# PRECIPITOUS TOWERS OF NORMAL FILTERS

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ABSTRACT. We prove that every tower of normal filters of height  $\delta$  ( $\delta$  supercompact) is precipitous assuming that each normal filter in the tower is the club filter restricted to a stationary set. We give an example to show that this assumption is necessary. We also prove that every normal filter can be generically extended to a well-founded V-ultrafilter (assuming large cardinals).

In this paper we investigate towers of normal filters. These towers were first used by Woodin in [W88]. Woodin proved that if  $\delta$  is a Woodin cardinal and  $\mathbb{P}$  is the full stationary tower up to  $\delta$  ( $\mathbf{P}_{<\delta}$ ) or the countable version ( $\mathbf{Q}_{<\delta}$ ) then the generic ultrapower is closed under  $< \delta$ sequences (so the generic ultrapower is well-founded). We show that if  $\mathbb{P}$  is a tower of height  $\delta$ ,  $\delta$  supercompact, and the filters generating  $\mathbb{P}$ are the club filter restricted to a stationary set, then  $\mathbb{P}$  is precipitous. We give an example (assuming large cardinals) of a non-precipitous tower. We also show that every normal filter can be extended to a V-ultrafilter with well-founded ultrapower in *some* generic extension of V (assuming large cardinals). Similarly for any tower of inaccessible height. This is accomplished by showing that there is a stationary set that projects to the filter or the tower and then forcing with  $\mathbf{P}_{<\delta}$  below this stationary set.

An important idea in our proof of precipitousness (Theorem 6.4) has the following form in Woodin's proof. If  $\mathcal{A}_{\flat} \subseteq \mathbf{P}_{<\delta}$  are maximal antichains ( $i \in \omega$  and  $\delta$  Woodin) then there is a  $\kappa < \delta$  such that

$$\begin{cases} a \prec V_{\kappa+1} \mid & |a| < \kappa \& (\forall i \in \omega) \exists b \prec V_{\kappa+1} \text{ such that} \\ 1) & a \subseteq b, b \text{ end extends } a \cap V_{\kappa} \\ 2) & \exists x \in \mathcal{A}_{i} \cap \mathcal{V}_{\kappa} \cap \lfloor (\lfloor \cap \cup \S \in \S) \end{cases} \end{cases}$$

contains a club (relative to  $|a| < \kappa$ ).

Before this, a similar idea was used in [FMS]. For example, let  $\mathcal{A} \subseteq \mathcal{NS}_{\omega_{\infty}}$  be a maximal antichain. If the sealing off forcing for  $\mathcal{A}$  is semiproper (this holds if we collapse a supercompact cardinal to  $\omega_2$ ) then

$$\{ a \prec H(\lambda) \mid |a| < \omega_1 \& (\exists b) b \cap \omega_1 = a \cap \omega_1 \& \exists A \in \mathcal{A} \cap \lfloor (\lfloor \cap \omega_\infty \in \mathcal{A}) \}$$

contains a club in  $\mathcal{P}_{\omega_{\infty}}(\mathcal{H}(\lambda))$  ( $\lambda >> \omega_1$ ). This can be used to show that  $NS_{\omega_1}$  is precipitous.

The basic facts about forcing that we use can be found in [J] or [K]. Throughout this paper generic mean set generic.

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### 1. NORMAL FILTERS

In this section we recall the basic definitions and facts about normal filters on  $\mathcal{P}(\mathcal{X})$ . The proofs of these facts are left to the reader—or see ([B]).

**Definition 1.1.** A set  $\mathcal{F} \subseteq \mathcal{PP}(\mathcal{X})$  is a normal filter on  $\mathcal{P}(\mathcal{X})$  iff

- 1. (Filter)  $\mathcal{A}, \mathcal{B} \in \mathcal{F} \Rightarrow \mathcal{A} \cap \mathcal{B} \in \mathcal{F}, \ \mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{P}(\mathcal{X}) \& \mathcal{A} \in \mathcal{F} \Rightarrow \mathcal{B} \in \mathcal{F}, \ and \ \emptyset \notin \mathcal{F}.$
- 2. (Fineness)  $\forall x \in X \{ a \subseteq X \mid x \in a \} \in \mathcal{F}.$
- 3. (Normality) If  $\mathcal{A}_{\S} \in \mathcal{F}$  (for  $\S \in \mathcal{X}$ ) then the diagonal intersection  $\triangle_{x \in X} \mathcal{A}_{\S} =_{[\{} \{ \dashv \subseteq \mathcal{X} \mid (\forall \S \in \dashv) \dashv \in \mathcal{A}_{\S} \} \in \mathcal{F}.$

If  $\mathcal{F}$  is a filter on  $\mathcal{P}(\mathcal{X})$  then  $I_{\mathcal{F}} =_{df} \{ \mathcal{A} \subseteq \mathcal{P}(\mathcal{X}) \mid \mathcal{P}(\mathcal{X}) \setminus \mathcal{A} \in \mathcal{F} \}$  is the dual ideal.  $I_{\mathcal{F}}^+ =_{df} \{ \mathcal{S} \subseteq \mathcal{P}(\mathcal{X}) \mid \mathcal{S} \notin \mathcal{I}_{\mathcal{F}} \}$ 

**Fact 1.2.** Let  $\mathcal{F}$  be a normal filter on  $\mathcal{P}(\mathcal{X})$ ,  $S \in I_{\mathcal{F}}^+$  and f a choice function on S (for all  $a \in S$ ,  $f(a) \in a$ ). Then there is an  $x \in X$  such that  $\{a \in S \mid f(a) = x\} \in I_{\mathcal{F}}^+$ .

*Remark*. It is easy to see that the conclusion of the above fact is actually equivalent to normality.

**Fact 1.3.** Let  $\mathcal{F}$  be a normal filter on  $\mathcal{P}(\mathcal{X})$ .

- 1. If  $Y \subseteq X$  then the projection of  $\mathcal{F}$  to  $\mathcal{P}(\mathcal{Y})$ ,  $\pi_{\mathcal{X},\mathcal{Y}}(\mathcal{F})$ , is a normal filter on  $\mathcal{P}(\mathcal{Y})$ , where  $\pi_{X,Y}(\mathcal{F}) = \{ \pi_{\mathcal{X},\mathcal{Y}}(\mathcal{A}) \mid \mathcal{A} \in \mathcal{F} \}$  and  $\pi_{\mathcal{X},\mathcal{Y}}(\mathcal{A}) = \{ \exists \cap \mathcal{Y} \mid \exists \in \mathcal{A} \}.$
- 2. If  $\mathcal{S} \in \mathcal{I}_{\mathcal{F}}^+$  then  $\mathcal{F} \upharpoonright \mathcal{S} = \{ \mathcal{B} \subseteq \mathcal{P}(\mathcal{X}) \mid (\exists \mathcal{A} \in \mathcal{F}) \mathcal{A} \cap \mathcal{S} \subseteq \mathcal{B} \}$  is a normal filter on  $\mathcal{P}(\mathcal{X})$ .

*Remark*. If  $Z \subseteq Y \subseteq X$  then  $\pi_{Y,Z} \circ \pi_{X,Y} = \pi_{X,Z}$ .

**Definition 1.4.** A set  $C \subseteq \mathcal{P}(\mathcal{X})$  is club (in  $\mathcal{P}(\mathcal{X})$ ) iff  $\exists f : X^{<\omega} \to X$ such that  $C = cl_{\{}$ , where  $cl_f = \{ a \subseteq X \mid f''a^{<\omega} \subseteq a \}$ . If  $a \subseteq X$  then  $cl_f(a) =_{df}$  the smallest set containing a that is closed under f.

