On a Glimm – Effros dichotomy and an Ulm–type classification in Solovay model

Vladimir Kanovei *†‡

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Abstract

We prove that in Solovay model every OD equivalence E on reals either admits an OD reduction to the equality on the set of all countable (of length $< \omega_1$) binary sequences, or continuously embeds E₀, the Vitali equivalence.

If E is a Σ_1^1 (resp. Σ_2^1) relation then the reduction in the "either" part can be chosen in the class of all Δ_1 (resp. Δ_2) functions.

The proofs are based on a topology generated by OD sets.

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^{*}Moscow Transport Engineering Institute

[†]kanovei@math.uni-wuppertal.de and kanovei@nw.math.msu.su

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Introduction

The solution of the continuum problem leaves open a variety of related questions. For instance, if one works in descriptive set theory one may be interested to know how different uncountable cardinals can be presented in the real line. This research line can be traced as far in the past as the beginning of the century; indeed Lebesgue [6] found such a presentation for \aleph_1 , the least uncountable cardinal.

The construction given in [6] worth to be briefly reviewed. One can associate, in an effective way, a set of rationals Q_x with each real x so that every set q of rationals has the form Q_x for some (perhaps, not unique) x. Let, for a countable ordinal α ,

 $X_{\alpha} = \{x : Q_x \text{ is welordered as a set of rationals and has the order type } \alpha\}.$

Then the sets X_{α} , $\alpha < \omega_1$, are nonempty and pairwise disjoint; therefore we present α_1 in the reals as the sequence of the sets X_{α} .

This reasoning is a particular case of a much more general construction.

Let E be an equivalence relation on the reals. Let κ be the cardinal of the set of all E-equivalence classes; then $\kappa \leq 2^{\aleph_0}$. One may think that the partition of the real line on the E-equivalence classes presents the cardinal κ in the reals.

For instance, in the Lebesgue's example, the equivalence can be defined as follows: $x \perp y$ iff either (1) both Q_x and Q_y are wellordered and have the same order type, or (2) both Q_x and Q_y are not wellordered. The L-equivalence classes are the sets X_{α} , $\alpha < \omega_1$, plus one more "default" class of all reals x such that Q_x is not wellordered.

Of course, one can present *every* cardinal $\kappa \leq 2^{\aleph_0}$ this way by a suitable equivalence. But the problem becomes much more difficult when one works in descriptive set theory and looks for an equivalence of a certain "effective" type. (Notice that the Lebesgue equivalence L is a Σ_1^1 relation.)

This leads us to the following question: given an equivalence relation E on reals, how many equivalence classes it has ?

The relevant question is then how to "count" the classes. Generally speaking, counting is a numbering of the given set of mathematical objects by mathematical objects of another type, usually more primitive in some sense. In particular, the obvious idea is to use ordinals (for instance natural numbers) to count the equivalence classes. This works well as long as one is not interested in the "effectivity" of the counting. Otherwise we face problems even with very simple relations. (Consider the equality as an equivalence relation. Then one cannot define in **ZFC** an "effective" in any reasonable sense counting of the equivalence classes, alias reals, by ordinals.)

The other natural possibility is to use *sets of ordinals* (for instance reals) to count the equivalence classes. ¹ Note that the next step, that is, counting by *sets of sets* of ordinals, would be silly because the classes themselves are of this type.

¹ There are known many mathematical examples, in probability and the measure theory, based on this type of enumeration of the equivalence classes, see Harrington, Kechris, and Louveau [1].

Definition 1 [Informal]

An equivalence relation is *discrete* iff it admits an "effective" enumeration of the equivalence classes by ordinals. An equivalence relation is *smooth* iff it admits an "effective" enumeration of the equivalence classes by *sets of* ordinals. \Box

Of course the definition has a definite meaning only provided one makes clear the meaning of the "effectivity". However in any reasonable case one has the following two counterexamples:

Example 1. The equality on a perfect set of reals is not discrete.

Example 2. The Vitali equivalence relation is not smooth.

(*Not* here means that one cannot prove in **ZFC** the existence of the required enumerations among the real–ordinal definable functions. However different additional axioms, e. g. the axiom of constructibility, make each equivalence discrete in certain sense.)

At the first look, there should be plenty of other counterexamples. However, in certain particular but quite representative cases one can prove a *dichotomy theorem* which says that an equivalence relation is not discrete (resp. smooth) iff it contains Example 1 (resp. Example 2). This is also the topic of this article, but to proceed with the reasoning we need to be more exact.

Notation

Let us review the basic notation of Harrington, Kechris, and Louveau [1]. See [1] or [3] for a more substantial review with details and explanations.

Let E and E' be equivalence relations on resp. sets X, X'.

A function $U: X \mapsto X'$ is a reduction of E to E' iff $x \mathsf{E} y \leftrightarrow U(x) \mathsf{E}' U(y)$ holds for all $x, y \in X$. An enumeration of the E -equivalence classes (by elements of X') is a reduction of E to the equality on X'. (Here, it is not assumed that all of elements of X' are involved.)

A 1-1 reduction is called an *embedding*. E' continuously embeds E iff there exists a continuous embedding E to E'. In the case when X is the Cantor set $\mathcal{D} = 2^{\omega}$ (with the product topology), E' continuously embeds E if and only if there exists a perfect set $P \subseteq X$ such that $\langle P; \mathsf{E} | P \rangle$ is homeomorphic to $\langle X'; \mathsf{E}' \rangle$. In other words, embedding E continuously means in this case that E' contains a homeomorphic copy of E.

In particular E' continuously embeds the equality on \mathcal{D} iff there exists a perfect set of E'-inequivalent points.

Finally, let E_0 denote the *Vitali equivalence* on $\mathcal{D} = 2^{\omega}$, defined as follows: $x \mathsf{E}_0 y$ iff x(n) = y(n) for almost all (i. e. all but finite) $n \in \omega$.

The main theorem

This paper intends to complete the diagram of the following three classical theorems on equivalence relations.

- Borel 1. Each Borel equivalence on reals, either has countably many equivalence classes or admits a perfect set of pairwise inequivalent points. (Silver [9], in fact for Π_1^1 -relations.)
- Borel 2. Each Borel equivalence relation on reals, either admits a Borel enumeration of the equivalence classes by reals ², or continuously embeds the Vitali equivalence E_0 . (The Glimm – Effros dichotomy theorem of Harrington, Kechris, and Louveau [1].)
- Solovay model 1. In Solovay model ³, each R-OD (real–ordinal definable) equivalence on \mathcal{N} either has $\leq \aleph_1$ equivalence classes and admits a R-OD enumeration of them, or admits a perfect set of pairwise inequivalent points. (Stern [11].)

Thus the results Borel - 1 and Solovay model - 1 say (informally) that an equivalence relation either is discrete or contains a continuous copy of Example 1 above. Similarly Borel - 2 says that an equivalence relation either is smooth or contains a continuous copy of Example 2 above.

Theorem 2 [Solovay model -2]

The following is true in Solovay model. Assume that E is an R-OD equivalence on \mathcal{N} . Then one and only one of the following two statements holds:

- (I) E admits a R-OD enumeration of the equivalence classes by elements of 2^{<ω1}. ⁴ If moreover E is a Σ¹₁ (resp. Σ¹₂) equivalence then the enumeration exists in the class Δ^{HC}₁ (resp. Δ^{HC}₂)⁵;
- (II) E continuously embeds E_0 .

This is the main result of this paper.

Remark 1. Hjorth [2] obtained a similar theorem in a strong determinacy hypothesis (AD holds in L[reals]), yet with a weaker part (I): an OD reduction to the equality

⁴ $2^{<\omega_1} = \bigcup_{\alpha < \omega_1} 2^{\alpha}$ denotes the set of all countable (of any length $< \omega_1$) binary sequences.

 $^{^2}$ That is, admits a Borel reduction to the equality on reals. Such an equivalence is called *smooth* in the notation of [1].

³ By Solovay model we mean a generic extension L[G] of L, the class of all constructible sets, by a generic over L subset of a certain notion of forcing $\mathcal{P}^{\Omega} \in L$ which provides a collapse of all cardinals in L, smaller than a fixed inaccessible cardinal Ω , to ω , see Solovay [10]. In this model, all projective sets are Lebesgue measurable.

⁵ Here by Δ_n^{HC} we denote the class of all subsets of HC (the family of all hereditarily countable sets) which are Δ_n in HC by formulas which may contain arbitrary *reals* as parameters.

on a set 2^{κ} , $\kappa \in \text{Ord}$.

Remark 2. The statements (I) and (II) are incompatible. Indeed otherwise there would exist an R-OD function $U: \mathcal{D} \longrightarrow 2^{<\omega}$ which reduces E_0 to the equality on $2^{<\omega}$. Let U be $OD[z], z \in \mathcal{D}$. Then for each $p \in \operatorname{ran} U (\subseteq 2^{<\omega}), F(p) = U^{-1}(p)$ is a E_0 -equivalence class, a countable OD[p, z] subset of \mathcal{D} . In Solovay model, this implies $F(p) \subseteq L[z, p]$ for all p. We obtain an OD[z] choice function $g: \operatorname{ran} U \longrightarrow \mathcal{D}$ such that $g(p) \in F(p)$ for all p. Then $\operatorname{ran} p$ is an R-OD selector for E_0 , hence a nonmeasurable R-OD set, contradiction with the known properties of the Solovay model.

Remark 3. $2^{\langle \omega_1 \rangle}$ cannot be replaced in Theorem 2 by an essentially smaller set. To see this consider the equivalence R on \mathcal{N}^2 defined as follows: $\langle z, x \rangle \mathsf{R} \langle z', x' \rangle$ iff

- either z and z' code the same countable ordinal and x and x' code, in the sense of z and z' respectively, the same subset of the ordinal,
- or both z and z' do not code an ordinal.

The relation \mathbb{R} admits an OD reduction <u>on</u>to $\Delta(2^{<\omega_1})$, the equality on $2^{<\omega_1}$, therefore does not embed E_0 continuously in Solovay model (see Remark 1). It follows that any set W such that \mathbb{R} admits a R-OD reduction to $\Delta(W)$ has a subset $W' \subseteq W$ which is in 1-1 R-OD correspondence with $2^{<\omega_1}$. In particular, the continuum \mathcal{N} does not satisfy this condition in Solovay model. (Indeed $2^{<\omega_1}$ has R-OD subsets of cardinality exactly \aleph_1 while \mathcal{N} does not have those in Solovay model.)

