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A LOGIC FOR APPROXIMATE REASONING

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§1. **Introduction.** Classical logic is not adequate to face the essential vagueness of human reasoning, which is approximate rather than precise in nature. The logical treatment of the concepts of vagueness and approximation is of increasing importance in artificial intelligence and related research. Consequently, many logicians have proposed different systems of many-valued logic as a formalization of approximate reasoning (see, for example, Goguen [G], Gerla and Tortora [GT], Novak [No], Pavelka [P], and Takeuti and Titani [TT]). As far as we know, all the proposals are obtained by extending the range of truth values of propositions. In these logical systems reasoning is still exact and to make a conclusion the antecedent clause of its rule must match its premise exactly. In addition, Wang [W] pointed out: "If we compare calculation with proving,... Procedures of calculation ... can be made so by fairly well-developed methods of approximation; whereas... we do not have a clear conception of approximate methods in theorem proving.... The concept of approximate proofs, though undeniably of another kind than approximations in numerical calculations, is not incapable of more exact formulation in terms of, say, sketches of and gradual improvements toward a correct proof" (see pp. 224–225). As far as the author is aware, however, no attempts have been made to give a conception of approximate methods in theorem proving.

The purpose of this paper is, unlike all the previous proposals, to develop a propositional calculus, a predicate calculus in which the truth values of propositions are still true or false exactly and in which the reasoning may be approximate and allow the antecedent clause of a rule to match its premise only approximately. In a forthcoming paper we shall establish set theory, based on the logic introduced here, in which there are $|L|$ binary predicates $\in_\lambda$, $\lambda \in L$ such that $R(\in, \in_\lambda) = \lambda$ where $\in$ stands for $\in_1$ and 1 is the greatest element in L, and $x \in_\lambda y$ is interpreted as that x belongs to y in the degree of λ, and relate it to intuitionistic fuzzy set theory of Takeuti and Titani [TT] and intuitionistic modal set theory of Lano [L]. In another forthcoming paper we shall introduce the resolution principle under approximate match and illustrate its applications in production systems of artificial intelligence.

It is very difficult to realize fast and real-time response in a classical artificial intelligence system which can reason. One of the main reasons for this difficulty is that the process of searching for applicable rules, which must match exactly the

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present database, consumes too much time (cf. Nilsson [N, Chapters 2 and 3]). If we comply with H. A. Simon's principle of bounded rationality and only desire that the applied rule match closely enough to present database, then lots of time may be saved. Therefore, the reasonings admitting approximate match possibly become effective tools to add the mechanism of fast and real-time response into classical artificial intelligence methodology, and the results in this paper have good prospects of applications in artificial intelligence.

§2. **Propositional calculus with approximate reasoning.** Let $T = \{F, \Rightarrow\}$, where F is a 0-ary operation and $\Rightarrow$ is a binary operation; let X be the set of propositional variables; and let $P(X)$ be the propositional algebra of the propositional calculus on X, i.e., the free T-algebra on X (cf. Barnes and Mack [BM, p. 12]). In addition, let L be a complete and infinitely distributive lattice, and let R be an L-valued similarity relation on X, i.e., a mapping from $X \times X$ into L such that $R(p,p) = 1$, $R(p,q) = R(q,p)$, and $R(p,q) \wedge R(q,r) \leq R(p,r)$ for any $p, q, r \in X$. Thus, R induces naturally an L-valued similarity relation $\overline{R}$ on $P(X)$ which is an extension of R, i.e., $\overline{R}|_{X\times X} = R$, and fulfills the following conditions: for any $p, q, r \in P(X)$,

$$\overline{R}(F,p) = \begin{cases} 1 & \text{if } p = F, \\ 0 & \text{if } p \neq F; \end{cases} \tag{1}$$

$$\overline{R}(p \Rightarrow q, r) = \begin{cases} \overline{R}(p,p') \wedge \overline{R}(q,q') & \text{if } r = p' \Rightarrow q', \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Furthermore, we may give an L-valued relation $\widetilde{R}$ on $2^{P(X)}$ as follows: for any $A, B \subseteq P(X)$,

$$\widetilde{R}(A,B) = \bigwedge_{q\in B} \bigvee_{p\in A} \overline{R}(p,q).$$

LEMMA 2.1. *Let $p = p(x_1, \dots, x_n) \in P(X)$, where the propositional variables $x_1, \dots, x_n$ all occur in p and the propositional variables occurring in p are all among $x_1, \dots, x_n$. Then*

$$\overline{R}(p,p') = \begin{cases} \bigwedge_{i=1}^{n} R(x_i, x_i') & \text{if } p' = p(x_1', \dots, x_n'), \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Induction on the length $l(p)$ of p. □