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**Fact 1.5.** The filter generated by the club sets in  $\mathcal{P}(\mathcal{X})$ ,  $\mathcal{C}_{\mathcal{X}}$ , is a normal filter on  $\mathcal{P}(\mathcal{X})$ .

**Definition 1.6.** A set  $S \subseteq \mathcal{P}(\mathcal{X})$  is stationary (in  $\mathcal{P}(\mathcal{X})$ ) iff  $S \in \mathcal{I}^+_{\mathcal{C}_{\mathcal{X}}}$ .  $\mathcal{S}$  is non-trivial iff  $X \notin \mathcal{S}$ . (Note that  $\{X\}$  is stationary).

*Remark*. If  $\kappa$  is regular and  $\lambda \geq \kappa$  then  $S = \{ a \subseteq \lambda \mid |a| < \kappa \& a \cap \kappa \in A \}$  $\kappa$  } is stationary and  $\mathcal{C}_{\lambda} \upharpoonright \mathcal{S}$  is the usual club filter on  $\mathcal{P}_{\kappa}(\lambda)$ ; if  $\lambda = \kappa$ then  $\mathcal{C}_{\lambda} \upharpoonright \mathcal{S}$  is the usual club filter on  $\lambda$ .

**Fact 1.7.** If  $\mathcal{F}$  is a normal filter on  $\mathcal{P}(\mathcal{X})$  then  $\mathcal{F}$  is countably complete.

**Fact 1.8.** If  $\mathcal{F}$  is a normal filter on  $\mathcal{P}(\mathcal{X})$  then  $\mathcal{F}$  contains the club filter  $\mathcal{C}_{\mathcal{X}}$ .

# 2. Towers of Normal Filters

We say a set  $\mathbb{P}$  is a *tower* if there is a limit ordinal  $\delta$  (the *height* of  $\mathbb{P}$ ) and a function  $\mathcal{F}^{\mathbb{P}} \colon \mathcal{V}_{\delta} \to \mathcal{V}$  such that for all  $X \in V_{\delta}$ ,  $\mathcal{F}^{\mathbb{P}}_{\mathcal{X}}$  is a normal filter on  $\mathcal{P}(\mathcal{X})$  and for all  $X \subseteq Y$  (both in  $V_{\delta}$ ),  $\mathcal{F}^{\mathbb{P}}_{\mathcal{Y}}$  projects to  $\mathcal{F}^{\mathbb{P}}_{\mathcal{X}}$  and  $\mathbb{P} = \{ S \in V_{\delta} \mid \exists X \in V_{\delta} \ S \in I^{+}_{\mathcal{F}^{\mathbb{P}}_{\mathcal{X}}} \}$ . (We often drop the superscript from  $\mathcal{F}^{\mathbb{P}}$ .) We define a partial order on  $\mathbb{P}$  by  $S_1 \leq S_2$  iff  $\cup S_1 \supseteq \cup S_2$ and  $(\forall a \in S_1) \ a \cap (\cup S_2) \in S_2$ .

In ([W88]) Woodin uses the full non-stationary tower  $\mathbf{P}_{<\delta}$  and the countable version  $\mathbf{Q}_{<\delta}$ . In the above notation  $\mathbf{P}_{<\delta}$  is the tower of height  $\delta$  with  $\mathcal{F}_{\mathcal{X}} = \mathcal{C}_{\mathcal{X}}$  (the club filter);  $\mathbf{Q}_{<\delta}$  is the tower of height  $\delta$  with  $\mathcal{F}_{\mathcal{X}} = \mathcal{C}_{\mathcal{X}} \upharpoonright \mathcal{S}_{\mathcal{X}}$  where  $S_X = \{ a \subseteq X \mid |a| \le \omega \}.$ 

**Lemma 2.1.** Assume  $\mathbb{P}$  is a tower of height  $\delta$  and  $X, Y \in V_{\delta}$  with  $X \subseteq Y$ . Let  $\pi : \mathcal{P}(\mathcal{Y}) \to \mathcal{P}(\mathcal{X})$  be the projection map  $(\pi(a) = a \cap X)$ . Then

- 1. If  $S \in I_{\mathcal{F}_{\mathcal{Y}}}^+$  then  $\pi''S \in I_{\mathcal{F}_{\mathcal{X}}}^+$ . 2. If  $S \in I_{\mathcal{F}_{\mathcal{X}}}^+$  then  $\pi^{-1}(S) \in I_{\mathcal{F}_{\mathcal{Y}}}^+$ .

*Proof.* To see (1) let  $S \in I^+_{\mathcal{F}_{\mathcal{Y}}}$  and  $C \in \mathcal{F}_{\mathcal{X}}$ . Then  $\pi^{-1}(C) \in \mathcal{F}_{\mathcal{Y}}$  (since  $\mathcal{F}_{\mathcal{Y}}$  projects to  $\mathcal{F}_{\mathcal{X}}$ ) so there is an  $a \in \pi^{-1}(C) \cap S$  and so  $\pi(a) \in C \cap \pi''S$ . The proof of (2) is similar. 

If we force with a tower then we can form a generic ultrapower:

**Lemma 2.2.** Assume  $\mathbb{P}$  is a tower of height  $\delta$  and  $G \subseteq \mathbb{P}$  is generic. For  $X \in V_{\delta}$  let  $G_X = \{ S \in I_{\mathcal{F}^+_{\mathcal{V}}} \mid S \in G \}$ . Then  $G_X$  is a V-normal ultrafilter on  $\mathcal{P}(\mathcal{X})$  extending  $\mathcal{F}_{\mathcal{X}}$ . If  $X \subseteq Y$  then  $G_Y$  projects to  $G_X$ .

Proof. Easy density arguments show (using the above Lemma) that each  $G_X$  is a V-normal ultrafilter on  $\mathcal{P}(\mathcal{X})$ . So by the definition of  $\mathbb{P}, G_X$  extends  $\mathcal{F}_{\mathcal{X}}$ . To see projection suppose  $X \subseteq Y$  are both in  $V_{\delta}$ (and  $\pi$  is the projection map). If  $S \in G_Y$  then  $S \leq \pi(S)$ , so  $\pi(S) \in G_X$ . Since they are V-ultrafilters,  $G_Y$  projects to  $G_X$ .

So if  $\mathbb{P}$  is a tower of height  $\delta$  and  $G \subseteq \mathbb{P}$  is generic then we may form the usual (direct limit) ultrapower (M, E): If  $f_i: \mathcal{P}(\mathcal{X}_{\delta}) \to \mathcal{V}$  ( $\delta \in \{\infty, \in\}, \{\delta \in \mathcal{V}, \mathcal{X}_{\delta} \in \mathcal{V}_{\delta}\}$  then  $f_1 \sim f_2$  iff for some (any)  $Z \in V_{\delta}$  with  $X_1 \cup X_2 \subseteq Z$ 

$$\{ a \subseteq Z \mid f_1(a \cap X_1) = f_2(a \cap X_2) \} \in G_Z.$$

M is the collection of all equivalence classes and  $[f_1]E[f_2]$  iff

$$\{ a \subseteq Z \mid f_1(a \cap X_1) \in f_2(a \cap X_2) \} \in G_Z.$$

As usual we get an elementary embedding  $j: V \to (M, E)$  and Loś' Theorem:  $j(x) = [c_x]$  where  $c_x$  is the constant function x with domain  $\mathcal{P}(\mathcal{Y})$  for some  $Y \in V_{\delta}$  and  $M \models \phi([f_1], \dots, [f_n])$  iff for some (any)  $Z \in V_{\delta}$  such that  $X_1 \cup \dots \cup X_n \subseteq Z$ 

$$\{ a \subseteq Z \mid \phi(f_1(a \cap X_1) \dots f_n(a \cap X_n)) \} \in G_Z.$$

Also note that by normality for all  $X \in V_{\delta}$ ,  $[f]E[\mathrm{id}_X]$  iff there is an  $x \in X$  such that [f] = j(x) (id<sub>X</sub> is the identity function with domain  $\mathcal{P}(\mathcal{X})$ ).