Remark 4. Even in the case of Σ_1^1 equivalence relations $2^{<\omega_1}$ cannot be replaced by \mathcal{N} in (I). Indeed the Σ_1^1 equivalence $x \in y$ iff either $x, y \in \mathcal{N}$ code the same (countable) ordinal or both x and y do not code an ordinal (Example 6.1 in Hjorth and Kechris [3]) neither admits a Δ_2^1 reduction to $\Delta(\mathcal{N})$ nor embeds E_0 via a Δ_2^1 function in **ZFC** plus $\forall x \in \mathcal{N}(\omega_1^{\mathrm{L}[x]} < \omega_1)$. (In Solovay model, Δ_2^1 can be strengthened to R-OD.) This shows that the Glimm – Effros theorem of Harrington, Kechris, and Louveau [1] (theorem Borel – 2 above) cannot be expanded from Borel to Σ_1^1 relations.

Remark 5. On the other hand, Σ_1^1 equivalence relations tend to satisfy a looser Ulm-type dichotomy. ⁶ In particular, Hjorth and Kechris [3] proved that every Σ_1^1 equivalence with Borel classes either admits a Δ_1 reduction to $\Delta(2^{<\omega_1})$, the equality on $2^{<\omega_1}$, or embeds E_0 continuously; furthermore in the assumption $\forall x \in \mathcal{N}(x^{\#} \text{ exists})$ the requirement that the E-classes are Borel can be dropped.

Thus Theorem 2 proves that the Ulm classification is available in the Solovay model. This gives a partial answer to the question posed by Hjorth and Kechris in [3].⁷

⁶ The notion introduced in [3]. Hjorth and Kechris refer to certain classification results in algebra, i.e. the Ulm classification of countable abelian *p*-groups.

⁷ "Is $\forall x \in \mathcal{N} (x^{\#} \text{ exists})$ needed to prove that" (I) (with a Δ_1^{HC} reduction) or (II) hold for Σ_1^1 relations; item 3) in Section 7: *Open problems* in a preprint version of [3]. Since the sharps hypothesis fails in Solovay model, we observe that the answer is: *not*.

It would be interesting to get the Ulm classification for $\, {\bf \Sigma}_1^1 \,$ relations in $\, {\bf ZFC} \, . \, {}^8$

A brief description of the exposition

- Section 1: We outline how the proof of Theorem 2 will go on. A topology \mathcal{T} generated by OD sets in Solovay model (a counterpart of the Gandy – Harrington topology) is introduced. Similarly to Harrington, Kechris, and Louveau [1], we have two cases: either the equivalence E of consideration is closed in the topology \mathcal{T}^2 or it is not closed. The plan of the proof of Theorem 2 is to demonstrate that the first case provides (I) while the second provides (II). We also review some important properties of the Solovay model.
- Section 2: We prove that in the case when $E = \overline{E}$ the equivalence E satisfies the requirements of Item (I) of Theorem 2. The argument for the "moreover" part of Item (I) includes the idea of forcing the equivalence of mutually generic reals over countable models, due to Hjorth and Kechris [3].
- Section 3: We begin to study the case when the given equivalence is not \mathcal{T}^2 -closed in Solovay model. We develop forcing notions \mathbb{X} and \mathbb{P} associated with \mathcal{T} and \mathcal{T}^2 respectively. In particular it is demonstrated that the intersection of a generic set is nonempty. The set $H = \{x : [x]_{\mathsf{E}} \subset [x]_{\mathsf{E}}\}$, nonempty as soon as we assume $\mathsf{E} \subseteq \overline{\mathsf{E}}$, is considered.
- Section 4: We accomplish the case when the given relation E is not \mathcal{T}^2 -closed. It is demonstrated that in this case E continuously embeds E_0 . The splitting construction is based on the principal idea of Harrington, Kechris, and Louveau [1], but the technical realization is quite different since we use straightforward forcing arguments rather than Choquet games, which makes the construction a little bit more elementary.

Important remark

It will be convenient to use the *Cantor set* $\mathcal{D} = 2^{\omega}$ rather than the Baire space $\mathcal{N} = \omega^{\omega}$ as the principal space in this paper.

 $^{^8}$ The author [4] proved the result assuming that each real belongs to a set–generic extension of $\,{\rm L}$.

1 Approach to the main theorem

In this section, we outline the proof of Theorem 2, the principal theorem of the paper. The idea of the proof has a semblance of the proof of the "Borel" Glimm – Effros theorem in Harrington, Kechris, and Louveau [1]; in particular the dichotomy will be determined by an answer to the question whether the given relation E is closed in a certain topology on \mathcal{D}^2 .

First of all we review the definition and some properties of Solovay model.

1–A The Solovay model

Let α be an ordinal. Then $\alpha^{<\omega}$ is the forcing to collapse α down to ω . We let $\mathcal{P}_{<\lambda}$ be the product of all sets $\alpha^{<\omega}$, $\alpha < \lambda$, with finite support; in other words, $\mathcal{P}_{<\lambda}$ is the set of all functions p defined on finite subsets of λ such that $p(\alpha) \in \alpha^{<\omega}$ for each $\alpha < \lambda$, $\alpha \in \operatorname{dom} p$.

The forcing notions $\alpha^{<\omega}$ and $\mathcal{P}_{<\lambda}$ are equivalent respectively to \mathcal{P}_{α} and \mathcal{P}^{λ} in Solovay [10]. We set $\mathcal{P}_{\leq\lambda} = \mathcal{P}_{<\lambda+1}$.

(Notice that the definitions of $\mathcal{P}_{<\lambda}$ and $\mathcal{P}_{\leq\lambda}$ are absolute.)

Let M be a transitive model of **ZFC**, a set or proper class, containing Ω , an inaccessible cardinal in L. By Ω -Solovay extension of M we shall understand a generic extension of the form M[G], where $G \subseteq \mathcal{P}_{<\Omega}$ is $\mathcal{P}_{<\Omega}$ -generic over M.

Definition 3 Ω -Solovay model axiom, Ω -SMA in brief, is the following hypothesis:

 Ω is inaccessible in L and the universe V is an Ω -Solovay extension of L.

1–B The dichotomy

As usual, we shall concentrate on the "lightface" case of an OD equivalence relation E ; the general case when E is $\mathrm{OD}[z]$ for a real z can be carried out similarly.

Thus let us consider an OD equivalence E in the assumption of Ω -SMA.

The relation E is fixed in the remainder of the proof of Theorem 2. The hypothesis Ω -SMA will be assumed during the proof of the theorem, but we shall not mind to specify Ω -SMA explicitly in all formulations of of lemmas etc.

For any set $X \subseteq \mathcal{D}$, we put $[X]_{\mathsf{E}} = \{y : \exists x \in X \ (x \in y)\}$, the E-saturation of X. Let \mathcal{T} be the topology generated on a given set X (for instance, $X = \mathcal{D} = 2^{\omega}$, the Cantor set) by all OD subsets of X. \mathcal{T}^2 is the product of two copies of \mathcal{T} , a topology on \mathcal{D}^2 .

 of X and Y), and then Y can be replaced by the complement of X, so that

 $x \ \overline{\mathsf{E}} \ y \ \longleftrightarrow \ \forall X \ [\ X \ \text{is OD} \ \& \ X \ \text{is E-invariant} \ \longrightarrow \ (x \in X \ \longleftrightarrow \ y \in X) \] \, .$

Therefore $\overline{\mathsf{E}}$ is an OD equivalence, too.

We now come to the key point of the dichotomy: either $\mathsf{E} = \overline{\mathsf{E}}$ or $\mathsf{E} \underset{\neq}{\subseteq} \overline{\mathsf{E}}$. This reduces the "lightface" case in Theorem 2 to the following form.

Theorem 4 Assume Ω -SMA. Let E be an OD equivalence on \mathcal{D} . Then

(I) If $\mathsf{E} = \overline{\mathsf{E}}$ then E admits an OD enumeration of the equivalence classes by elements of $2^{<\omega_1}$.

If moreover E is a Σ_1^1 (resp. Σ_2^1) equivalence then the enumeration exists in the class Δ_1^{HC} (resp. Δ_2^{HC});

(II) If $\mathsf{E} \underset{\neq}{\subseteq} \overline{\mathsf{E}}$ then E continuously embeds E_0 .

In the case when the relation E is $\mathrm{OD}[z]$ (resp. $\Sigma_1^1[z]$, $\Sigma_2^1[z]$) for a real z, the z uniformly enters the reasoning, not causing any problem. (In particular one considers $\mathcal{T}[z]$, the topology generated by $\mathrm{OD}[z]$ sets, rather than \mathcal{T} .)

We prove part (I) of the theorem in the next section. Part (II) will be considered in the two following sections. The rest of this section presents different properties of the Solovay model.

1–C Weak sets in Solovay model

A set x will be called Ω -weak over M iff x belongs to an $\alpha^{<\omega}$ -generic extension of M for some $\alpha < \Omega$.

Proposition 5 Assume Ω -SMA. Then $\Omega = \omega_1$. Furthermore, suppose that $S \subseteq \text{Ord}$ is Ω -weak over L. Then

- 1. Ω is inaccessible in L[S] and V is an Ω -Solovay extension of L[S].
- 2. If Φ is a sentence containing only sets in L[S] as parameters then Λ (the empty function) decides Φ in the sense of $\mathcal{P}_{<\Omega}$ as a forcing notion over L[S].
- 3. If a set $X \subseteq L[S]$ is OD[S] then $x \in L[S]$.

(OD[S] = S-ordinal definable, that is, definable by an \in -formula containing S and ordinals as parameters.)

The proof (a copy of the proof of Theorem 4.1 in Solovay [10]) is based on several lemmas, including the following crucial lemma:

Lemma 6 (Lemma 4.4 in [10])

Let M be a transitive model of **ZFC**, $\lambda \in \text{Ord} \cap M$. Suppose that M' is a $\lambda^{<\omega}$ -generic extension of M and M'' is a $\lambda^{<\omega}$ -generic extension of M'. Let $S \in M'$, $S \subseteq \text{Ord}$. Then M'' is a $\lambda^{<\omega}$ -generic extension of M[S].

Proof of the proposition.

Item 1. By definition, S belongs to a $\alpha^{<\omega}$ -generic extension of L for some $\alpha < \Omega$. Then in fact $S \in L[x]$ for a real x. It follows (Corollary 3.4.1 in [10]) that there exists an ordinal $\lambda < \Omega$ such that S belongs to the model $M_{\lambda} = L[G_{\leq \lambda}]$ for some $\lambda < \Omega$, where $G_{\leq \lambda} = G \cap \mathcal{P}_{\leq \lambda}$.

Notice that $G_{\leq\lambda}$ is $\mathcal{P}_{\leq\lambda}$ -generic over L. Therefore by Lemma 4.3 in Solovay [10], $M' = M_{\lambda}$ is a $\lambda^{<\omega}$ -generic extension of L.

Let us consider the next step $\lambda + 1$. Obviously the model $M_{\lambda+1} = L[G_{\leq \lambda+1}]$ is a $(\lambda+1)^{<\omega}$ -generic extension of M_{λ} . Since $(\lambda+1)^{<\omega}$ is order isomorphic to the product $\lambda^{<\omega} \times (\lambda+1)^{<\omega}$, ⁹ we conclude that $M_{\lambda+1}$ is a $(\lambda+1)^{<\omega}$ -generic extension of a certain $\lambda^{<\omega}$ -generic extension M'' of $M' = M_{\lambda}$.