DEFINITION 2.2. Let $\mathscr{A} = \mathscr{A}_1 \cup \mathscr{A}_2 \cup \mathscr{A}_3$, where

$$\mathscr{A}_1 = \{p \Rightarrow (q \Rightarrow p) : p, q \in P(X)\},$$

$$\mathscr{A}_2 = \{(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r)) : p, q, r \in P(X)\},$$

$$\mathscr{A}_3 = \{\sim\sim P \Rightarrow p : p \in P(X)\},$$

and $\sim p = p \Rightarrow F$ for any $p \in P(X)$. Let $A \subseteq P(X)$ and $w \in P^+(X) = \bigcup_{n=1}^{\infty} P(X)^n$. Then the degree $d(A, w)$ to which w is a deduction from A is defined by induction on $l(w)$.

(1) If $l(w) = 1$, then

$$d(A, w) = \bigvee\{\overline{R}(w, p) \colon p \in \mathscr{A} \cup A\}.$$

(2) If $w = w_1 \cdots w_n$, then

$$\begin{aligned} d(A, w) = &\bigvee\{\overline{R}(w_n, p) \colon p \in \mathscr{A} \cup A\} \\ &\vee \bigvee\{d(A, w_1 \cdots w_i) \wedge d(A, w_1 \cdots w_j) \wedge \overline{R}(w_i, p) \wedge \overline{R}(w_j, q) \colon i, j < n, \\ &\qquad p = q \Rightarrow w_n, \text{ and } p, q \in P(X)\}. \end{aligned}$$

DEFINITION 2.3. Let $A \subseteq P(X)$ and $q \in P(X)$. Then the degree $\mathrm{Ded}(A, q)$ to which A syntactically implies q is defined by

$$\mathrm{Ded}(A, q) = \bigvee\{d(A, w) \colon w_{l(w)} = q,\ q \in P^+(X)\}.$$

DEFINITION 2.4. Let $A \subseteq P(X)$ and $q \in P(X)$. Then the degree $\mathrm{Con}(A, q)$ to which A semantically implies q is defined by

$$\mathrm{Con}(A, q) = \bigvee\{\widetilde{R}(\mathscr{A} \cup A, B) \colon B \subseteq P(X),\ B \vDash q\},$$

where $B \vDash q$ means that B semantically implies q in two-valued propositional calculus.

THEOREM 2.5 (The Soundness and Adequacy Theorem). *Let* $A \subseteq P(X)$ *and* $q \in P(X)$. *Then* $\mathrm{Con}(A, q) = \mathrm{Ded}(A, q)$.

PROOF. With soundness and adequacy of $P(X)$ in two-valued propositional calculus (see Barnes and Mack [BM, Theorems III.2.1 and 13]) it suffices to show that

$$\mathrm{Ded}(A, q) = \bigvee\{\widetilde{R}(\mathscr{A} \cup A, B) \colon B \vdash q\},$$

where $A \vdash p$ stands for "A syntactically implies p in two-valued propositional calculus".

(1) If $p, q \in P(X)$ and there exist $x_1, \ldots, x_m, y_1, \ldots, y_m \in X$ such that $p = p(x_1, \ldots, x_m)$ and $q = p(y_1, \ldots, y_m)$, then we say p is similar to q. Let $w = w_1 \cdots w_n \in P^+(X)$, $w_n = q$, and $A \subseteq P(X)$. Then we have

$$d(A, w) = \bigvee_z d'(A, w, z)$$

by simple induction on $n = l(w)$, where z is a function, $n \in \mathrm{domain}(z) \subseteq \{1, 2, \ldots, n\}$ and for each $m \in \mathrm{doman}(z)$ $z(m)$ says either w_m is similar to $w'_m \in \mathscr{A} \cup A$ or

$$(*) \qquad \begin{cases} w_i \text{ is similar to } r \Rightarrow w_m, \\ w_j \text{ is similar to } r. \end{cases}$$

and

$$d'(A, w, z) = \bigwedge\{\overline{R}(p, s) \colon z \text{ asserts } p \text{ is similar to } s\}.$$

Now given such A, w, and z, for each $m \in \mathrm{domain}(z)$ if $z(m)$ says w_m is similar to $w'_m \in \mathscr{A} \cup A$, then do nothing. On the other hand, in case $(*)$ as above insert between w_i and w_{i+1} in w