Given any  $X \in V_{\delta}$  we can also form the ultrapower using only  $G_X$  to get an elementary embedding  $j_X : V \to \text{Ult}(V, G_X)$ . As usual, there is an elementary embedding  $k : \text{Ult}(V, G_X) \to (M, E)$  defined by  $k([f]_X) = [f]$ . Note that if X is transitive then  $k \upharpoonright X = \text{id}$ .

**Definition 2.3.** A tower  $\mathbb{P}$  is precipitous if the generic ultrapower in  $V^{\mathbb{P}}$  is well-founded.

**Definition 2.4.** Let  $\mathbb{P}$  be a tower of height  $\delta$ . If  $A, B \subseteq \mathbb{P}$  are antichains then  $A \prec B$  means  $(\forall p \in A)(\exists q \in B) \ p \leq q$ . For  $p, q \in \mathbb{P}$   $p \sim q$  means  $p \Vdash ``q \in G"$  and  $q \Vdash ``p \in G"$ . We say p and q are disjoint if for any  $Z \in V_{\delta}$  with  $\bigcup p, \bigcup q \subseteq Z, \ \pi_{Z, \cup p}^{-1}(p) \cap \pi_{Z, \cup q}^{-1}(q) = \emptyset$ . An antichain is disjoint if every pair of elements from it are disjoint.

For a normal filter  $\mathcal{F}$  being able to refine every antichain in  $I_{\mathcal{F}}^+$  to a disjoint antichain is a strong statement (see [F86]). But for towers we have the following.

**Lemma 2.5.** Assume  $\mathbb{P}$  is a tower of height  $\delta$ ,  $\delta$  inaccessible. If  $A \subseteq \mathbb{P}$  is a maximal antichain then there is a disjoint maximal antichain  $B \prec A$ . Moreover, if  $q \in B$  and  $p \in A$  and  $q \leq p$  then  $q \sim p$ .

*Proof.* Let  $A \subseteq \mathbb{P}$  be a maximal antichain and  $\langle S_{\alpha} : \alpha \in \lambda \rangle$  a 1-1 listing of A (so  $\lambda \leq \delta$ ). It is enough to define by induction a disjoint sequence  $\langle S'_{\alpha} : \alpha \in \lambda \rangle$  such that  $S'_{\alpha} \leq S_{\alpha}$  and  $S'_{\alpha} \sim S_{\alpha}$ . To define  $S'_{\beta}$  ( $\beta < \lambda$ ) choose  $\nu < \delta$  such that  $\nu > \beta$  and for all  $\alpha < \beta, \cup S'_{\alpha} \subseteq V_{\nu}$  and  $\cup S_{\beta} \subseteq V_{\nu}$ . Let  $\pi_{\alpha} = \pi_{V_{\nu}, \cup S'_{\alpha}}$  (for  $\alpha < \beta$ ) and  $\pi_{\beta} = \pi_{V_{\nu}, \cup S_{\beta}}$ . Since each  $S'_{\alpha} \sim S_{\alpha}$  we have for each  $\alpha < \beta$  a set  $C_{\alpha,\beta} \in \mathcal{F}_{\mathcal{V}_{\nu}}$  such that  $C_{\alpha,\beta} \cap \pi_{\alpha}^{-1}(S'_{\alpha}) \cap \pi_{\beta}^{-1}(S_{\beta}) = \emptyset$ . Let

$$S'_{\beta} = \{ a \subseteq V_{\nu} \mid a \in \pi_{\beta}^{-1}(S_{\beta}) \& (\forall \alpha \in a \cap \beta) a \in C_{\alpha,\beta} \& \beta \in a \}.$$
  
is clearly works.

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The equivalence of (1) and (2) in the following is standard (see [JMMP] and [F86]). Their equivalence with (3) uses the above Lemma.

**Lemma 2.6.** Let  $\mathbb{P}$  be a tower of height  $\delta$ ,  $\delta$  inaccessible. Then the following are equivalent.

- 1.  $\mathbb{P}$  is precipitous.
- 2. Player I does not have a winning strategy in the following game: I and II alternately play elements of  $\mathbb{P}$  such that  $p_0 \geq p_1 \geq \cdots \geq$  $p_n \cdots$  and II wins iff  $\exists a \subseteq \bigcup_{n \in \omega} (\cup p_n)$  such that  $(\forall n) \ a \cap (\cup p_n) \in$  $p_n$ .
- 3. If  $p \in \mathbb{P}$  and  $A_n \subseteq \mathbb{P}$  are maximal antichains  $(n \in \omega)$  then

$$\{ a \subseteq V_{\delta} \mid a \cap \cup p \in p \& (\forall n) (\exists q \in A_n) a \cap \cup q \in q \} \neq \emptyset.$$

*Proof.* (1)  $\Longrightarrow$  (2). Assume that (2) fails. Let  $\sigma$  be a winning strategy for I. Define a tree  $T: \langle (p_0, a_0), \dots ((p_n, a_n)) \rangle \in T$  iff

- 1.  $\langle p_0, \ldots p_n \rangle$  is according to  $\sigma$ .
- 2.  $(\forall i \in n) a_i \in p_i$ .
- 3.  $(\forall i < j \leq n) a_i \cap (\cup p_i) = a_i$ .

Since  $\sigma$  is a winning strategy for I, T is well-founded. Let  $G \subseteq \mathbb{P}$  be generic with  $\sigma(\emptyset) \in G$ . So there exists (in V[G]) a sequence  $\langle p_0, p_1, \ldots \rangle$ according to  $\sigma$  such that every  $p_i \in G$ . Let  $j: V \to M \subseteq V[G]$  be the generic embedding. If M is well founded then j(T) is well founded in V[G]. But

$$\langle (j(p_0), j'' \cup p_0), \dots (j(p_n), j'' \cup p_n), \dots \rangle$$

is an infinite descending chain in j(T).

 $(2) \Longrightarrow (3)$ . Let  $p, A_n$  witness the failure of (3). A winning strategy for I is to let  $\sigma(\emptyset) = p$  and at the *n*th move play something below an element of  $A_n$  (and below II's last move).

(3)  $\implies$  (1). Assume (1) fails. So there is a  $p \in \mathbb{P}$  and names  $\tau_n$ such that  $(\forall n \in \omega) \ p \Vdash ``\tau_n, \tau_{n+1} \in \text{Ord } \& \tau_n > \tau_{n+1}$ ''. Let  $A_{-1}$  be any

disjoint maximal antichain with  $p \in A_{-1}$ . Inductively construct disjoint maximal antichains  $A_n$  and functions  $T_n$  such that

- 1.  $A_{-1} \succ A_0 \succ A_1 \ldots$
- 2.  $(\forall n \in \omega) \ \forall q \in A_n$ , if  $q \leq p$  then  $T_n(q) : q \to \text{Ord such that}$  $q \Vdash \tau_n = [T_n(q)].$
- 3. Suppose  $p \ge q_1 \ge q_2$  and  $q_1 \in A_{n_1}, q_2 \in A_{n_2}$ , and  $n_2 > n_1$ . Then  $(\forall a \in q_2) T_{n_2}(q_2)(a) < T_{n_1}(q_1)(a \cap (\cup q_1))$ .