Lemma 6 says that M'' is a $\lambda^{<\omega}$ -generic extension of L[S], therefore a $\mathcal{P}_{\leq \lambda}$ -generic extension of L[S] as well by Lemma 4.3 in [10].

It follows that $M_{\lambda+1}$ is a $\mathcal{P}_{\leq \lambda+1}$ -generic extension of L[S].

Finally M = L[G] is a $\mathcal{P}_{\geq \lambda+2}$ -generic extension of $M_{\leq \lambda+1} = L[G_{\leq \lambda+1}]$. This ends the proof of item 1 of the proposition.

Items 2 and 3. It suffices to refer to item 1 and apply resp. Lemma 3.5 and Corollary 3.5 in [10] for L[S] as the initial model.

1–D Coding of reals and sets of reals in the model

If $G \subseteq \alpha^{<\omega}$ is $\alpha^{<\omega}$ -generic over a transitive model M (M is a set or a class) then $f = \bigcup G$ maps ω onto α , so that α is countable in M[G] = M[f]. Functions $f: \omega \longrightarrow \alpha$ obtained this way will be called $\alpha^{<\omega}$ -generic over M.

We let $\mathbb{F}_{\alpha}(M)$ be the set of all $\alpha^{<\omega}$ -generic over M functions $f \in \alpha^{\omega}$. We put $\mathbb{F}_{\alpha}[S] = \mathbb{F}_{\alpha}(\mathcal{L}[S])$ and $\mathbb{F}_{\alpha} = \mathbb{F}_{\alpha}(\mathcal{L}) = \mathbb{F}_{\alpha}[\emptyset]$.

We recall that $\mathcal{D} = 2^{\omega}$ is the principal descriptive space in this research. The following definitions intend to give a useful coding system for reals and sets of reals in Solovay model.

Let $\alpha \in \text{Ord.}$ By Term_{α} we denote the set of all "terms" — indexed sets $t = \langle \alpha, \langle t_n : n \in \omega \rangle \rangle$ such that $t_n \subseteq \alpha^{<\omega}$ for each n.

We put Term = $\bigcup_{\alpha < \omega_1} \text{Term}_{\alpha}$. (Recall that $\omega_1 = \Omega$ assuming Ω -SMA.)

⁹ Here $\lambda^{<\omega} \times (\lambda+1)^{<\omega}$ is the set of all pairs $\langle p,q \rangle$ such that $S \in \lambda^{<\omega}$, $q \in (\lambda+1)^{<\omega}$, and dom $p = \operatorname{dom} q$.

"Terms" $t \in \text{Term}_{\alpha}$ are used to code functions $C : \alpha^{\omega} \longrightarrow \mathcal{D} = 2^{\omega}$; namely, for every $f \in \alpha^{\omega}$ we define $x = \mathsf{C}_t(f) \in \mathcal{D}$ by: x(n) = 1 iff $f \mid m \in t_n$ for some m.

Assume that $t = \langle \alpha, \langle t_n : n \in \omega \rangle \rangle \in \operatorname{Term}_{\alpha}$, $u \in \alpha^{<\omega}$, M arbitrary. We introduce the sets $\mathsf{X}_{tu}(M) = \{\mathsf{C}_t(f) : u \subset f \in \mathbb{F}_{\alpha}(M)\}$ and $\mathsf{X}_t(M) = \mathsf{X}_{t\Lambda}(M) = \mathsf{C}_t \ \mathbb{F}_{\alpha}(M)$. As above, we let $\mathsf{X}_t[S] = \mathsf{X}_t(\operatorname{L}[S])$ and $\mathsf{X}_t = \mathsf{X}_t[\emptyset] = \mathsf{X}_t(\operatorname{L})$; the same for X_{tu} .

Proposition 7 Assume Ω -SMA. Let $S \subseteq \text{Ord}$ be Ω -weak over L. Then

- 1. If $\alpha < \Omega$, $F \subseteq \mathbb{F}_{\alpha}[S]$ is OD[S], and $f \in F$, then there exists $m \in \omega$ such that each $f' \in \mathbb{F}_{\alpha}[S]$ satisfying f' | m = f | m belongs to F.
- 2. For each $x \in \mathcal{D}$, there exist $\alpha < \Omega = \omega_1$, $f \in \mathbb{F}_{\alpha}[S]$, and $t \in \operatorname{Term}_{\alpha} \cap L[S]$ such that $x = \mathsf{C}_t(f)$.
- 3. Each OD[S] set $X \subseteq \mathcal{D}$ is a union of sets of the form $X_t[S]$, where $t \in Term_{\alpha} \cap L[S]$ for some $\alpha < \Omega = \omega_1$.
- 4. Suppose that $t \in \text{Term}_{\alpha} \cap L[S]$, $\alpha < \Omega = \omega_1$, and $u \in \alpha^{<\omega}$. Then every OD[S]set $X \subseteq \mathsf{X}_{tu}[S]$ is a union of sets of the form $\mathsf{X}_{tv}[S]$, where $u \subseteq v \in \alpha^{<\omega}$.

Proof Item 1. We observe that $F = \{f' \in \alpha^{\omega} : \Phi(p, f')\}$ for an \in -formula Φ . Let $\Psi(p, f')$ denote the formula: " $\Lambda \mathcal{P}_{\leq \Omega}$ -forces $\Phi(p, f')$ over the universe", so that

$$F = \{ f' \in \alpha^{\omega} : \Psi(p, f') \text{ is true in } L[S, f'] \}.$$

by Proposition 5 (items 1 and 2). Therefore, since $f \in F \subseteq \mathbb{F}_{\alpha}[S]$, there exists $m \in \omega$ such that the restriction u = f | m (then $u \in \alpha^{<\omega}$) $\alpha^{<\omega}$ -forces $\Psi(p, \hat{f})$ over L[S], where \hat{f} is the name of the α -collapsing function. The m is as required.

Item 2. Since the universe is a Solovay extension of L[S] (Proposition 5), x belongs to an $\alpha^{<\omega}$ -generic extension of L[S], for some $\alpha < \Omega$. Thus $x \in L[S, f]$ where $f \in \mathbb{F}_{\alpha}[S]$. Let \hat{x} be the name of x. We put $t_n = \{u \in \alpha^{<\omega} : u \text{ forces } \hat{x}(n) = 1\}$.

Item 3. Let $x \in X$. We use item 2 to get $\alpha < \Omega$, $f \in \mathbb{F}_{\alpha}[S]$, and $t \in \operatorname{Term}_{\alpha} \cap L[S]$ such that $x = \mathsf{C}_t(f)$. Then we apply item 1 to the $\operatorname{OD}[S]$ set

$$F = \{ f' \in \mathbb{F}_{\alpha}[S] : \mathsf{C}_t(f') \in X \}$$

and the given function f. This results in a condition $u = f | m \in \alpha^{<\omega}$ ($m \in \omega$) such that $x \in X_{tu}[S] \subseteq X$. Finally the set $X_{tu}[S]$ is equal to $X_{t'}[S]$ for some other $t' \in \operatorname{Term}_{\alpha} \cap L[S]$.

Item 4. Similar to the previous item.

2 The case of a closed relation

In this section, we prove item (I) of Theorem 4. Thus let us suppose Ω -SMA and consider an OD equivalence relation E on \mathcal{D} satisfying $\mathsf{E} = \overline{\mathsf{E}}$.

First of all we obtain a characterization for $\overline{\mathsf{E}}$.

We recall that $\Omega = \omega_1$ in the assumption Ω -SMA, and Term = $\bigcup_{\alpha < \omega_1} \operatorname{Term}_{\alpha}$.

Let us fix an OD enumeration Term $\cap L = \{t(\xi) : \xi < \omega_1\}$ such that each "term" $t \in \text{Term} \cap L$ has uncountably many numbers ξ , and $t(\xi) \in \text{Term}_{\alpha}$ for some $\alpha \leq \xi$ whenever $\xi < \omega_1 = \Omega$.

Lemma 8 Assume Ω -SMA and $\mathsf{E} = \overline{\mathsf{E}}$. Let $x, y \in \mathcal{D}$. Then $x \mathsf{E} y$ is equivalent to each of the following two conditions :

(i)
$$x \in [\mathsf{X}_{t(\xi)}(\mathbf{L})]_{\mathsf{E}} \iff y \in [\mathsf{X}_{t(\xi)}(\mathbf{L})]_{\mathsf{E}} \text{ for each } \xi < \omega_1;$$

$$(ii) \quad x \in [\mathsf{X}_{t(\xi)}(\mathsf{L}_{\xi})]_{\mathsf{E}} \quad \longleftrightarrow \quad y \in [\mathsf{X}_{t(\xi)}(\mathsf{L}_{\xi})]_{\mathsf{E}} \quad for \ each \quad \xi < \omega_1$$

Proof $x \in y$ implies both (i) and (ii) because the sets $X_{t(\xi)}(L)$ and $X_{t(\xi)}(L_{\xi})$ are OD. Let us prove the opposite direction.

Let $\gamma = \alpha^{++}$ in L, so that $\gamma < \omega_1 = \Omega$ and $\mathbb{F}_{\alpha}(L) = \mathbb{F}_{\alpha}(L_{\gamma})$. Then the "term" $t' = \langle \gamma, \langle t_n : n \in \omega \rangle \rangle$ belongs to $\operatorname{Term}_{\gamma} \cap L$, and $\mathsf{X}_t(L) = \mathsf{X}_{t'}(L_{\xi})$ whenever $\gamma \leq \xi < \omega_1$. Finally, $t' = t(\xi)$ for some ξ , $\gamma \leq \xi < \omega_1$, and then $\mathsf{X}_t(L) = \mathsf{X}_{t(\xi)}(L_{\xi})$.

2–A The OD subcase

We have to prove that $\mathsf{E} = \overline{\mathsf{E}}$ admits an OD enumeration of the equivalence classes by elements of $2^{<\omega_1}$.

For every $x \in \mathcal{D}$, we define $\Xi(x) = \{\xi < \omega_1 : x \in [X_{t(\xi)}(L)]_E\}$ and let $\phi_x \in 2^{\omega_1}$ be the characteristic function of $\Xi(x)$. Lemma 8 implies that the OD map $x \mapsto \phi_x$ enumerates the E-classes by elements of 2^{ω_1} . To get an enumeration by elements of $2^{<\omega_1}$, we prove

Lemma 9 Assume Ω -SMA. If $h \in 2^{\omega_1}$ is R-OD then there exists $\gamma < \omega_1$ such that $h \in L[h|\gamma]$.