1′. $(w_i \Rightarrow ((w_i \Rightarrow w_i) \Rightarrow w_i)) \Rightarrow ((w_i \Rightarrow (w_i \Rightarrow w_i)) \Rightarrow (w_i \Rightarrow (r \Rightarrow w_m))$ (similar to an axiom),

2′. $w_i \Rightarrow ((w_i \Rightarrow w_i) \Rightarrow w_i)$ axiom,

3′. $(w_i \Rightarrow (w_i \Rightarrow w_i)) \Rightarrow (w_i \Rightarrow (r \Rightarrow w_m))$ (mp, 1′, 2′),

4′. $w_i \Rightarrow (w_i \Rightarrow w_i)$ axiom,

5′. $w_i \Rightarrow (r \Rightarrow w_m)$ (mp, 3′, 4′),

6′. $r \Rightarrow w_m$ (mp, w_i. 5′).

Do something similar after w_j to get r so that w_m now correctly follows from $r \Rightarrow w_m$ and r. Thus, we form an expansion w^* of w (and a corresponding z^*) so that

$$d'(A, w, z) = d'(A, w^*, z^*),$$

while for steps in w^* which z^* says are justified by Modus Ponens truly are justified by Modus Ponens (without resort to similarities). Let B be the set of formulae in w^* which are justified by being similar to elements of $\mathscr{A} \cup A$. Then z^* provides the required (standard) proof of q from B and

$$\begin{aligned} d'(A, w^*, z^*) &= \bigwedge \{\overline{R}(p, s) \colon z^* \text{ asserts } p \text{ is similar to } s \in \mathscr{A} \cup A\} \\ &\leq \widetilde{R}(\mathscr{A} \cup A, B). \end{aligned}$$

Therefore,

$$\mathrm{Ded}(A, q) \leq \bigvee \{\widetilde{R}(\mathscr{A} \cup A, B) \colon B \vdash q\}.$$

(2) We want to show that

$$\widetilde{R}(\mathscr{A} \cup A, B) \leq \mathrm{Ded}(A, q)$$

provided $B \vdash q$. In fact, if $B \vdash q$, then there exists a proof w of q from B. Since $\mathrm{Ded}(A, q) \geq d(A, w)$, it suffices to show that $d(A. w) \geq \widetilde{R}(\mathscr{A} \cup A, B)$. We proceed by induction on $l(w)$. If $l(w) = 1$, then $q = w \in \mathscr{A} \cup B$. If $q \in \mathscr{A}$, then

$$d(A, w) = \bigvee_{p \in \mathscr{A} \cup A} \overline{R}(q. p) \geq \overline{R}(q, q) = 1.$$

and if $q \in B$, then

$$d(A. w) = \bigvee_{p \in \mathscr{A} \cup A} \overline{R}(q, p) \geq \widetilde{R}(\mathscr{A} \cup A, B).$$

Assume $w = w_1 \cdots w_n$ and the conclusion always holds for u with $l(u) < n$. We want to show that the conclusion holds also for w. If $w_n \in \mathscr{A} \cup B$, it is as above. If there exist $i, j < n$ such that $w_i = w_j \Rightarrow w_n$, then from (2) in Definition 2.2 we obtain

$$d(A.w) \geq d(A.w_1 \cdots w_i) \wedge d(A.w_1 \cdots w_j).$$

By the induction hypothesis and noticing that $B \vdash w_i$ and $B \vdash w_j$, it is known that

$$d(A, w_1 \cdots w_i) \geq \widetilde{R}(\mathscr{A} \cup A, B), \qquad d(A, w_1 \cdots w_j) \geq \widetilde{R}(\mathscr{A} \cup A, B),$$

and

$$d(A.w) \geq \widetilde{R}(\mathscr{A} \cup A, B). \qquad \square$$

COROLLARY 2.6 (The Deduction Theorem). *Let $A \subseteq P(X)$, and let $p, q \in P(X)$. Then* $\mathrm{Ded}(A, p \Rightarrow q) = \mathrm{Ded}(A \cup \{p\}, q)$.