But now  $p, \langle A_n : n \in \omega \rangle$  witnesses the set defined in (3) is empty: Suppose  $a \subseteq V_{\delta}$  is in this set. Then  $a \cap (\cup p) \in p$  and  $(\forall n) \exists q_n \in A_n$  such that  $a \cap (\cup q_n) \in q_n$ . Our construction gives that  $p \ge q_0 \ge q_1 \dots$ , and so  $T_0(q_0)(a \cap (\cup q_0)) > T_1(q_1)(a \cap (\cup q_1)) > \dots$ 

Remarks . The equivalence of (1) and (2) does not use that  $\delta$  is inaccessible.

We can also add to the above list:

(4) If  $p \in \mathbb{P}$  and  $A_n \subseteq \mathbb{P}$  are maximal antichains  $(n \in \omega)$  then

$$\{ a \subseteq V_{\delta} \mid p \in a \& a \cap \cup p \in p \& (\forall n) (\exists q \in A_n \cap a) a \cap \cup q \in q \} \neq \emptyset.$$

If  $I(\delta)$  is the above set then  $I(\delta) \neq \emptyset$  is also equivalent to  $\exists \kappa < \delta$  such that  $I(\kappa) \neq \emptyset$  (since  $\operatorname{cof}(\delta) > \omega$ ). This form of precipitous is used in [W88]. Also note that if for all  $p \in \mathbb{P}$  and all maximal antichains  $A_n \subseteq \mathbb{P}$   $(n \in \omega) \exists \kappa < \delta$  such that  $I(\kappa) \in I_{\mathcal{F}_{\mathcal{V}_{\kappa}}}^+$  then the generic ultrapower is closed under  $\omega$ -sequences in V[G]. In fact, this is equivalent to being closed under  $\omega$ -sequences. So, for example, if  $\mathbb{P}$  is the tower of height  $\delta$  where  $\mathcal{F}_{\mathcal{X}} = \mathcal{C}_{\mathcal{X}} \upharpoonright \mathcal{S}_{\mathcal{X}}$  and  $S_{\mathcal{X}} = \{ a \subseteq \mathcal{X} \mid |a| < \aleph_{\omega} \}$  then  $\mathbb{P}$  is precipitous (if  $\delta$  is Woodin) but the generic ultrapower is not closed under  $\omega$ -sequences.

### 3. Well-founded extensions of filters

In this section we show that any normal filter is part of a tower of arbitrarily large height. So (assuming large cardinals) every normal filter can be generically extended to a well-founded V-ultrafilter — although the filter itself may not be precipitous. We also prove a similar result for any tower on  $V_{\delta}$ , assuming that  $\delta$  is inaccessible.

We use the following Lemma—Foreman proved this when when  $\mathcal{F}$  is an ultrafilter.

**Lemma 3.1.** Assume  $\mathcal{F}$  is a normal filter on  $\mathcal{P}(\mathcal{Y})$ . Assume that  $|X| \geq 2^{2^{|Y|}}$  and  $Y \subseteq X$ . Then there is a stationary set S in  $\mathcal{P}(\mathcal{X})$  such that the club filter on  $\mathcal{P}(\mathcal{X})$  restricted to S projects to  $\mathcal{F}$ .

*Proof.* Let  $\pi$  be the projection map from  $\mathcal{P}(\mathcal{X})$  to  $\mathcal{P}(\mathcal{Y})$ . Let  $\langle C_x : x \in X \rangle$  be a listing of all the elements in  $\mathcal{F}$ . Let

$$S = \{ a \subseteq X \mid \forall x \in a \ (a \in \pi^{-1}(C_x)) \}.$$

We need to see that S is stationary and that  $\pi(\mathcal{C}_{\mathcal{X}} \upharpoonright S) = \mathcal{F}$ . For this, it is enough to show that  $\forall f : X^{<\omega} \to X$ ,  $\pi(S \cap cl_f) \in \mathcal{F}$ . Fix such an f. Let  $\overline{f} : Y^{<\omega} \to Y$  be such that if  $a \subseteq Y$  is closed under  $\overline{f}$  then  $cl_f(a) \cap Y = a$  (set  $\overline{f}(y_1 \dots y_{\langle i,j \rangle})$ ) = the j'th element of Y in  $cl_f(y_1 \dots y_i)$ , where  $\langle \cdot, \cdot \rangle$  is some simple pairing function on  $\omega$ ).

Claim . Suppose that  $g: Y^{<\omega} \to \mathcal{P}_{\omega_{\infty}}(\mathcal{F})$ . Then

$$\{ a \subseteq Y \mid \forall \tau \in a^{<\omega} \, \forall C \in g(\tau) (a \in C) \} \in \mathcal{F}.$$

Proof of Claim. Let  $h: Y^{<\omega} \to Y$  be a bijection. For  $y \in Y$  let  $D_y = \bigcap g(h^{-1}(y))$  (so  $D_y \in \mathcal{F}$ ). Then  $\operatorname{cl}_h \cap \triangle_{y \in Y} D_y \subseteq \{ a \subseteq Y \mid \forall \tau \in a^{<\omega} \forall C \in g(\tau) (a \in C) \}$ , so this set is in  $\mathcal{F}$ .  $\Box$  Claim.

Now given  $\tau \in Y^{<\omega}$  let  $g(\tau) = \{ C_x \mid x \in cl_f(\tau) \}$ . So

$$C = \{ a \subseteq Y \mid a \in \operatorname{cl}_{\bar{f}} \& \forall \tau \in a^{<\omega} \forall C \in g(\tau) \ (a \in C) \} \in \mathcal{F}.$$

But if  $a \in C$  then  $cl_f(a) \in S \cap cl_f$  and  $cl_f(a) \cap Y = a$ . Therefore  $\pi(S \cap cl_f) \in \mathcal{F}$ .

As a corollary to this and a Theorem of Woodin we get the following.

**Corollary 3.2.** Assume  $\mathcal{F}$  is a normal filter on  $\mathcal{P}(\mathcal{X})$  and there is a Woodin cardinal > |X|. Then in some generic extension of V, there is a V-ultrafilter extending  $\mathcal{F}$  with well-founded ultrapower.

*Proof.* We use the result of Woodin ([W]) that if  $\delta$  is Woodin and  $G \subseteq \mathbf{P}_{<\delta}$  is generic then the direct limit ultrapower is well-founded (and so the ultrapower using any measure from G is well-founded). Let S be a stationary set on some  $\mathcal{P}(\mathcal{Y})$  ( $Y \in V_{\delta}$ ) such that  $\mathcal{C}_{\mathcal{Y}} \upharpoonright S$  projects to  $\mathcal{F}$  (we may assume that X is an ordinal, so  $X \in V_{\delta}$ ). Let  $G \subseteq \mathbf{P}_{<\delta}$  be generic with  $S \in G$ . Then

$$\{ S' \mid S' \in G \& S' \text{ is stationary in } \mathcal{P}(\mathcal{X}) \}$$

is a V-ultrafilter extending  $\mathcal{F}$  with well-founded ultrapower.

**Theorem 3.3.** Assume  $\mathbb{P}$  is a tower on  $V_{\delta}$ ,  $\delta$  inaccessible. Then there is a stationary set S in  $\mathcal{P}(\mathcal{V}_{\delta})$  such that for all  $X \in V_{\delta}$ ,  $\mathcal{C}_{\mathcal{V}_{\delta}} \upharpoonright S$  projects to  $\mathcal{F}_{\mathcal{X}}$ .