Proof of the lemma. By Ω -SMA, there exists $\alpha < \omega_1$ such that $h \in L[f]$ for a $\alpha^{<\omega}$ -generic over L function $f \in \alpha^{\omega}$. Let <u>h</u> be a name for h in this forcing.

We argue in L. We define $H_{\xi} = \{s \in \alpha^{<\omega} : s \text{ forces } \underline{h}(\xi) = 1\}$ for all $\xi < \Omega$. (We recall that $\Omega = \omega_1$ in the universe but Ω is inaccessible in L under the assumption

 Ω -SMA.) Since $\alpha < \Omega$, we have $(< \Omega)$ -many different sets H_{ξ} . Therefore there exist an ordinal $\gamma < \Omega$ and a function $\tau : \Omega \longrightarrow \gamma$ such that $H_{\xi} = H_{\tau(\xi)}$ for all $\xi < \Omega$.

In the universe, this implies $h \in L[h|\lambda]$, as required.

To continue the proof of the theorem, we let λ_x denote the least ordinal $\lambda < \Omega = \omega_1$ such that $L[\phi_x] = L[\phi_x | \lambda_x]$, for each $x \in \mathcal{D}$.

Unfortunately, the map $x \mapsto \psi_x = \phi_x | \lambda_x$ does not enumerate $\overline{\mathsf{E}}$ -classes by elements of $2^{<\omega_1}$. (The equality $\psi_x = \psi_y$ is not sufficient for $x \mathsf{E} y$.)

We utilize a more tricky idea.

Let $x \in \mathcal{D}$. Then $\psi_x = \phi_x | \lambda_x \in 2^{\lambda_x}$. The set $[x]_{\overline{\mathsf{E}}} = \{x' : \phi_x = \phi_{x'}\}$ is $OD[\phi_x]$, therefore $OD[\psi_x]$ because $\phi_x \in L[\psi_x]$. It follows, by Proposition 7, that $[x]_{\overline{\mathsf{E}}}$ includes a nonempty subset of the form $X_t(L[\psi_x])$, where $t \in \operatorname{Term} \cap L[\psi_x]$.

Let t_x be the least, in the principal $OD[\psi_x]$ wellordering of $L[\psi_x]$, among the "codes" $t \in \text{Term} \cap L[\psi_x]$ such that $\emptyset \neq X_t(L[\psi_x]) \subseteq [x]_{\overline{\mathsf{E}}}$.

The map $x \mapsto \langle \psi_x, t_x \rangle$ is OD, of course. Since the definition is $\overline{\mathsf{E}}$ -invariant and $\mathsf{E} = \overline{\mathsf{E}}$, we have $\psi_x = \psi_y$ and $t_x = t_y$ whenever $x \mathsf{E} y$.

Assume now that $\psi_x = \psi_y$ and $t_x = t_y$. In this case one and the same nonempty set $\mathsf{X}_{t_x}(\mathsf{L}[\psi_x]) = \mathsf{X}_{t_y}(\mathsf{L}([\psi_y]))$ is a subset of both $[x]_\mathsf{E}$ and $[y]_\mathsf{E}$, so $x \mathsf{E} y$.

Hence the map $x \mapsto \langle \psi_x, t_x \rangle$ enumerates the E-classes by elements of the set

 $\{\langle \psi, t \rangle : \psi \in 2^{<\omega_1} \text{ and } t \in \operatorname{Term} \cap L[\psi_x]\}$

This set admits an OD injection in $2^{<\omega_1}$. Therefore we can obtain an OD enumeration of the E-equivalence classes by elements of $2^{<\omega_1}$. This ends the proof of the principal assertion in item (I) of Theorem 4.

2–B The Σ_2^1 and Σ_1^1 subcases

Let us consider the case when E is a Σ_2^1 (resp. Σ_1^1) equivalence relation in item (I) of Theorem 4. We have to engineer a Δ_2^{HC} (resp. Δ_1^{HC}) enumeration of the E-equivalence classes by elements of $2^{<\omega_1}$.

The most natural plan would be to prove that the OD enumeration $x \mapsto \langle \psi_x, t_x \rangle$ defined above is e.g. Δ_2^{HC} provided E is Σ_2^1 . However there is no idea how to convert the definition of ψ_x to Δ_2^{HC} , or even to formalize it in HC. Fortunately we do not neet in fact the minimality of $\psi_x = \phi_x | \lambda_x$; all that we exploited is the existence of a term $t \in \text{Term} \cap L[\psi_x]$ such that $\emptyset \neq \mathsf{X}_t(\mathsf{L}[\psi_x]) \subseteq [x]_{\mathsf{E}}$.

We could now define $\psi_x = \phi_x | \lambda$, where $\lambda = \lambda_x$ is the least ordinal $\lambda < \omega_1$ such that Term $\cap L[\psi_x]$ contains the required term. This can be formalized in HC, but hardly as a Δ_2^{HC} definition: indeed, e. g. the condition $X_t(L[\psi_x]) \subseteq [x]_{\overline{\mathsf{E}}}$ does not look like better than Π_2^{HC} .

The correct plan includes one more idea, originally due to Hjorth and Kechris [3]: certain requirements are eliminated by forcing over a countable submodel.

Let us consider details. We recall that Ω -SMA is assumed.

Let $x \in \mathcal{D}$. We define $\varphi_x \in 2^{\omega_1}$ by $\varphi_x(\xi) = 1$ iff $x \in [\mathsf{X}_{t(\xi)}(\mathsf{L}_{\xi})]_{\mathsf{E}}$ for all $\xi < \omega_1$. (Pay attention on the similarity and the difference between φ_x and ϕ_x above.)

Definition 10 We let T_{E} be the set of all triples $\langle x, \psi, t \rangle$ such that $x \in \mathcal{D}$, $\psi \in 2^{<\omega_1}$, $t \in \operatorname{Term}_{\alpha} \cap L_{\gamma}[\psi]$, where $\alpha < \gamma = \operatorname{dom} \psi < \omega_1$, and the following conditions (a) through (d) are satisfied.

- (a) $L_{\gamma}[\psi]$ models **ZFC**⁻ (minus power set) so that ψ can occur as an extra class parameter in Replacement and Separation.
- (b) It is true in $L_{\gamma}[\psi]$ that $\langle \Lambda, \Lambda \rangle$ forces $C_t(\hat{f}) \in C_t(\hat{g})$ in the sense of $\alpha^{<\omega} \times \alpha^{<\omega}$ as the forcing, where \hat{f} and \hat{g} are the names for the generic functions in α^{ω} .
- (c) $\psi = \varphi_x | \gamma$.
- (d) x belongs to $[X_t(L_{\gamma}[\psi])]_{\mathsf{E}}$.

A point $x \in \mathcal{D}$ is E -classifiable iff there exist ψ and t such that $\langle x, \psi, t \rangle \in T_{\mathsf{E}}$. \Box

Lemma 11 Assume Ω -SMA. If E is a Σ_2^1 equivalence and $\mathsf{E} = \overline{\mathsf{E}}$ then all points $x \in \mathcal{D}$ are E -classifiable.

Proof Let $x \in \mathcal{D}$. Then φ_x is OD[x], therefore $\varphi_x \in L[x]$ by Proposition 5. Since $\mathsf{E} = \overline{\mathsf{E}}$, Lemma 8 implies that the set $[x]_{\mathsf{E}}$ is $OD[\varphi_x]$. Therefore by Proposition 7 we have $x \in \mathsf{X}_t(\mathrm{L}[\varphi_x]) \subseteq [x]_{\mathsf{E}}$ for some $t \in \operatorname{Term}_{\alpha} \cap \mathrm{L}[\varphi_x]$, $\alpha < \Omega = \omega_1$.

The model $L_{\omega_1}[\varphi_x]$ has an elementary submodel $L_{\gamma}[\psi]$, where $\gamma < \omega_1$ and $\psi = \varphi_x | \gamma$, containing t and α . We prove that $\langle x, \psi, t \rangle \in T_{\mathsf{E}}$. Since conditions (a) and (c) of Definition 10 obviously hold for $L_{\gamma}[\psi]$, let us check requirements (b), (d).

We check (b). Indeed otherwise there exist conditions $u, v \in \alpha^{<\omega}$ such that $\langle u, v \rangle$ forces $\mathsf{C}_t(\hat{f}) \not\models \mathsf{C}_t(\hat{g})$ in $\mathsf{L}_{\gamma}[\psi]$ in the sense of $\alpha^{<\omega} \times \alpha^{<\omega}$ as the notion of forcing. Then $\langle u, v \rangle$ also forces $\mathsf{C}_t(\hat{f}) \not\models \mathsf{C}_t(\hat{g})$ in $\mathsf{L}_{\omega_1}[\varphi_x]$. Let us consider an $\alpha^{<\omega} \times \alpha^{<\omega}$ -generic over $\mathsf{L}[\varphi_x]$ pair $\langle f, g \rangle \in \alpha^{\omega} \times \alpha^{\omega}$ such that $u \subset f$ and $v \subset g$. Then both $y = \mathsf{C}_t(f)$ and $z = \mathsf{C}_t(g)$ belong to $\mathsf{X}_t(\mathsf{L}[\varphi_x])$, so $y \mathsf{E} z$ because $\mathsf{X}_t(\mathsf{L}[\varphi_x]) \subseteq [x]_{\mathsf{E}}$.

Notice that $\langle f, g \rangle$ also is generic over $L_{\omega_1}[\varphi_x]$. We observe that $y \in z$ is false in $L_{\omega_1}[\varphi_x, f, g]$, that is, in $L[\varphi_x, f, g]$, by the choice of u and v. But $y \in z$ is a Σ_2^1 formula, therefore absolute for transitive models containing all ordinals, contradiction.

We check (d). Take any $\alpha^{<\omega}$ -generic over $L[\varphi_x]$ function $f \in \alpha^{\omega}$. Then $y = C_t(f)$ belongs to $X_t(L[\varphi_x])$, hence $y \in x$. On the other hand, f is generic over $L_{\gamma}[\psi]$.

Thus $\langle x, \psi, t \rangle \in T_{\mathsf{E}}$. This means that x is E-classifiable, as required.

Thus, for each E-classifiable x, all countable sequences ψ which satisfy $T_{\mathsf{E}}(x, \psi, t)$ for some t, are restrictions of one and the same sequence $\varphi_x \in 2^{\omega_1}$, defined above.

Definition 12 Let $x \in \mathcal{D}$. It follows from Lemma 11 that there exists the least ordinal $\gamma = \gamma_x < \omega_1$ such that $T_{\mathsf{E}}(x, \varphi_x | \gamma, t)$ for some t. We put $\psi_x = \varphi_x | \gamma$ and let t_x denote the least, in the sense of the $OD[\psi_x]$ wellordering of $L_{\gamma}[\psi_x]$, "term" $t \in \operatorname{Term}[\psi_x] \cap L_{\gamma}[\psi_x]$ which satisfies $T_{\mathsf{E}}(x, \psi_x, t)$. We put $U(x) = \langle \psi_x, t_x \rangle$. \Box

Lemma 13 Assume Ω -SMA. If E is a Σ_2^1 equivalence and $\mathsf{E} = \overline{\mathsf{E}}$ then the map U enumerates the E -classes.