PROOF. From (1) in the proof of Theorem 2.5 it suffices to show that

$$\bigvee \left\{ \bigvee_{z} d'(A, w^*, z^*) \colon w_{l(w)} = p \Rightarrow q \right\}$$
$$= \bigvee \left\{ \bigvee_{z'} d'(A \cup \{p\}, w'^*, z'^*) \colon w'_{l(w')} = q \right\}.$$

However, the above equality is immediate from the (standard) Deduction Theorem (see Barnes and Mack [BM, Theorem III.2.4]). $\square$

§3. **Predicate calculus with approximate reasoning.** Let V be an infinite set of individuals, let $\mathscr{R}$ be a set of predicate symbols, and let $ar \colon \mathscr{R} \to N$ be the arity function on $\mathscr{R}$, and let $P = P(V, \mathscr{R})$ be the full predicate algebra on $(V, \mathscr{R})$, i.e., the free algebra on the set $\{r(x_1, \dots, x_{ar(r)}) \colon r \in \mathscr{R},\ x_i \in V\ (i = 1, \dots, ar(r))\}$ of type $\{F, \Rightarrow\} \cup \{(\forall x) \colon x \in V\}$, where F is an 0-ary operation, $\Rightarrow$ binary and each $(\forall x)$ unary. In addition, let R_n be an L-valued similarity relation on $\mathscr{R}_n = \{r \in \mathscr{R} \colon ar(r) = n\}$ for any $n \in N$. Then they induce an L-valued similarity relation $\overline{R}$ on P and an L-valued relation $\widetilde{R}$ on 2^P. $\widetilde{R}$ is given as in the second section, and $\overline{R}$ is defined as follows:

(1) For any $r_1, r_2 \in \mathscr{R}$ and $x_1^1, \dots, x_{ar(r_1)}^1, x_1^2, \dots, x_{ar(r_2)}^2 \in V$,

$$\overline{R}(r_1(x_1^1, \dots, x_{ar(r_1)}^1), r_2(x_1^2, \dots, x_{ar(r_2)}^2))$$
$$= \begin{cases} R_{ar(r_1)}(r_1, r_2) & \text{if } ar(r_1) = ar(r_2) \text{ and } x_i^1 = x_i^2\ (i = 1, \dots, ar(r_1)), \\ 0 & \text{otherwise.} \end{cases}$$

(2) For any $\varphi \in P$,

$$\overline{R}(F, \varphi) = \begin{cases} 1 & \text{if } \varphi = F, \\ 0 & \text{if } \varphi \neq F. \end{cases}$$

(3) For any $\varphi, \psi, \theta \in P$,

$$\overline{R}(\varphi \Rightarrow \psi, \theta) = \begin{cases} \overline{R}(\varphi, \varphi') \wedge \overline{R}(\psi, \psi') & \text{if } \theta = \varphi' \Rightarrow \psi', \\ 0 & \text{otherwise.} \end{cases}$$

(4) For any $\varphi, \psi \in P$,

$$\overline{R}((\forall x)\varphi, \psi) = \begin{cases} \overline{R}(\varphi, \varphi'(x/y)) & \text{if } \psi = (\forall x)\varphi', \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 3.1. *Let* $\varphi = \varphi(P_1, \dots, P_n) \in P$, *where* $P_1, \dots, P_n \in \mathscr{R}$. *Assume* $P_1, \dots, P_n$ *all occur in* φ *and all predicate symbols occurring in* φ *are among* $P_1, \dots, P_n$. *Then*

$$\overline{R}(\varphi, \varphi') = \begin{cases} \bigwedge_{i=1}^{n} R_{ar(P_i)}(P_i, P_i') & \text{if } \varphi' = \varphi(P_1', \dots, P_n'), \\ 0 & \text{otherwise,} \end{cases}$$

where $P_1', \dots, P_n' \in \mathscr{R}$ *and* $\varphi' \in P$ *is obtained by replacing* P_i *in* φ *with* P_i' *for any* $i \in \{1, \dots, n\}$.

DEFINITION 3.2. Let $\mathscr{A} = \bigcup_{i=1}^{5} \mathscr{A}_i$, where $\mathscr{A}_1, \mathscr{A}_2$, and $\mathscr{A}_3$ are as in Definition 2.2,

$$\mathscr{A}_4 = \{(\forall x)(p \Rightarrow q) \Rightarrow (p \Rightarrow (\forall x)q) \colon p, q \in P,\ x \notin \operatorname{var}(p)\}.$$
$$\mathscr{A}_5 = \{(\forall x)p(x) \Rightarrow p(y) \colon p(x) \in P,\ y \in V\},$$

$\operatorname{var}(p)$ is the set of free variables occurring in p. Let $A \subseteq p$ and $w \in P^+ = \bigcup_{n=1}^{\infty} P^n$. Then the degree $d(A, w)$ to which w is a deduction from A is defined by induction on $l(w)$.