Proof. Let  $S = \{ a \subseteq V_{\delta} \mid (\forall X \in a) (\forall C \in \mathcal{F}_{\mathcal{X}} \cap \dashv) \dashv \cap \mathcal{X} \in \mathcal{C} \}$ . It is enough to see that for every  $X \in V_{\delta}$  and every  $f : V_{\delta}^{<\omega} \to V_{\delta}$ ,  $\pi_{V_{\delta},X}(\mathrm{cl}_{f} \cap S) \in \mathcal{F}_{\mathcal{X}}$ . Fix X and f. Since  $\delta$  is inaccessible,  $\exists \beta < \delta$  such that  $V_{\beta}$  is closed under f (and  $X \in V_{\beta}$ ). But then

 $\overline{S} = \{ a \subseteq V_{\beta} \mid a \in cl_{f} \& (\forall Y \in a) (\forall C \in \mathcal{F}_{\mathcal{Y}} \cap \dashv) \dashv \cap \mathcal{Y} \in \mathcal{C} \} \in \mathcal{F}_{\mathcal{V}_{\beta}}$ (Since  $\mathcal{F}_{\mathcal{V}_{\beta}}$  projects to  $\mathcal{F}_{\mathcal{Y}}$  for all  $Y \in V_{\beta}$ .) So we are done: the projection of  $cl_{f} \cap S$  to  $V_{\beta}$  contains  $\overline{S}$  and the projection of  $\overline{S}$  to X is in  $\mathcal{F}_{\mathcal{X}}$ .

## 4. Examples of non-precipitous towers

In this section we give examples (which were suggested by Woodin) of non-precipitous towers (assuming the existence of a supercompact). These examples use Lemma 4.1 below, which says that under certain conditions (precipitousness and moving its height) towers are not in their ultrapowers. We do not know if these conditions are necessary nor if the supercompact is needed for these examples.

The proof of the following Lemma is based on a proof of the fact that ultrafilters are not in their ultrapowers.

**Lemma 4.1.** Assume  $\mathbb{P}$  is a precipitous tower of height  $\delta$ ,  $\delta$  inaccessible, and  $V^{\mathbb{P}} \models i_G(\delta) > \delta$ . Then  $\mathbb{P}$  is not in its generic ultrapower.

*Proof.* Assume the Lemma fails. Let  $G \subseteq \mathbb{P}$  be generic and  $j: V \to M \subseteq V[G]$  the generic embedding with  $j(\delta) > \delta$  and  $\mathbb{P} \in M$ .

Note that G is also generic over  $L(\mathbb{P})$  and that  $V_{\delta} \in L(\mathbb{P})$ , so  $\exists p \in \mathbb{P}$ such that  $L(\mathbb{P}) \models "p \Vdash i_G(\delta) \geq j(\delta)"$  (since G witnesses it). Let  $[d] = \delta$ and  $[p] = \mathbb{P}$ . We may assume dom(d) = dom(p) =  $V_{\alpha+1}$  ( $\alpha < \delta$ ) and (since  $j(\delta) > \delta$ ) ( $\forall a \subseteq V_{\alpha}$ )  $d(a) < \delta$  and  $p(a) \subseteq V_{d(a)}$  is a tower in L(p(a)) of height d(a). But then  $L(p(a))^{p(a)} \models "i_G(d(a)) < \delta"$  (since  $\delta$ is inaccessible). So in M,  $L(\mathbb{P})^{\mathbb{P}} \models i_G(\delta) < j(\delta)$ . Contradiction.  $\Box$ 

Now assume that  $\kappa$  is supercompact and  $\delta > \kappa$  is inaccessible. We will define a tower of height  $\delta$  that is not precipitous. Let

 $A_0 = \{ \mu \mid (\exists X \in V_{\delta}) \ \mu \text{ is a supercompactness measure on } \mathcal{P}_{\kappa}(\mathcal{X}) \}$ 

For  $\mu \in A_0$  let  $\operatorname{supp}(\mu) =$  the unique X such that  $\mu$  is a supercompact measure on  $\mathcal{P}_{\kappa}(\mathcal{X})$ . Inductively define  $A_{\lambda}$ : for limit  $\lambda$ ,  $A_{\lambda} = \bigcap_{\alpha < \lambda} A_{\alpha}$ ; given  $A_{\lambda}$ , let  $A_{\lambda+1} = \{ \mu \in A_{\lambda} \mid (\forall Y \in V_{\delta}) \text{ if } Y \supseteq \operatorname{supp}(\mu) \text{ then } (\exists \nu \in A_{\lambda}) \operatorname{supp}(\nu) = Y \& \nu \text{ projects to } \mu \}$ . Let  $A = \bigcap A_{\alpha}$ . Note that A is non-empty: the projections of any supercompactness measure on  $\mathcal{P}_{\kappa}(\mathcal{V}_{\delta})$  are in A. By construction  $(\forall \mu \in A)(\forall Y \in V_{\delta})$  if  $Y \supseteq$  $\operatorname{supp}(\mu)$  then  $(\exists \nu \in A) \operatorname{supp}(\nu) = Y \& \nu$  projects to  $\mu$ . Also note that the measures in A are closed under projection. Given  $X \in V_{\delta}$ , let  $\mathcal{F}_{\mathcal{X}}$  be the filter on  $\mathcal{P}(\mathcal{X})$  generated by  $\{ \mathcal{C} \subseteq \mathcal{P}_{\kappa}(\mathcal{X}) \mid (\forall \mu \in \mathcal{A}) \text{ if } \operatorname{supp}(\mu) =$ 

 $\mathcal{X}$  then  $\mathcal{C} \in \mu$  }. These are normal filters that project to one another; let  $\mathbb{P}$  be the associated tower. Assume that  $\mathbb{P}$  is precipitous. Note that  $i_G(\kappa) \geq \delta$  and so  $i_G(\delta) > \delta$ . We will get a contradiction by showing that  $\mathbb{P}$  is in the generic ultrapower. Let  $G \subseteq \mathbb{P}$  be generic and  $j: V \to M \subseteq V[G]$  the generic embedding.

Claim.  $(\forall X \in V_{\delta}) \exists \mu \in A \text{ such that the generic ultrafilter } G_X = \mu.$ 

Proof of Claim. Fix  $X \in V_{\delta}$ . We may assume  $X = V_{\alpha}$  for some  $\alpha$ . Fix  $S \in \mathbb{P}$ . By extending S if necessary we may assume that  $\cup S = V_{\beta} \supseteq V_{\alpha+2}$  and  $S \subseteq \mathcal{P}_{\kappa}(\mathcal{V}_{\beta})$ . Since  $S \in \mathbb{P}$  there is a  $\mu \in A$  such that  $\operatorname{supp}(\mu) = V_{\beta}$  with  $S \in \mu$ . By Lemma 3.1 there is a stationary S' in  $\mathcal{P}(\mathcal{V}_{\beta})$  such that  $\mathcal{C}_{\mathcal{V}_{\beta}} \upharpoonright S'$  projects to  $\mu \upharpoonright V_{\alpha}$ . The proof of Lemma 3.1 shows that  $S' \in \mu$ . Hence  $S \cap S' \in \mathbb{P}$  and  $S \cap S' \Vdash G_{V_{\alpha}} = \mu \upharpoonright V_{\alpha}$ .  $\Box$  Claim.