Proof If $x \in y$ then U(x) = U(y) because Definition 10 is E-invariant for x.

Let us prove the converse. Assume that U(x) = U(y), that is, in particular, $\psi_x = \psi_y = \psi \in 2^{<\omega}$ and $t_x = t_y = t \in \operatorname{Term}_{\alpha}[\psi] \cap L_{\gamma}[\psi]$, where $\alpha < \gamma = \operatorname{dom} \psi < \omega_1 = \Omega$.

By (d) we have $C_t(f) \to x$ and $C_t(g) \to y$ for some $\alpha^{<\omega}$ -generic over $L_{\gamma}[\psi]$ functions $f, g \in \alpha^{\omega}$. Let us consider an $\alpha^{<\omega}$ -generic over both $L_{\gamma}[\psi, f]$ and $L_{\gamma}[\psi, g]$ function $h \in \alpha^{\omega}$. Then, by (b), $C_t(h) \to C_t(f)$ holds in $L_{\gamma}[\psi, f, h]$, therefore in the universe because E is Σ_2^1 . Similarly, we have $C_t(h) \to C_t(g)$. It follows that $C_t(f) \to C_t(g)$, hence $x \to y$, as required.

Lemma 14 Suppose that E is Σ_2^1 (resp. Σ_1^1) and $\mathsf{E} = \overline{\mathsf{E}}$. Then U is a function of class Δ_2^{HC} (resp. Δ_1^{HC}).

Proof It suffices to check that the set T_{E} is Δ_2^{HC} (resp. Δ_1^{HC}).

Notice that conditions (a) and (b) in Definition 10 are Δ_1^{HC} because they reflect truth within $L_{\gamma}[\psi]$ and the enumeration $t(\xi)$ was chosen in Δ_1^{HC} .

Suppose that E is Σ_2^1 , that is, Σ_1^{HC} . Then condition (d) is obviously Σ_1^{HC} . Condition (c) can be converted to Δ_2^{HC} (in fact a bounded quantifier $\forall \beta < \gamma$ over a conjunction of Σ_1^{HC} and Π_1^{HC} relations). Indeed we observe that (c) is equivalent to

$$\forall \xi < \gamma \ (\psi(\xi) = 1 \iff x \in [\mathsf{X}_{t(\xi)}(\mathsf{L}_{\xi})]_{\mathsf{E}}) \tag{(*)}$$

by Lemma 8.

The case when E belongs to Σ_1^1 is more difficult.

Let us first consider condition (d). Immediately, it is Σ_2^1 , therefore Σ_1^{HC} , so it remains to convert it also to a Π_1^{HC} form. Notice that in the assumption of (a) and (b), the set $X = \mathsf{X}_t(\mathsf{L}_{\gamma}[\psi])$ consists of pairwise E-equivalent points: this was actually showed in the proof of Lemma 13. Therefore, since obviously $\mathsf{X}_t(\mathsf{L}_{\gamma}[\psi]) \neq \emptyset$, (d) is equivalent to $\forall y \in \mathsf{X}_t(\mathsf{L}_{\gamma}[\psi])$ ($x \in y$). This is clearly Π_1^{HC} provided E is Π_2^1 .

Let us consider (c). The right-hand side of the equivalence (*) is Σ_1^1 with inserted Δ_1^{HC} functions, therefore Δ_1^{HC} . It follows that (*) itself is Δ_1^{HC} , as required. \Box

This completes the proof of the additional part (Σ_1^1 and Σ_2^1 relations) in item (I) of Theorem 4.

3 OD topology and the forcing

This section starts the proof of item (II) of Theorem 4 for a given OD equivalence relation E in the assumption Ω -SMA.

We have to embed E_0 in E continuously. The embedding will be defined in the next section; here we obtain some useful preliminary results related to the defined above topology \mathcal{T} , an associated forcing and the relevant product forcing. At the end of the section, we introduce the set H of all points $x \in \mathcal{D}$ whose \overline{E} -classes are bigger than E-classes; H is nonempty in the assumption $E \subseteq \overline{E}$.

The reasoning is based on special properties of the OD topology \mathcal{T} , having a semblance of the Gandy – Harrington topology (even in a simplified form because some specific Σ_1^1 details vanish). In particular, the topology is strongly Choquet. However we shall not utilize this property (and shall not prove it). We take indeed another way. The reasoning will be organized as a sequence of straight forcing arguments. This manner of treatment of equivalence relations was taken from Miller [8].

3–A Topology and the forcing

The topology \mathcal{T} obviously does not have a countable base; but it has one in a local sense. A set X will be called \mathcal{T} -separable if the OD power set $\mathcal{P}^{\text{OD}}(X) = \mathcal{P}(X) \cap \text{OD}$ has only countably many different OD subsets.

Lemma 15 Assume Ω -SMA. Let $\alpha < \Omega$ and $t \in \text{Term}_{\alpha} \cap L$. Then $X = X_t(L)$ is \mathcal{T} -separable.

Proof By Proposition 7 every OD subset of X is uniquely determined by an OD subset of $\alpha^{<\omega}$. Since each OD set $S \subseteq \alpha^{<\omega}$ is constructible (Proposition 5), we obtain an OD map $h : \alpha^+$ onto $\mathcal{P}^{\text{OD}}(X)$, where α^+ is the least cardinal in L bigger than α . Therefore $\mathcal{P}^{\text{OD}}(X)$ has $\leq \alpha^{++}$ -many OD subsets. It remains to notice that $\alpha^{++} < \Omega$ because Ω is inaccessible in L.

Let $\mathbf{X} = \{X \subseteq \mathcal{D} : X \text{ is OD and nonempty}\}.$

Let us consider \mathbb{X} as a forcing notion (smaller sets are stronger conditions) for generic extensions of L in the assumption Ω -SMA. Of course formally $\mathbb{X} \notin L$, but \mathbb{X} is OD order isomorphic to a partially ordered set in L. (Indeed it is known that there exists an OD map ϕ : ordinals onto the class of all OD sets. Since \mathbb{X} itself is OD, \mathbb{X} is a 1–1 image of an OD set \mathbb{X}' of ordinals via ϕ . By Proposition 5 both \mathbb{X}' and the ϕ -preimage of the order on \mathbb{X} belong to L.)

It also is true that a set $G \subseteq \mathbb{X}$ is \mathbb{X} -generic over L iff it nonempty intersects every dense OD subset of \mathbb{X} .

Corollary 16 Assume Ω -SMA. If a set $X \in \mathbb{X}$ is nonempty then there exists an \mathbb{X} -generic over L set $G \subseteq \mathbb{X}$ containing X.

Proof We can suppose, by Proposition 7, that $X = X_t(L)$ where $t \in \text{Term}_{\alpha} \cap L$ and $\alpha < \Omega$. Now apply Lemma 15.

Lemma 17 Assume Ω -SMA. If $G \subseteq \mathbb{X}$ is a generic over L set then the intersection $\cap G$ is a singleton $\{a\} = \{a_G\}$.

Proof Assume that this is not the case. Let $\mathbf{X}' \in \mathbf{L}$ be a constructible p. o. set order isomorphic \mathbf{X} via an OD function $f: \mathbf{X}'$ onto \mathbf{X} . Then $G' = \phi^{-1}(G)$ is \mathbf{X}' -generic over L. We assert that the statement that $\bigcap G$ is not a singleton can be converted to a sentence relativized to $\mathbf{L}[G']$.

(Indeed, it follows from the reasoning in the proof of Lemma 15 that L[G'] is in fact a *P*-generic extension of L for a certain set $P \in L$, $P \subseteq \mathbb{X}'$ of a cardinality $\alpha < \Omega$ in L. The next L-cardinal α^+ is $< \Omega$. Therefore G' belongs to a $\alpha^{+<\omega}$ generic extension of L, so G' is Ω -weak over L. Then by Proposition 5 the universe is a Solovay extension of L[G']. This is enough to convert any statement about G' in V – like the statement: $\bigcap \phi^{"}G'$ is not a singleton – to a sentence relativized to L[G'].)

Then there exists $X \in \mathbb{X}$ such that $\bigcap G$ is not a singleton for every generic over L set $G \subseteq \mathbb{X}$ containing X. We can assume that $X = \mathsf{X}_t(\mathsf{L})$, where $t \in \operatorname{Term}_{\alpha} \cap \mathsf{L}$, $\alpha < \Omega$. Then X is \mathcal{T} -separable; let $\{\mathcal{X}_n : n \in \omega\}$ be an enumeration of all OD dense subsets of $\mathcal{P}^{\mathrm{OD}}(X)$. Using Proposition 7 (item 1), we obtain an increasing $\alpha^{<\omega}$ -generic over L sequence $u_0 \subseteq u_1 \subseteq u_2 \subseteq \ldots$ of $u_n \in \alpha^{<\omega}$ such that $X_n = \mathsf{X}_{tu_n}(\mathsf{L}) \in \mathcal{X}_n$. Obviously this gives an \mathbb{X} -generic over L set $G \subseteq \mathbb{X}$ containing X and all X_n .

Now let $f = \bigcup_{n \in \omega} u_n$; $f \in \alpha^{\omega}$ and f is $\alpha^{<\omega}$ -generic over L. Then $x = \mathsf{C}_t(f) \in X_n$ for all n, so $x \in \bigcap G$. Since $\bigcap G$ obviously cannot contain more than one point, it is a singleton, so we get a contradiction with the choice of X.

Reals a_G will be called OD-generic over L.

3–B The product forcing

We recall that E is an OD equivalence on \mathcal{D} and $\overline{\mathsf{E}}$ is the \mathcal{T}^2 -closure of E.

For a set $P \subseteq \mathcal{D}^2$, we put $\operatorname{pr}_1 P = \{x : \exists y \ P(x, y)\}$ and $\operatorname{pr}_2 P = \{y : \exists x \ P(x, y)\}$. Notice that if P is OD, so are $\operatorname{pr}_1 P$ and $\operatorname{pr}_2 P$.

The classical reasoning in Harrington, Kechris, and Louveau [1] plays on interactions between E and \overline{E} . In the forcing setting, we have to fix a restriction by \overline{E} directly in the definition of the product forcing. Thus we consider

$$\mathbb{P} = \mathbb{P}(\overline{\mathsf{E}}) = \{ P \subseteq \overline{\mathsf{E}} : P \text{ is OD and nonempty and } P = (\mathsf{pr}_1 P \times \mathsf{pr}_2 P) \cap \overline{\mathsf{E}} \}$$

as a forcing notion. As above for X, the fact that formally \mathbb{P} does not belong to L does not cause essential problems.

