb{Q}$-correctness $V[\tau] \models \phi(\tau)$.
(ii) $n=1$ follows from Shoenfield's Absoluteness Lemma. Suppose (ii) is true for $n \geq 1$ and assume $V$ is $\Sigma_{n+2}^{1}-\mathbb{Q}$-absolute and $\Sigma_{n+1}^{1}-\mathbb{Q}$-correct. By (i) $V$ is also $\Sigma_{n+1}^{1}-\mathbb{P}$-correct. Let $\phi$ be a $\Sigma_{n+2}^{1}$-sentence, $\phi=\exists x \psi(x)$, where $\psi$ is $\Pi_{n+1}^{1}$. Suppose first that $V \models \phi$; i.e. $V \models \psi(a)$ for some $a \in V$. By induction $V^{\mathbb{P}} \models \psi(a)$; thus $V^{\mathbb{P}} \models \phi$.

Assume now that $V^{\mathbb{P}} \models \phi$; i.e. $V^{\mathbb{P}} \models \psi(\tau)$ for some $\mathbb{P}$-name $\tau$. By $\Sigma_{n+1}^{1}-\mathbb{P}-$ correctness $V[\tau] \models \psi(\tau)$. Hence $\Sigma_{n+1}^{1}-\mathbb{Q}$-correctness implies $V^{\mathbb{Q}} \models \phi$. Thus $V \models \phi$ by $\Sigma_{n+2}^{1}-\mathbb{Q}$-absoluteness.

Lemma 3 (Truss $[\operatorname{Tr} 1,6.5]) . \mathbb{D}<_{c} \mathbb{A}$.

Definition (Judah - Shelah [JS 1, § 0]). A ccc notion of forcing $(\mathbb{P}, \leq$ ) is called Souslin iff it can be thought of as a $\Sigma_{1}^{1}$-subset of the reals $\mathbb{R}$ with both $\leq$ and $\perp$ (incompatibility) being $\Sigma_{1}^{1}$-relations (in the plane $\mathbb{R}^{2}$ ).

Note that all p.o.s discussed in this paper are Souslin.
Theorem 3 (Judah [Ju, § 2]). Assume that $\forall a \in \mathbb{R}\left(\omega_{1}^{L[a]}<\omega_{1}^{V}\right)$, and $\mathbb{P} \in V$ is a Souslin forcing. Then $V$ is $\Sigma_{3}^{1}-\mathbb{P}$-correct.

THEOREM 4. $\Sigma_{4}^{1}-\mathbb{A}$-absoluteness implies that $\forall a \in \mathbb{R}\left(\omega_{1}^{L[a]}<\omega_{1}^{V}\right)$.
Corollary 3. $\Sigma_{4}^{1}-\mathbb{A}$-absoluteness implies $\Sigma_{3}^{1}-\mathbb{A}$-correctness, $\Sigma_{4}^{1}-\mathbb{D}$-absoluteness, and $\Sigma_{3}^{1}-\mathbb{D}$-correctness.

Theorem 5. $\Sigma_{4}^{1}-\mathbb{A}$-absoluteness implies $\Sigma_{3}^{1}(\mathcal{L})$ and $\Sigma_{3}^{1}(\mathcal{B})$.
The proof of Theorem 4 follows the lines of the proof of 2.6 in [BJS]. Theorem 5 is a consequence of Theorem 4 and a result in $[\mathrm{Ju}, \S 2]$. We give the proof here for completeness' sake. - Note that $\Sigma_{3}^{1}-\mathbb{D}$-absoluteness is equivalent to $\Sigma_{2}^{1}(\mathcal{B})[\mathrm{Ju}, \S 2]$. Thus the implication $\Sigma_{3}^{1}-\mathbb{A}$-absoluteness $\Longrightarrow \Sigma_{3}^{1}-\mathbb{D}$-absoluteness (immediate from Lemmata 2 and 3) is just another version of the Raisonnier-Stern Theorem; and Corollary 3 may be thought of as the corresponding result for $\Sigma_{4}^{1}$.

Proof of Theorem 4. Suppose there is an $a \in \mathbb{R}$ such that $\omega_{1}^{L[a]}=\omega_{1}^{V}$. By $\Sigma_{3}^{1}-\mathbb{A}-$ absoluteness we have $\Sigma_{2}^{1}(\mathcal{L})$; i.e. $\forall b \in \mathbb{R}(\mu(R a(L[b]))=1$ ) (see the beginning of this section). Note that $x \in R a(L[b])$ is equivalent to

$$
\forall c(c \notin L[b] \cap B C \vee \hat{c} \text { is not null } \vee x \notin \hat{c}),
$$

where $B C$ is the set of Borel codes which is $\Pi_{1}^{1}[J e$, Lemma 42.1], and for $c \in B C, \hat{c}$ is the set coded by $c$. As $L[b]$ is $\Sigma_{2}^{1}$ [Je, Lemma 41.1], $R a(L[b])$ is a $\Pi_{2}^{1}$-set. Hence $\forall b \in \mathbb{R}(\mu(R a(L[b]))=1)$ which is equivalent to