Claim .  $M_{\delta} = V_{\delta}$ 

Proof of Claim. It is always the case that  $V_{\delta} \subseteq M_{\delta}$ . So let  $\mathcal{A} \in \mathcal{M}_{\delta}$ . Since M is the direct limit of the  $j_X : V \to \text{Ult}(V, G_X)$ , there is an  $\alpha < \delta$ such that  $\mathcal{A} \in \mathcal{M}_{\alpha}$  and an  $\overline{\mathcal{A}} \in \text{Ult}(V, G_{V_{\alpha}})$  such that  $k(\overline{\mathcal{A}}) = \mathcal{A}$  (where k is the canonical map from  $\text{Ult}(V, G_{V_{\alpha}})$  into M). But  $k \upharpoonright V_{\alpha} = \text{id}$ , so  $k(\overline{\mathcal{A}}) = \overline{\mathcal{A}}$ . Hence  $\mathcal{A} \in \text{Ult}(\mathcal{V}, \mathcal{G}_{\mathcal{V}_{\alpha}}) \subseteq \mathcal{V}$ .  $\Box$  Claim.

But our construction of  $\mathbb{P}$  is absolute to  $L(V_{\delta})$ , so  $\mathbb{P} \in L(V_{\delta}) \subseteq M$ . Contradiction, so  $\mathbb{P}$  is not precipitous.

## 5. Large Cardinals

In this section we will describe the large cardinal we use. A cardinal  $\delta$  is  $\lambda$ -supercompact if there is an elementary embedding  $j: V \to M$  such that c.p. $(j) = \delta$  and  $M^{\lambda} \subseteq M$ . For A a set of ordinals we say  $\delta$  is  $[\lambda] A$ -superstrong if there is an elementary embedding  $j: V \to M$  such that c.p. $(j) = \delta$ ,  $j(A) \cap j(\delta) = A \cap j(\delta)$  and  $j'' \lambda \in M$ .

**Theorem 5.1.** Assume  $\delta$  is  $|V_{\delta+\omega+2}|$ -supercompact. Then for all  $A \subseteq \delta$  there are stationary many  $\kappa < \delta$  such that  $\kappa$  is  $[|V_{\kappa+\omega}|] A$ -superstrong.

*Remarks*. 1. Actually, all we need (in Theorem 6.4) is some small amount of " $[|V_{\kappa+\omega}|]$  *A*-strong" for a certain *A*.

2. The proof of this Theorem follows the proof of "if  $\delta$  is  $2^{\delta}$ -supercompact then  $\delta$  is a Woodin cardinal" (see [MS]).

3. In fact, if we let  $\lambda(\alpha) = |V_{\alpha+\omega}|$  (or other simple functions like  $\lambda(\alpha) = 2^{\alpha}$  or  $\lambda(\alpha) = \alpha$ ) then this same method of proof gives the following:

a. If  $\delta$  is supercompact then there are stationary many  $\kappa < \delta$  such that  $\kappa$  is  $[\lambda(\kappa)]$  superstrong.

b. If  $\delta$  is  $[\lambda(\delta)]$  superstrong then there are stationary many  $\kappa < \delta$  such that  $\kappa$  is  $[\lambda(\kappa)]$  Shelah and  $\delta$  is  $[\lambda(\delta)]$  Shelah (where  $\kappa$  is  $[\lambda(\kappa)]$  Shelah has the obvious definition).

c. If  $\delta$  is  $[\lambda(\delta)]$  Shelah then there are stationary many  $\kappa < \delta$  such that  $\kappa$  is  $[\lambda(\kappa)]$  Woodin (and  $\delta$  is  $[\lambda(\delta)]$  Woodin).

The case  $\lambda(\alpha) = \alpha$  is what is proved in [MS].

4. We can strengthen these results by requiring that  $M^{\lambda} \subseteq M$  rather than just  $j'' \lambda \in M$ .

Proof of Theorem 5.1. Assume the Theorem fails. So there is an  $A \subseteq \delta$ and a club  $C \subseteq \delta$  such that if  $\kappa \in C$  then  $\kappa$  is not  $[|V_{\kappa+\omega}|] A$ -superstrong. Let  $j: V \to M$  with  $\operatorname{cp}(j) = \delta$  and  $M^{V_{\delta+\omega+2}} \subseteq M$ . So

(\*) 
$$M \models "\delta \text{ is not } [|V_{\delta+\omega}|] j(A) - \text{superstrong"}$$

Let *E* be the sequence of measures derived from j with support  $S = j(\delta) \cup \{j''V_{\delta+\omega}\}$ .  $(E = \langle \mu_{\tau} : \tau \in S^{<\omega} \rangle$  where  $\mu_{\tau}(X) = 1$  iff  $\tau \in j(X)$ .) We can form the (direct limit) ultrapower using *E*:  $(f, \tau) \sim (g, \sigma)$  (for  $\tau, \sigma \in S^{<\omega}$ ) iff  $j(f)(\tau) = j(g)(\sigma)$  and  $[f, \tau]E[g, \sigma]$  iff  $j(f)(\tau) \in j(g)(\sigma)$  (so the ultrapower is well-founded). Let  $i_E : V \to \text{Ult}(V, E)$  be the canonical embedding. We will show that  $E \in M$  and use this to contradict (\*).

Claim. Define  $id^*(\langle a \rangle) = a$ . For any  $\alpha < j(\delta), [id^*, \langle \alpha \rangle] = \alpha$ .

*Proof of Claim.* Easy, by induction on  $\alpha$ .  $\Box$  Claim.

Claim .  $i_E(A) = j(A)$ .

Proof of Claim. Using the above claim it is easy to see that  $i_E(\delta) = j(\delta)$ . So  $\alpha \in i_E(A)$  iff  $[id^*, \langle \alpha \rangle] \in i_E(A)$  iff  $\alpha \in j(A)$ .  $\Box$  Claim.

Claim .  $E \in M$ .

Proof of Claim. Since for every  $\tau \in S^{<\omega}$ ,  $\mu_{\tau}(V_{\delta+\omega+1}^{<\omega}) = 1$ , E is defined from its support and  $j \upharpoonright \mathcal{P}(\mathcal{V}_{\delta+\omega+\infty}^{<\omega})$ . Since M is closed under  $|V_{\delta+\omega+2}|$ sequences, E is in M.  $\Box$  Claim.

So we can form the ultrapower of M by E and get an elementary embedding  $i_E^M: M \to \text{Ult}(M, E)$ . This ultrapower is well-founded:

Claim . 
$$[f, \tau]_M \in [g, \sigma]_M$$
 (for  $f, g \in M$ ) iff  $j(f)(\tau) \in j(g)(\sigma)$ .

Proof of Claim.  $[f, \tau]_M \in [g, \sigma]_M$  iff (by definition)  $\{a_1 \cap a_2 \mid f(a_1) \in f(a_2)\} \in \mu_{\tau \cap \sigma}$  iff  $j(f)(\tau) \in j(g)(\sigma)$ .  $\Box$  Claim.

Finally, the following claim contradicts (\*).

Claim .  $i_E^M$  has critical point  $\delta$ ,  $i_E^M(\delta) = j(\delta)$ ,  $i_E^M(A) = j(A)$  and  $i_E^{M''}V_{\delta+\omega} \in \mathrm{Ult}(M, E)$ 

Proof of Claim. M and V have the same  $V_{\delta+\omega+2}$  so they have the same functions from  $V_{\delta+\omega+1}^{<\omega}$  into  $V_{\delta+20}$ . Hence c.p. $(i_E^M) = \delta$ ,  $i_E^M(\delta) = i_E(\delta)$ and  $i_E^M(A) = i_E(A)$ . Finally, it is easy to see that  $[id^*, \langle j''V_{\delta+\omega}\rangle]_M =$  $i_E^{M''}V_{\delta+\omega}.$ 

# 6. Proof of Precipitousness

**Theorem 6.1.** Assume that  $\mathbb{P}$  is a tower of height  $\delta$  where  $\delta$  is  $|V_{\delta+\omega+2}|$ supercompact and for all  $X \in V_{\delta}$ ,  $\mathcal{F}_{\mathcal{X}}^{\mathbb{P}} = \mathcal{C}_{\mathcal{X}} \upharpoonright \mathcal{S}_{\mathcal{X}}$  for some stationary set  $S_X$ . Then  $\mathbb{P}$  is precipitous.