The following assertion connects \mathbb{P} and \mathbb{X} .

Assertion 18 Assume Ω -SMA. Then

- 1. If $P \in \mathbb{P}$ then $\operatorname{pr}_1 P$ and $\operatorname{pr}_2 P$ belong to \mathbb{X} .
- 2. If $X, Y \in \mathbb{X}$ and $P = (X \times Y) \cap \overline{\mathsf{E}} \neq \emptyset$ then $P \in \mathbb{P}$.
- 3. If $P \in \mathbb{P}$, $X \in \mathbb{X}$, $X \subseteq \operatorname{pr}_1 P$, then there exists $Q \in \mathbb{P}$, $Q \subseteq P$, such that $X = \operatorname{pr}_1 Q$. Similarly for pr_2 .

Proof Set $Q = \{ \langle x, y \rangle \in P : x \in X \& y \overline{\mathsf{E}} x \}$ in item 3.

A set $P \in \mathbb{P}$ is \mathbb{P} -separable if the set $\mathbb{P}_{\subseteq P} = \{Q \in \mathbb{P} : Q \subseteq P\}$ has only countably many different OD subsets.

Lemma 19 Assume Ω -SMA. Suppose that $P = (X \times Y) \cap \overline{\mathsf{E}}$ is nonempty, where $X = \mathsf{X}_t(\mathsf{L}), Y = \mathsf{X}_{t'}(\mathsf{L}), \text{ and } t, t' \in \operatorname{Term} \cap \mathsf{L}$. Then $P \in \mathbb{P}$ and P is \mathbb{P} -separable.

Proof We have $P \in \mathbb{P}$ by Assertion 18. A proof of the \mathbb{P} -separability can be obtained by a minor modification of the proof of Lemma 15.

Lemma 20 Assume Ω -SMA. Let $G \subseteq \mathbb{P}$ be a \mathbb{P} -generic over \mathbb{L} set. Then the intersection $\bigcap G$ contains a single point $\langle a, b \rangle$ where a and b are OD-generic over \mathbb{L} and $a \ \overline{\mathsf{E}} b$.

Proof By Assertion 18, both $G_1 = \{ \mathsf{pr}_1 P : P \in G \}$ and $G_2 = \{ \mathsf{pr}_1 P : P \in G \}$ are OD-generic over L subsets of \mathbb{X} , so that there exist unique OD-generic over L points $a = a_{G_1}$ and $b = a_{G_2}$. It remains to show that $\langle a, b \rangle \in \overline{\mathsf{E}}$.

Suppose not. There exists an E -invariant OD set A such that we have $x \in A$ and $y \in B = \mathcal{D} \setminus A$. Then $A \in G_1$ and $B \in G_2$ by the genericity. There exists a condition $P \in G$ such that $\mathsf{pr}_1 P \subseteq A$ and $\mathsf{pr}_2 B \subseteq B$, therefore $P \subseteq (A \times B) \cap \overline{\mathsf{E}} = \emptyset$, which is impossible.

Pairs $\langle a, b \rangle$ as in Lemma 20 will be called **P**-generic and denoted by $\langle a_G, b_G \rangle$.

For sets X and Y and a binary relation R, let us write X R Y if and only if $\forall x \in X \exists y \in Y (x R y)$ and $\forall y \in Y \exists x \in X (x R y)$.

Lemma 21 Assume Ω -SMA. Suppose that $P_0 \in \mathbb{P}$, points $a, a' \in X_0 = \operatorname{pr}_1 P_0$ are OD-generic over L, and $a \overline{\mathsf{E}} a'$. There exists a point b such that both $\langle a, b \rangle$ and $\langle a', b \rangle$ belong to P_0 and are \mathbb{P} -generic pairs.

Proof By Lemma 19 and Proposition 7 there exists a \mathbb{P} -separable set $P_1 \subseteq P_0$ such that $a \in X_1 = \mathsf{pr}_1 P_1$. We put $Y_1 = \mathsf{pr}_2 P_1$; then $X_1 \overline{\mathsf{E}} Y_1$, and $P_1 = (X_1 \times Y_1) \cap \overline{\mathsf{E}}$.

We let $P' = \{ \langle x, y \rangle \in P_0 : y \in Y_1 \}$. Then $P' \in \mathbb{P}$ and $P_1 \subseteq P' \subseteq P_0$. Furthermore $a' \in X' = \mathsf{pr}_1 P'$. (Indeed, since $a \in X_1$ and $X_1 \overline{\mathsf{E}} Y_1$, there exists $y \in Y_1$ such that

 $a \overline{\mathsf{E}} y$; then $a' \overline{\mathsf{E}} y$ as well because $a \overline{\mathsf{E}} a'$, therefore $\langle a', y \rangle \in P'$.) As above there exists a \mathbb{P} -separable set $P'_1 \subseteq P'$ such that $a' \in X'_1 = \mathsf{pr}_1 P'_1$. Then $Y'_1 = \mathsf{pr}_2 P'_1 \subseteq Y_1$.

It follows from the choice of P and P' that \mathbb{P} admits only countably many different dense OD sets below P_1 and below P'_1 . Let $\{\mathcal{P}_n : n \in \omega\}$ and $\{\mathcal{P}'_n : n \in \omega\}$ be enumerations of both families of dense sets. We define sets $P_n, P'_n \in \mathbb{P}$ $(n \in \omega)$ satisfying the following conditions:

(i) $a \in X_n = \operatorname{pr}_1 P_n$ and $a' \in X'_n = \operatorname{pr}_1 P'_n$;

(ii)
$$Y'_n = \operatorname{pr}_2 P'_n \subseteq Y_n = \operatorname{pr}_2 P_n$$
 and $Y_{n+1} \subseteq Y'_n$;

(iii) $P_{n+1} \subseteq P_n$, $P'_{n+1} \subseteq P'_n$, $P_n \in \mathcal{P}_{n-2}$, and $P'_n \in \mathcal{P}'_{n-2}$.

By (iii) both sequences $\{P_n : n \in \omega\}$ and $\{P'_n : n \in \omega\}$ are \mathbb{P} -generic over L, so by Lemma 20 they result in two generic pairs, $\langle a, b \rangle \in P_0$ and $\langle a', b \rangle \in P_0$, having the first terms equal to a and a' by (i) and second terms equal to each other by (ii). Thus it suffices to conduct the construction of P_n and P'_n .

The construction goes on by induction on n.

Assume that P_n and P'_n have been defined. We define P_{n+1} . By (ii) and Assertion 18, the set $P = (X_n \times Y'_n) \cap \overline{\mathsf{E}} \subseteq P_n$ belongs to \mathbb{P} and $a \in X = \mathsf{pr}_1 P$. (Indeed, $\langle a, b \rangle \in P$, where b satisfies $\langle a', b \rangle \in P'_n$, because $a \overline{\mathsf{E}} a'$.) However \mathcal{P}_{n-1} is dense in \mathbb{P} below $P \subseteq P_0$; therefore $\mathsf{pr}_1 \mathcal{P}_{n-1} = \{\mathsf{pr}_1 P' : P' \in \mathcal{P}_{n-1}\}$ is dense in \mathbb{X} below $X = \mathsf{pr}_1 P$. Since a is generic, we have $a \in \mathsf{pr}_1 P'$ for some $P' \in \mathcal{P}_{n-1}$, $P' \subseteq P$. It remains to put $P_{n+1} = P'$, and then $X_{n+1} = \mathsf{pr}_1 P_{n+1}$ and $Y_{n+1} = \mathsf{pr}_2 P_{n+1}$.

After this, to define P'_{n+1} we let $P = (X'_n \times Y_{n+1}) \cap \overline{\mathsf{E}}$, etc. \Box

3–C The key set

We recall that Ω -SMA is assumed, E is an OD equivalence on \mathcal{D} , and $\overline{\mathsf{E}}$ is the \mathcal{T}^2 -closure of E in \mathcal{D}^2 . We shall also suppose that $\mathsf{E} \subseteq \overline{\mathsf{E}}$, in accordance to item (II) of Theorem 4. Then there exist $\overline{\mathsf{E}}$ -classes which include more than one E -class. (In fact we shall have no other use of the hypothesis $\mathsf{E} \subsetneq \overline{\mathsf{E}}$.) We define the union of all those $\overline{\mathsf{E}}$ -classes,

$$H = \{ x \in \mathcal{D} : \exists y \in \mathcal{D} \ (x \,\overline{\mathsf{E}} \, y \& x \not\not\in y) \},\$$

the "key set" from the title. The role of this set in the reasoning below is entirely similar to the role of the corresponding set V in Harrington, Kechris, and Louveau [1].

Obviously H is OD, nonempty, and E-invariant. Furthermore, $H' = H^2 \cap \overline{\mathsf{E}} \neq \emptyset$ (in fact both projections of H^* are equal to H), so that in particular $H' \in \mathbb{P}$ by Assertion 18.

Lemma 22 Assume Ω -SMA. If $a, b \in H$ and $\langle a, b \rangle$ is \mathbb{P} -generic over \mathbb{L} then $a \not\in b$.

Proof Otherwise there exists a set $P \in \mathbb{P}$, $P \subseteq H \times H$ such that $a \in b$ holds for all \mathbb{P} -generic $\langle a, b \rangle \in P$. We conclude that then $a \in a' \longrightarrow a \in a'$ for all OD-generic points $a, a' \in X = \operatorname{pr}_1 P$; indeed, take b such that both $\langle a, b \rangle \in P$ and $\langle a', b \rangle \in P$ are \mathbb{P} -generic, by Lemma 21. In other words the relations E and $\overline{\mathsf{E}}$ coincide on the set $Y = \{x \in X : x \text{ is OD-generic over } \mathsf{L}\} \in \mathbb{X}$. $(Y \neq \emptyset$ by corollaries 16 and 17.)

Moreover, E and $\overline{\mathsf{E}}$ coincide on the set $Z = [Y]_{\mathsf{E}}$. Indeed if $z, z' \in Z, z \,\overline{\mathsf{E}} \, z'$, then let $y, y' \in Y$ satisfy $z \,\mathsf{E} \, y$ and $z' \,\mathsf{E} \, y'$. Then $y \,\overline{\mathsf{E}} \, y'$, therefore $y \,\mathsf{E} \, y'$, which implies $z \,\mathsf{E} \, z'$.

We conclude that $Y \cap H = \emptyset$.

(Indeed, suppose that $x \in Y \cap H$. Then by definition there exists $y \in \mathcal{D}$ such that $x \overline{\mathsf{E}} y$ but $x \not\in y$. Then $y \notin Z$ because E and $\overline{\mathsf{E}}$ coincide on Z. Thus the pair $\langle x, y \rangle$ belongs to the OD set $P = Y \times (\mathcal{D} \setminus Z)$. Notice that P does not intersect E by definition of Z. Therefore $\langle x, y \rangle$ cannot belong to the closure $\overline{\mathsf{E}}$ of E , contradiction.) But $\emptyset \neq Y \subseteq X \subseteq H$, contradiction.