$$
\forall b \exists c(c \in B C \wedge \hat{c} \text { is null } \wedge \forall x(x \in \hat{c} \vee x \in R a(L[b])))
$$

is a $\Pi_{4}^{1}$-sentence. So it is true in $V^{\mathbb{A}}$ by $\Sigma_{4}^{1}$-absoluteness; in particular $R a(L[a][r])$ (where $r$ is Amoeba over $V$ ) has measure one in $V[r]$ which implies that there is a random real in $V[r]$ over $L[a][r]$, contradicting Corollary 1 in $\S 1$.

Proof of Corollary 3. Follows from Theorems 3 and 4 and Lemmata 2 and 3.
Proof of Theorem 5 (Judah). Let $\phi(x)$ be a $\Sigma_{3}^{1}$-formula and $A=\{x ; \phi(x)\}$. We shall show that $A$ is measurable in $V$. First note that the sentence $A$ has measure zero is
equivalent to

$$
\exists c(c \in B C \wedge \mu(\hat{c})=0 \wedge \forall x(\neg \phi(x) \vee x \in \hat{c})),
$$

which is $\Sigma_{4}^{1}$. So by $\Sigma_{4}^{1}-\mathbb{A}$-absoluteness, if $A$ is null in $V^{\mathbb{A}}$, it is also null in $V$.
Hence assume that $A$ is not null in $V^{\mathbb{A}}$. As $\mu(R a(V))=1$ in $V^{\mathbb{A}}$, there is $r \in R a(V) \cap A$ in $V^{\mathbb{A}}$; i.e. $V^{\mathbb{A}} \models \phi(r)$. By $\Sigma_{3}^{1}-\mathbb{A}$-correctness $V[r] \models \phi(r)$. Now let $\phi(x)=\exists y \psi(x, y)$, where $\psi$ is $\Pi_{2}^{1}$. Then there is an $s \in V[r]$ such that $V[r] \vDash \psi(r, s)$. If $a \in V$ codes the parameters of $\psi$ and of the name of $s$, we have by Shoenfield's Absoluteness Lemma $L[a][r] \models \psi(r, s)$. Let $\breve{r}$ be the $\mathbb{B}$-name for the random real $r$ and $s(\breve{r})$ a $\mathbb{B}$-name for $s$. Then the Boolean value $\|\psi(\breve{r}, s(\breve{r}))\|$ is non-zero. Furthermore, if $r^{\prime} \in\|\psi(\breve{r}, s(\breve{r}))\| \cap V$ is random over $L[a]$, then $L[a]\left[r^{\prime}\right] \models \psi\left(r^{\prime}, s\left(r^{\prime}\right)\right)$ and - by absoluteness - $V \models \psi\left(r^{\prime}, s\left(r^{\prime}\right)\right)$; in particular $V \models \phi\left(r^{\prime}\right)$.

By $\Sigma_{3}^{1}-\mathbb{A}$-absoluteness we have that $\mu(R a(L[a]))=1$ in $V$ (cf Introduction). And the previous discussion gives us that $R a(L[a]) \cap\|\psi(\breve{r}, s(\breve{r}))\| \subseteq A$. This shows that any non-null $\Sigma_{3}^{1}$-set has positive inner measure; and it is easy to conclude from this that any $\Sigma_{3}^{1}$-set is indeed measurable.

Finally, $\Sigma_{3}^{1}(\mathcal{B})$ follows along the same lines because $\mathbb{A}$ adds a comeager set of Cohen reals.

Questions. 1) Does $\Sigma_{3}^{1}(\mathcal{L})$ imply $\Sigma_{4}^{1}-\mathbb{A}$-absoluteness?
2) Does $\Sigma_{4}^{1}$-Amoeba-meager-absoluteness (or $\Sigma_{4}^{1}-\mathbb{D}$-absoluteness) imply $\Sigma_{3}^{1}(\mathcal{B})$ ? (cf [ $\operatorname{Tr} 1, \S 5]$ for Amoeba-meager forcing - the problem here is whether $\Sigma_{4}^{1}$-Amoeba-meagerabsoluteness implies $\forall a \in \mathbb{R}\left(\omega_{1}^{L[a]}<\omega_{1}^{V}\right)$; cf [BJS, §2] for $\mathbb{D}$ - the problem here is that $\mathbb{D}$ does not add a comeager set of Cohen reals)
3) Does $\forall n$ ( $V$ is $\Sigma_{n}^{1}-\mathbb{A}$-absolute ) imply projective measurability?
4) (Judah) Does $\Sigma_{3}^{1}(\mathcal{L})$ imply $\Sigma_{3}^{1}(\mathcal{B})$ ? (cf Corollary 3)

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