**Definition 6.2.** A set b end extends a if for all  $x \in a, a \cap x = b \cap x$ .

**Definition 6.3.** Let  $\mathbb{P}$  be a tower of height  $\delta$ . For  $\kappa < \delta$ ,  $A \subseteq \mathbb{P} \cap V_{\kappa}$ ,  $\lambda$  any ordinal >  $\kappa^+$  and s any set in  $V_{\kappa^+}$  we define  $C_s(\kappa, \lambda, A)$  to be the set of all  $a \prec V_{\kappa+\omega}$  such that: given any  $a^*$  (with  $a^* \prec a, |a^*| < a$  $\kappa, a \text{ end extends } a^* \cap V_{\kappa}, A \in a^*)$  there exists b such that

- 1.  $b \prec V_{\lambda}$  with  $\mathcal{F}_{\mathcal{V}_{\kappa+\omega}} \in \lfloor, s \in b \text{ and } a^* \subseteq b$ . 2.  $\forall C \in \mathcal{F}_{\mathcal{V}_{\kappa+\omega}} (C \in \lfloor \implies \lfloor \cap \mathcal{V}_{\kappa+\omega} \in C)$ .
- 3. b end extends  $a^* \cap V_{\kappa}$ .
- 4.  $(\exists x \in b \cap A) \ b \cap \cup x \in x$ .

**Theorem 6.4.** Assume  $\mathbb{P}$  is a tower of height  $\delta$  and  $\delta$  is  $|V_{\delta+\omega+2}|$ supercompact. Then there are stationary many inaccessible cardinals  $\kappa < \delta$  such that for any ordinal  $\lambda > \kappa^+$ , any  $s \in V_{\kappa^+}$  and any maximal antichain  $A \subseteq \mathbb{P} \cap V_{\kappa}$ , the set  $C_s(\kappa, \lambda, A)$  is in  $\mathcal{F}_{\mathcal{V}_{\kappa+\omega}}$ .

*Proof.* Assume the Theorem fails. So there is a club  $C \subseteq \delta$  such that if  $\kappa$  is inaccessible and in C then there is a  $\lambda > \kappa^+$ ,  $s \in V_{\kappa^+}$  and a maximal antichain  $A \subseteq \mathbb{P} \cap V_{\kappa}$  such that  $C_s(\kappa, \lambda, A) \notin \mathcal{F}_{\mathcal{V}_{\kappa+\omega}}$ . By Theorem 5.1 there is a  $\kappa \in C$  and an elementary embedding  $j: V \to J$ M with critical point  $\kappa$  such that  $j''V_{\kappa+\omega} \in M$ ,  $V_{\kappa+\omega+\omega} \subseteq M$  and  $j(\mathbb{P}) \cap V_{\kappa+\omega+\omega} = \mathbb{P} \cap V_{\kappa+\omega+\omega}$ . Fix a  $\lambda$ , s and A for this  $\kappa$ , so

$$p = \{ a \prec V_{\kappa+\omega} \mid a \notin C_s(\kappa, \lambda, A) \} \in I^+_{\mathcal{F}_{\mathcal{V}_{\kappa+\omega}}}.$$

So  $p \in M$  and  $p \in \mathbb{P} \cap V_{\kappa+\omega+\omega}$  and therefore  $p \in j(\mathbb{P} \cap V_{\kappa})$ . So there is a  $q \in j(A)$  and  $r \in j(\mathbb{P} \cap V_{\kappa})$  such that  $r \leq p, q$ .

Let  $b \prec V_{i(\lambda)}^M, b \in M$ , such that 1.  $j(\kappa), p, q, r, j(\mathcal{F})_{\mathcal{V}_{|(\kappa)+\omega|}}, |(f), | \upharpoonright \mathcal{V}_{\kappa+\omega} \in [$ 

2. 
$$\forall C \in j(\mathcal{F})_{\mathcal{V}_{|(\kappa)+\omega}} (\mathcal{C} \in [] \implies [] \cap |(\mathcal{V}_{\kappa+\omega}) \in \mathcal{C}).$$
  
3.  $b \cap \cup r \in r.$ 

Since  $r \leq p, a =_{df} b \cap V_{\kappa+\omega} \in p$ . Therefore  $\exists a^* \prec a(|a^*| < \kappa \& a \text{ end extends } a^* \cap V_{\kappa} \& A \in a^*)$  such that no *b* satisfies conditions 1-4 in the definition of  $C_s(\kappa, \lambda, A)$ . Fix a witness  $a^*$ . So in *M* the same must be true of  $j(a^*)$  and conditions 1-4 in  $C_{j(s)}(j(\kappa), j(\lambda), j(A))$ . Now use *b* from above to get a contradiction in *M*. Note that  $j(a^*) = j''a^*$  (since  $|a^*| < \kappa$ ) so  $j(a^*) \subseteq b$  (since  $j \upharpoonright V_{\kappa+\omega} \in b$ ) and *b* end extends  $j(a^*) \cap j(V_{\kappa})$  (since  $b \cap V_{\kappa+\omega} = a$  which end extends  $a^* \cap V_{\kappa}$ ). Finally, we get condition (4) since  $q \in b \cap j(A)$  and  $b \cap \cup q \in q$  (since  $r \leq q$ ).

Remark . We can get by with much weaker assumptions on  $\delta$  if we drop condition (2) from the definition of  $C_s(\kappa, \lambda, A)$ . For example, using the notation from the above proof, if  $M^{<\kappa} \subseteq M$  and the set r is stationary in V, then we also reach a contradiction: We find  $b \prec V_{j(\lambda)}^M$  with  $b \in V$ as above (except we drop condition (2)). Then  $\operatorname{Hull}^M(j(a^*) \cup \{q\} \cup$  $\{j(s)\} \cup \{\mathcal{F}_{\mathcal{V}_{\kappa+\omega}}\} \cup (\lfloor \cap \cup \amalg)) \in \mathcal{M}$  plays the role of b.

So if we let  $C_s^*(\kappa, \lambda, A)$  be defined like  $C_s(\kappa, \lambda, A)$  but we drop condition (2), then if  $\delta$  is Woodin and  $\langle A_{\alpha} : \alpha \in \delta \rangle$  is a sequence of maximal antichains in  $\mathbb{P}$ , then there are stationary many inaccessibles  $\kappa < \delta$ such that  $(\forall \alpha < \kappa)(\forall \lambda > \kappa^+)(\forall s \in V_{\kappa^+}) \ C_s^*(\kappa, \lambda, A_{\alpha} \cap V_{\kappa}) \in \mathcal{F}_{\mathcal{V}_{\kappa+\omega}}$ . If the tower has a "simple" definition (for example  $\mathbf{P}_{<\delta}$  or  $\mathbf{Q}_{<\delta}$ ) then condition (2) in  $C(\kappa, \lambda, A)$  holds automatically, so the proof of precipitousness below goes through. Note that the above Theorem does not need the assumption that  $\mathcal{F}_{\mathcal{X}} = \mathcal{C}_{\mathcal{X}} \upharpoonright \mathcal{S}_{\mathcal{X}}$ .