Lemma 22 is a counterpart of the proposition in Harrington, Kechris, Louveau [1] that E|H is meager in $\overline{E}|H$. But in fact the main content of this argument in [1] was implicitly taken by Lemma 21.

Lemma 23 Assume Ω -SMA. Let $X, Y \subseteq H$ be nonempty OD sets and $X \overline{\mathsf{E}} Y$. There exist nonempty OD sets $X' \subseteq X$ and $Y' \subseteq Y$ such that $X' \cap Y' = \emptyset$ but still $X' \overline{\mathsf{E}} Y'$.

Proof There exist points $x_0 \in X$ and $y_0 \in Y$ such that $x_0 \neq y_0$ but $x_0 \overline{\mathsf{E}} y_0$. (Otherwise X = Y, and $\overline{\mathsf{E}}$ is the equality on X, which is impossible, see the previous proof.) Let U and V be disjoint Baire intervals in \mathcal{D} containing resp. x_0 and y_0 . The sets $X' = X \cap U \cap [Y \cap V]_{\overline{\mathsf{E}}}$ and $Y' = Y \cap V \cap [X \cap U]_{\overline{\mathsf{E}}}$ are as required. \Box

4 The embedding

In this section we accomplish the proof of Theorem 4, therefore Theorem 2 (see Section 1). Thus we prove, assuming Ω -SMA and $\mathsf{E} \subsetneqq \overline{\mathsf{E}}$, that E , the given OD equivalence on \mathcal{D} , continuously embeds E_0 .

4–A The embedding

By the assumption the set H of Subsection 3–C is nonempty; obviously H is OD. By lemmas 15 and 19 there exists a nonempty \mathcal{T} -separable OD set $X_0 \subseteq H$ such that the set $P_0 = (X_0 \times X_0) \cap \overline{\mathsf{E}}$ belongs to \mathbb{P} and is \mathbb{P} -separable; $\mathsf{pr}_1 P_0 = \mathsf{pr}_2 P_0 = X_0$.

We define a family of sets X_u $(u \in 2^{<\omega})$ satisfying

(a) $X_u \subseteq X_0$, X_u is nonempty and OD, and $X_{u^{\wedge}i} \subseteq X_u$, for all u and i.

In addition to the sets X_u , we shall define sets R_{uv} for some pairs $\langle u, v \rangle$, to provide important interconnections between branches in $2^{\langle \omega \rangle}$.

Let $u, v \in 2^n$. We say that $\langle u, v \rangle$ is a *neighbouring pair* iff $u = 0^{k \wedge} 0^{\wedge} r$ and $v = 0^{k \wedge} 1^{\wedge} r$ for some k < n (0^k is the sequence of k terms each equal to 0) and some $r \in 2^{n-k-1}$ (possibly k = n - 1, that is, $r = \Lambda$).

Thus we define sets $\mathsf{R}_{uv} \subseteq X_u \times X_v$ for all neighbouring pairs $\langle u, v \rangle$, so that the following requirements (b) and (c) will be satisfied.

- (b) R_{uv} is OD, $\mathsf{pr}_1\mathsf{R}_{uv} = X_u$, $\mathsf{pr}_2\mathsf{R}_{uv} = X_v$, and $\mathsf{R}_{u^{\wedge}i, v^{\wedge}i} \subseteq \mathsf{R}_{uv}$ for every neighbouring pair $\langle u, v \rangle$ and each $i \in \{0, 1\}$.
- (c) For any k, the set $\mathsf{R}_k = \mathsf{R}_{0^{k} \wedge 0, 0^{k} \wedge 1}$ is \mathcal{T} -separable, and $\mathsf{R}_k \subseteq \mathsf{E}$.

Notice that if $\langle u, v \rangle$ is neighbouring then $\langle u^{\wedge}i, v^{\wedge}i \rangle$ is neighbouring, but $\langle u^{\wedge}i, v^{\wedge}j \rangle$ is not neighbouring for $i \neq j$ (unless $u = v = 0^k$ for some k).

This implies $X_u \mathsf{R}_{uv} X_v$, therefore $X_u \mathsf{E} X_v$, for all neighbouring pairs u, v.¹⁰

Remark 24 Every pair of $u, v \in 2^n$ can be tied in 2^n by a finite chain of neighbouring pairs. It follows that $X_u \mathsf{E} X_v$ and $X_u \overline{\mathsf{E}} X_v$ hold for all pairs $u, v \in 2^n$. \Box

Three more requirements will concern genericity.

Let $\{\mathcal{X}_n : n \in \omega\}$ be a fixed (not necessarily OD) enumeration of all dense in \mathbb{X} below X_0 subsets of \mathbb{X} . Let $\{\mathcal{P}_n : n \in \omega\}$ be a fixed enumeration of all dense in \mathbb{P} below P_0 subsets of \mathbb{P} . It is assumed that $\mathcal{X}_{n+1} \subseteq \mathcal{X}_n$ and $\mathcal{P}_{n+1} \subseteq \mathcal{P}_n$. Note that $\mathcal{X}' = \{P \in \mathbb{P} : P \subseteq P_0 \& \operatorname{pr}_1 P \cap \operatorname{pr}_2 P = \emptyset\}$ is dense in \mathbb{P} below P_0 by Lemma 23, so we can suppose in addition that $\mathcal{P}_0 = \mathcal{X}'$.

In general, for any \mathcal{T} -separable set S let $\{\mathcal{X}_n(S) : n \in \omega\}$ be a fixed enumeration of all dense subsets in the algebra $\mathcal{P}^{^{\mathrm{OD}}}(S) \setminus \{\emptyset\}$. It is assumed that $\mathcal{X}_{n+1}(S) \subseteq \mathcal{X}_n(S)$.

¹⁰ We recall that $X \mathrel{\mathsf{R}} Y$ means that $\forall x \in X \exists y \in Y (x \mathrel{\mathsf{R}} y)$ and $\forall y \in Y \exists x \in X (x \mathrel{\mathsf{R}} y)$.

- (g1) $X_u \in \mathcal{X}_n$ whenever $u \in 2^n$.
- (g2) If $u, v \in 2^n$ and $u(n-1) \neq v(n-1)$ (that is, the last terms of u, v are different), then $P_{uv} = (X_u \times X_v) \cap \overline{\mathsf{E}} \in \mathcal{P}_n$. In fact this implies (g1).
- (g3) If $\langle u, v \rangle = \langle 0^{k \wedge} 0^{\wedge} r, 0^{k \wedge} 1^{\wedge} r \rangle \in (2^n)^2$ then $\mathsf{R}_{uv} \in \mathcal{X}_n(\mathsf{R}_k)$.

In particular (g1) implies by Corollary 17 that for any $a \in 2^{\omega}$ the intersection $\bigcap_{n \in \omega} X_{a|n}$ contains a single point, denoted by $\phi(a)$, which is **X**-generic over L, and the map ϕ is continuous in the Polish sense.

Assertion 25 ϕ is a continuous 1–1 embedding E_0 into E.

Proof Let us prove that ϕ is 1–1. Suppose that $a \neq b \in 2^{\omega}$. Then $a(n-1) \neq b(n-1)$ for some n. Let $u = a \mid n, v = b \mid n$, so that we have $x = \phi(a) \in X_u$ and $y = \phi(b) \in X_v$. But then the set $P = (X_u \times X_v) \cap \overline{\mathsf{E}}$ belongs to \mathcal{P}_n by (g2), therefore to \mathcal{P}_0 . This implies $X_u \cap X_v = \emptyset$ by definition of \mathcal{P}_0 , hence $\phi(a) \neq \phi(b)$ as required.

Let us finally verify that $a \mathsf{E}_0 b$ implies $\phi(a) \mathsf{E} \phi(b)$. It is sufficient to prove that $\phi(0^{k} \circ 0^{k} c) \mathsf{E} \phi(0^{k} \circ 1^{k} c)$ holds for all $k \in \omega$ and $c \in 2^{\omega}$, simply because every pair of $u, v \in 2^n$ is tied in 2^n by a chain of neighbouring pairs.

The sequence of sets $W_m = \mathsf{R}_{0^k \wedge 0^{\wedge}c \mid m, 0^k \wedge 1^{\wedge}c \mid m}$ $(m \in \omega)$ is then generic over L by (g3) in the sense of the forcing $\mathcal{P}^{\text{OD}}(\mathsf{R}_k) \setminus \{\emptyset\}$ (we recall that $\mathsf{R}_k = \mathsf{R}_{0^k \wedge 0, 0^{k \wedge 1}}$), which is simply a copy of \mathbb{X} , so that by Corollary 17 the intersection of all sets W_m is a singleton. Obviously the singleton can be only equal to $\langle \phi(0^{k \wedge 0^{\wedge}c}), \phi(0^{k \wedge 1^{\wedge}c}) \rangle$. We conclude that $\phi(0^{k \wedge 0^{\wedge}c}) \in \phi(0^{k \wedge 1^{\wedge}c})$, as required. \Box

4–B Restriction lemma

Thus the theorem is reduced to the construction of sets X_u and R_{uv} (in the assumption Ω -SMA). Before the construction starts, we prove the principal combinatorial fact.

Lemma 26 Let $n \in \omega$ and X_u be nonempty OD for each $u \in 2^n$. Assume that an OD set $\mathsf{R}_{uv} \subseteq \mathcal{D}^2$ is given for every neighbouring pair of $u, v \in 2^n$ so that $X_u \mathsf{R}_{uv} X_v$.

- 1. If $u_0 \in 2^n$ and $X' \subseteq X_{u_0}$ is OD and nonempty then there exists a system of OD nonempty sets $Y_u \subseteq X_u$ $(u \in 2^n)$ such that $Y_u \mathsf{R}_{uv} Y_v$ holds for all neighbouring pairs u, v, and in addition $Y_{u_0} = X'$.
- 2. Suppose that $u_0, v_0 \in 2^n$ is a neighbouring pair and nonempty OD sets $X' \subseteq X_{u_0}$ and $X'' \subseteq X_{v_0}$ satisfy $X' \mathsf{R}_{u_0v_0} X''$. Then there exists a system of OD nonempty sets $Y_u \subseteq X_u$ $(u \in 2^n)$ such that $Y_u \mathsf{R}_{uv} Y_v$ holds for all neighbouring pairs u, v, and in addition $Y_{u_0} = X'$, $Y_{v_0} = X''$.