(1) If $l(w) = 1$, then

$$d(A, w) = \bigvee \{\overline{R}(w, p) \colon p \in \mathscr{A} \cup A\}.$$

(2) If $w = w_1 \cdots w_n$, then

$$\begin{aligned} d(A, w) = {} & \bigvee \{\overline{R}(w_n, p) \colon p \in \mathscr{A} \cup A\} \\ & \vee \bigvee \{d(A, w_1 \cdots w_i) \wedge d(A, w_1 \cdots w_j) \wedge \overline{R}(w_i, p) \wedge \overline{R}(w_j, q) \colon \\ & \qquad i, j < n,\ p = q \Rightarrow w_n, \text{ and } p, q \in P\} \\ & \vee \bigvee \{d(B, w_{k_1} \cdots w_{k_l}) \wedge \overline{R}(w_{k_l}, q) \colon k_1 < \cdots < k_l < n.\ q \in P, \\ & \qquad w_n = (\forall x)q,\ B \subseteq A, \text{ and } x \notin \operatorname{var}(B)\}, \end{aligned}$$

where $\operatorname{var}(B) = \bigcup_{p \in B} \operatorname{var}(p)$.

REMARK. In usual (two-valued or many-valued) logical systems, we only discuss proofs (deductions) in which the antecedent clause of a rule must match the premise exactly (see, for example, Barnes and Mack [BM, Definitions II.4.1 and IV.3.2], where Modus Ponens and Generalisation are used in the sense of exact match). In Definitions 2.2 and 3.2, however, we give up the condition of exact match and deal with general $w \in P^+(X)$ or $w \in P^+(V.\mathscr{R})$ instead and the degree of match of premises and antecedent clauses of rules are considered in the evaluation of the degree to which w is a deduction.

THEOREM 3.3 (The Soundness and Adequacy Theorem). *Let* $A \subseteq P$ *and* $q \in P$. *Then* $\operatorname{Con}(A, q) = \operatorname{Ded}(A, q)$, *where* $\operatorname{Con}(A. q)$ *is defined as in Definition* 2.4 (*now,*

$B \models q$ stands for "B semantically implies q in two-valued predicate calculus") and $\mathrm{Ded}(A.q)$ *is as in Definition* 2.3.

PROOF. Given the soundness and adequacy of two-valued $\mathrm{Pred}(V, \mathscr{R})$ (see Barnes and Mack [BM, Theorems IV.4.7 and 16]) it suffices to show that

$$\mathrm{Ded}(A.q) = \bigvee\{\widetilde{R}(\mathscr{A} \cup A, B)\colon B \vdash q\},$$

where $B \vdash q$ means that B syntactically implies q in two-valued predicate calculus. The idea of (1) in the proof of Theorem 2.5 still works by making the following change: z is now more complicated since it has to specify subproof, i.e., we must consider the additional case:

"$w_m = (\forall x)\lambda(x)$ is justified because w_{k_l} is similar to $\lambda(x)$ and x can be quantified over".

In this case we insert into the proof between w_{k_l} and w_{k_l+1}

1′. $(\forall x)w_{k_l}(x)$ ($= p$, say; generalisation, w_{k_l}),

2′. $(p \Rightarrow ((p \Rightarrow p)) \Rightarrow p(p \Rightarrow (p \Rightarrow p)) \Rightarrow (p \Rightarrow (\forall x)\lambda(x)))$ (similar to an axiom),

...

7′. $p \Rightarrow (\forall x)\lambda(x)$.

Conversely, we only need to add the following paragraph at the end of (2) in the proof of Theorem 2.5.

If there exist $k_1, \dots, k_l < n$ and $C \subseteq B$ such that $k_1 < \cdots < k_l$, $x \notin \mathrm{var}(C)$, $w_{k_1}, \dots, w_{k_l}$ is a proof of w_{k_l} from C and $w_n = (\forall x)w_{k_l}$, then from (2) in Definition 3.2 and the induction hypothesis, we have

$$\widetilde{R}(\mathscr{A} \cup A, B) \leq \widetilde{R}(\mathscr{A} \cup A, C) \leq d(A, w_{k_1} \cdots w_{k_l}) \leq d(A, w). \qquad \square$$

REMARK. We also have the Deduction Theorem for the predicate calculus (cf. Corollary 2.6).

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