Proof of Theorem 6.1. Let  $\mathbb{P}$  be a tower of height  $\delta$ , where  $\delta$  is  $|V_{\delta+\omega+2}|$ supercompact. Assume  $(\forall X \in V_{\delta}) \mathcal{F}_{\mathcal{X}}^{\mathbb{P}} = \mathcal{C}_{\mathcal{X}} \upharpoonright \mathcal{S}_{\mathcal{X}}$  for some stationary
set  $S_X$ . We will verify condition (3) of Lemma 2.6. So let  $p \in \mathbb{P}$  and  $A_n \subseteq \mathbb{P}$  be maximal antichains  $(n \geq 1)$ . Since there are club many  $\kappa < \delta$  such that  $\forall n \ (A_n \cap V_{\kappa})$  is a m.a.c. in  $\mathbb{P} \cap V_{\kappa}$ , by Theorem 6.4
there is an inaccessible  $\kappa < \delta$  such that  $p \in V_{\kappa}$  and (letting  $B_n = A_n \cap V_{\kappa}$ and  $s = S_{V_{\kappa+\omega}}$ )  $\forall n \geq 1$ ,  $C_s(\kappa, \delta, B_n) \in \mathcal{F}_{\mathcal{V}_{\kappa+\omega}}$ .

Let  $\nu >> \delta$  (say  $\nu$  is strong limit,  $cf(\nu) > \delta$ ). Choose  $a_0 \prec V_{\nu}$ such that  $p, \kappa, \delta, \mathbb{P}, s, A_1, \dots \in a_0$  and  $a_0 \cap \cup p \in p$  and  $a_0 \cap V_{\kappa+\omega} \in C_s(\kappa, \delta, B_1)$ . Let  $\eta_0 < \kappa$  be a limit ordinal in  $a_0$  such that  $p \in V_{\eta_0}$ . Assume inductively that we have defined  $a_0, \dots a_n$  and  $\eta_0 < \dots < \eta_n < \kappa$  such that  $a_n \prec V_{\nu}$  and  $p, \kappa, \delta, \mathbb{P}, s, \eta_0 \dots \eta_n, A_1, \dots \in a_n$  and for all  $i \leq n$  we have (letting  $B_0 = \{p\}$ )  $\exists x_i \in a_i \cap B_i$   $(a_i \cap \cup x_i \in x_i \& x_i \in V_{\eta_i})$ and if i < n then  $a_i \cap V_{\eta_i} = a_{i+1} \cap V_{\eta_i}$ . Also,  $a_n \cap V_{\kappa+\omega} \in C_s(\kappa, \delta, B_{n+1})$ .

If we can keep going with this construction then we are done since  $\bigcup_{n\in\omega}(a_n\cap V_{\eta_n})$  witnesses that the set in part (3) of Lemma 2.6 is nonempty.

To define  $a_{n+1}$  let  $a_n^* \prec a_n$  with  $|a_n^*| < \kappa$ , and  $p, \kappa, \delta, \mathbb{P}, s, \eta_0, \ldots, \eta_n, A_1, \cdots \in a_n^*$ , and  $a_n$  end extends  $a_n^* \cap V_{\kappa}$ , and  $a_n \cap V_{\eta_n} \subseteq a_n^*$ . Since  $a_n \cap V_{\kappa+\omega} \in C_s(\kappa, \delta, B_{n+1})$  and  $a_n^* \cap V_{\kappa+\omega}$  has the required properties, there exists a b such that

- 1.  $b \prec V_{\delta}$  with  $\mathcal{F}_{\mathcal{V}_{\kappa+\omega}} \in [, s \in b \text{ and } a_n^* \cap V_{\kappa+\omega} \subseteq b.$
- 2.  $\forall C \in \mathcal{F}_{\mathcal{V}_{\kappa+\omega}} \ (\mathcal{C} \in [ \implies ] \cap \mathcal{V}_{\kappa+\omega} \in \mathcal{C}).$
- 3. b end extends  $a_n^* \cap V_{\kappa}$ .
- 4.  $(\exists x_{n+1} \in b \cap B_{n+1}) \ b \cap \cup x_{n+1} \in x_{n+1}.$

Let  $a_{n+1} = \{ f(\tau) \mid f \in a_n^* \& \tau \in b \cap V_{\kappa+\omega} \}$ . We verify the inductive assumptions for  $a_{n+1}$ .

Claim . 
$$a_n^* \subseteq a_{n+1} \prec V_{\nu}$$
 and  $a_{n+1} \cap V_{\kappa+\omega} = b \cap V_{\kappa+\omega}$ .

Proof of Claim. Clearly  $a_n^* \subseteq a_{n+1}$ . To see that  $a_{n+1} \prec V_{\nu}$  suppose  $V_{\nu} \models \exists x \phi(x, f_1(\tau_1), \dots, f_n(\tau_n))$ ". Since  $\nu$  is large

$$V_{\nu} \models \exists g \forall x_1, \dots, x_n \in V_{\kappa+\omega} [\exists x \phi(x, f_1(x_1), \dots) \implies \phi(g(\langle x_1, \dots, x_n \rangle), f_1(x_1), \dots)]$$

So there is such a g in  $a_n^*$  and hence  $a_{n+1} \prec V_{\nu}$ .

To see  $a_{n+1} \cap V_{\kappa+\omega} \subseteq b \cap V_{\kappa+\omega}$  suppose that  $f(\tau) \in V_{\kappa+\omega}$ . Since  $\tau \in V_{\kappa+\omega}$  we may assume  $f: V_{\kappa+n} \to V_{\kappa+n}$  for some  $n \in \omega$ . But  $a_n^* \cap V_{\kappa+\omega} \subseteq b$  so  $f \in b$  and hence  $f(\tau) \in b$ . The other inclusion is clear.  $\Box$  Claim.

Now we need to check three conditions and to define  $\eta_{n+1}$ :

- 1.  $a_n \cap V_{\eta_n} = a_{n+1} \cap V_{\eta_n}$ : This holds since  $a_n \cap V_{\eta_n} = a_n^* \cap V_{\eta_n} = b \cap V_{\eta_n}$ (since  $\eta_n \in a_n^* \cap V_{\kappa}$ ) and  $b \cap V_{\eta_n} = a_{n+1} \cap V_{\eta_n}$ .
- 2.  $(\exists x_{n+1} \in a_{n+1} \cap B_{n+1}) a_{n+1} \cap \cup x_{n+1} \in x_{n+1}$ : This holds since there is such an  $x_{n+1}$  in b and  $b \cap V_{\kappa+\omega} = a_{n+1} \cap V_{\kappa+\omega}$ . Let  $\eta_{n+1} < \kappa$  be a limit ordinal in  $a_{n+1}$  such that  $a_{n+1} \in V_{\eta_{n+1}}$  and  $\eta_{n+1} > \eta_n$ .
- 3.  $a_{n+1} \cap V_{\kappa+\omega} \in C_s(\kappa, \delta, B_{n+2})$ . Since  $C_s(\kappa, \delta, B_{n+2}) \in \mathcal{F}_{\mathcal{V}_{\kappa+\omega}}$ , there is an  $f: V_{\kappa+\omega}^{<\omega} \to V_{\kappa+\omega}$  such that  $s \cap \operatorname{cl}_f \subseteq C_s(\kappa, \delta, B_{n+2})$ . Since  $C_s(\kappa, \delta, B_{n+2}) \in a_{n+1}$ , such an f is in  $a_{n+1}$  so  $a_{n+1} \cap V_{\kappa+\omega} \in$  $\operatorname{cl}_f$ . Since  $s \in b$ , we have (by property (2))  $b \cap V_{\kappa+\omega} \in s$ . But  $a_{n+1} \cap V_{\kappa+\omega} = b \cap V_{\kappa+\omega}$  so  $a_{n+1} \cap V_{\kappa+\omega} \in C_s(\kappa, \delta, B_{n+2})$ .

This completes the construction and the proof.

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