Proof Notice that 1 follows from 2. Indeed take arbitrary v_0 such that either $\langle u_0, v_0 \rangle$ or $\langle v_0, u_0 \rangle$ is neighbouring, and put respectively $X'' = \{ y \in X_{v_0} : \exists x \in X' (x \mathsf{R}_{u_0v_0} y) \}$, or $X'' = \{ y \in X_{v_0} : \exists x \in X' (y \mathsf{R}_{v_0u_0} x) \}$.

To prove item 2, we use induction on n.

For n = 1 — then $u_0 = \langle 0 \rangle$ and $v_0 = \langle 1 \rangle$ — we take $Y_{u_0} = Y'$ and $Y_{v_0} = Y''$.

The step. We prove the lemma for n + 1 provided it has been proved for $n; n \ge 1$. 1. The principal idea is to divide 2^{n+1} on two copies of 2^n , minimally connected by neighbouring pairs, and handle them more or less separately using the induction hypothesis. The two "copies" are $U_0 = \{s^{\wedge}0 : s \in 2^n\}$ and $U_1 = \{s^{\wedge}1 : s \in 2^n\}$.

The only neighbouring pair that connects U_0 and U_1 is the pair of $\hat{u} = 0^{n\wedge}0$ and $\hat{v} = 0^{n\wedge}1$. If in fact $u_0 = \hat{u}$ and $v_0 = \hat{v}$ then we apply the induction hypothesis (item 1) independently for the families of sets $\{X_u : u \in U_0\}$ and $\{X_u : u \in U_1\}$ and the given sets $X' \subseteq X_{u_0}$ and $X'' \subseteq X_{v_0}$. Assembling the results, we get nonempty OD sets $Y_u \subseteq X_u$ ($u \in 2^{n+1}$) such that $Y_u \mathbb{R}_{uv} Y_v$ for all neighbouring pairs u, v, perhaps with the exception of the pair of $u = u_0 = \hat{u}$ and $v = v_0 = \hat{v}$, and in addition $Y_{u_0} = X'$ and $Y_{v_0} = X''$. Thus finally $Y_{\hat{u}} \mathbb{R}_{\hat{u}\hat{v}} Y_{\hat{v}}$ by the choice of X' and Y'.

It remains to consider the case when both u_0 and v_0 belong to one and the same domain, say to U_0 . Then we first apply the induction hypothesis (item 2) to the family $\{X_u : u \in U_0\}$ and the sets $X' \subseteq X_{u_0}$ and $X'' \subseteq X_{v_0}$. This results in a system of nonempty OD sets $Y_u \subseteq X_u$ $(u \in U_0)$; in particular we get an OD nonempty set $Y_{\hat{u}} \subseteq X_{\hat{u}}$. It remains to put $Y_{\hat{v}} = \{y \in X_{\hat{v}} : \exists x \in Y_{\hat{u}} (x \mathsf{R}_{\hat{u}\hat{v}} y)\}$, so that $Y_{\hat{u}} \mathsf{R}_{\hat{u}\hat{v}} Y_{\hat{v}}$, and to apply the induction hypothesis (item 1) to the family $\{X_u : u \in U_1\}$ and the set $Y_{\hat{v}} \subseteq X_{\hat{v}}$.

4–C The construction

We put $X_{\Lambda} = X_0$.

Now assume that the sets X_s $(s \in 2^n)$ and relations R_{st} for all neighbouring pairs of $s, t \in 2^{\leq n}$ have been defined, and expand the construction at level n+1.

We first put $A_{s^{\wedge}i} = X_s$ for all $s \in 2^n$ and $i \in \{0, 1\}$. We also define $Q_{uv} = \mathsf{R}_{st}$ for any neighbouring pair of $u = s^{\wedge}i$, $v = t^{\wedge}i$ in 2^{n+1} other than the pair $\hat{u} = 0^{n^{\wedge}}0$, $\hat{v} = 0^{n^{\wedge}}1$. For the latter one (notice that $A_{\hat{u}} = A_{\hat{v}} = X_{0^n}$) we put $Q_{\hat{u}\hat{v}} = \overline{\mathsf{E}}$, so that $A_u Q_{uv} A_v$ holds for all neighbouring pairs of $u, v \in 2^{n+1}$ including the pair $\langle \hat{u}, \hat{v} \rangle$.

The sets A_u and Q_{uv} will be reduced in several steps to meet requirements (a), (b), (c) and (g1), (g2), (g3) of Subsection 4–A.

Part 1. After 2^{n+1} steps of the procedure of Lemma 26 (item 1) we obtain a system of nonempty OD sets $B_u \subseteq A_u$ ($u \in 2^{n+1}$) such that still $B_u \mathbb{Q}_{uv} B_v$ for all neighbouring pairs u, v in 2^{n+1} , but $B_u \in \mathcal{X}_{n+1}$ for all u. Thus (g1) is fixed.

Part 2. To fix (g2), consider an arbitrary pair of $u_0 = s_0^{\wedge}0$, $v_0 = t_0^{\wedge}1$, where $s_0, t_0 \in 2^n$. By Remark 24 and density of the set \mathcal{P}_{n+1} there exist nonempty OD

sets $B' \subseteq B_{u_0}$ and $B'' \subseteq B_{v_0}$ such that $P = (B' \times B'') \cap \overline{\mathsf{E}} \in \mathcal{P}_{n+1}$ and $\operatorname{pr}_1 P = B'$, $\operatorname{pr}_2 P = B''$, so in particular $B' \overline{\mathsf{E}} B''$. Now we apply Lemma 26 (item 1) separately for the two systems of sets, $\{B_{s\wedge 0}: s \in 2^n\}$ and $\{B_{t\wedge 1}: t \in 2^n\}$ (compare with the proof of Lemma 26 !), and the sets $B' \subseteq B_{s_0\wedge 0}$, $B'' \subseteq B_{t_0\wedge 1}$ respectively. This results in a system of nonempty OD sets $B'_u \subseteq B_u$ $(u \in 2^{n+1})$ satisfying $B'_{u_0} = B'$ and $B'_{v_0} = B''$, so that we have $(B'_{u_0} \times B'_{v_0}) \cap \overline{\mathsf{E}} \in \mathcal{P}_{n+1}$, and still $B'_u \operatorname{Q}_{uv} B'_v$ for all neighbouring pairs $u, v \in 2^{n+1}$, perhaps with the exception of the pair of $\hat{u} = 0^{n\wedge 0}$, $\hat{v} = 0^{n\wedge 1}$, which is the only one that connects the two domains. To handle this exceptional pair, note that $B'_{\hat{u}} \overline{\mathsf{E}} B'_{u_0}$ and $B'_{\hat{v}} \overline{\mathsf{E}} B''_{v_0}$ (Remark 24 is applied to each of the two domains), so that $B'_{\hat{u}} \overline{\mathsf{E}} B'_{\hat{v}}$ since $B' \overline{\mathsf{E}} B''$. We observe that $Q_{\hat{u}\hat{v}}$ is so far equal to $\overline{\mathsf{E}}$.

After 2^{n+1} steps (the number of pairs u_0, v_0 to be considered here) we get a system of nonempty OD sets $C_u \subseteq B_u$ $(u \in 2^{n+1})$ such that $(C_u \times C_v) \cap \overline{\mathsf{E}} \in \mathcal{P}_{n+1}$ whenever $u(n) \neq v(n)$, and still $C_u \mathsf{Q}_{uv} C_v$ for all neighbouring pairs $u, v \in 2^{n+1}$. Thus (g2) is fixed.

Part 3. We fix (c) for the exceptional neighbouring pair of $\hat{u} = 0^{n\wedge}0$, $\hat{v} = 0^{n\wedge}1$. Since E is \mathcal{T}^2 -dense in $\overline{\mathsf{E}}$, and $C_{\hat{u}} \overline{\mathsf{E}} C_{\hat{v}}$, the set $\mathsf{R} = (C_{\hat{u}} \times C_{\hat{v}}) \cap \mathsf{E}$ is nonempty. Then some nonempty OD set $\mathsf{Q} \subseteq \mathsf{R}$ is \mathcal{T} -separable by Lemma 15. Consider the OD sets $C' = \mathsf{pr}_1 \mathsf{Q} \ (\subseteq C_{\hat{u}})$ and $C'' = \mathsf{pr}_2 \mathsf{Q} \ (\subseteq C_{\hat{v}})$; obviously $C' \mathsf{Q} C''$, so that $C' \mathsf{Q}_{\hat{u}\hat{v}} C''$. (We recall that at the moment $\mathsf{Q}_{\hat{u}\hat{v}} = \overline{\mathsf{E}}$.) Using Lemma 26 (item 2) again, we obtain a system of nonempty OD sets $Y_u \subseteq C_u \ (u \in 2^{n+1})$ such that still $Y_u \mathsf{Q}_{uv} Y_v$ for all neighbouring pairs u, v in 2^{n+1} , and $Y_{\hat{u}} = C'$, $Y_{\hat{v}} = C''$. We re-define $\mathsf{Q}_{\hat{u}\hat{v}}$ by $\mathsf{Q}_{\hat{u}\hat{v}} = \mathsf{Q}$, but this keeps $Y_{\hat{u}} \mathsf{Q}_{\hat{u}\hat{v}} Y_{\hat{v}}$.

Part 4. We fix (g3). Consider a neighbouring pair u_0, v_0 in 2^{n+1} . Then $u_0 = 0^{k\wedge}0^{\wedge}r$, $v_0 = 0^{k\wedge}1^{\wedge}r$ for some $k \leq n$ and $r \in 2^{n-k}$. We observe that the temporary relation $Q' = Q_{u_0v_0} \cap (Y_{u_0} \times Y_{v_0})$ is a nonempty (because $Y_{u_0} Q_{u_0v_0} Y_{v_0}$) OD subset of $\mathsf{R}_k = \mathsf{R}_{0^{k\wedge}0,0^{k\wedge}1}$ by the construction. Let $\mathsf{Q} \subseteq \mathsf{Q}'$ be a nonempty OD set in $\mathcal{X}_{n+1}(\mathsf{R}_k)$. Now put $Y' = \mathsf{pr}_1 \mathsf{Q}$ and $Y'' = \mathsf{pr}_2 \mathsf{Q}$ (then $Y' \mathsf{Q} Y''$ and $Y' \mathsf{Q}_{u_0v_0} Y''$) and run Lemma 26 (item 2) for the system of sets Y_u ($u \in 2^{n+1}$) and the sets $Y' \subseteq Y_{u_0}$, $Y'' \subseteq Y_{v_0}$. After this define the "new" $\mathsf{Q}_{u_0v_0}$ by $\mathsf{Q}_{u_0v_0} = \mathsf{Q}$.

Do this consequtively for all neighbouring pairs; the finally obtained sets – let us denote them by X_u $(u \in 2^{n+1})$ – are as required. The final relations R_{uv} $(u, v \in 2^{n+1})$ can be obtained as the restrictions of the relations Q_{uv} to $X_u \times X_v$.

This ends the construction.

This also ends the proof of theorems 4 and 2. \Box

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