# Kolmogorov complexity and symmetric relational structures

W.L. Fouché & P.H. Potgieter

Department of Quantitative Management, University of South Africa PO Box 392, Unisarand 0003, Pretoria, South Africa e-mail: {fouchwl,potgiph}@unisa.ac.za

November 11, 2018

#### Abstract

We study partitions of Fraïssé limits of classes of finite relational structures where the partitions are encoded by infinite binary strings which are random in the sense of Kolmogorov-Chaitin.

### **1** Introduction

This paper follows on [5] where a study was made of the properties of combinatorial configurations which are encoded or generated by infinite binary strings which are random in the sense of Kolmogorov-Chaitin [14], [2] (to be referred to as KC-strings). We shall study countable homogeneous structures from this point of view. A relational structure X is homogeneous if any isomorphism  $f: A \to B$  between finite substructures of X can be extended to an automorphism of X. This is perhaps the strongest symmetry condition one can impose on a relational structure. Our aim is to depict various situations where this kind of symmetry will be seen to be preserved by an arbitrary KC-string. Our work is based on Fraïssé's well-known characterisation of countable homogeneous structures [7].

A well-known example of a countable homogeneous structure is the random graph R of Rado [17]. We now illustrate some of the results of this paper with respect to the graph R. For a finite graph  $\beta$ , write  $[R,\beta]$  for the set of copies of  $\beta$  in R. We call a subset Y of  $[R,\beta]$  a  $\beta$ -organisation when Y is exactly the set of all copies of  $\beta$  in some subgraph R' of R, where R' is isomorphic to R. Now, R has a simple recursive representation of the form  $(\omega, E)$  where E is a recursive subset of the set of 2-subsets of  $\omega$ . This implies that one can find a recursive enumeration  $(\beta_j|j < \omega)$ , without repetition, of the set  $[R,\beta]$ . Let  $\varepsilon = \prod_{j=0}^{\infty} \varepsilon_j$  be a KC-string. If we define a 2-colouring  $\chi_{\varepsilon} : [R,\beta] \to \{0,1\}$  by giving each  $\beta_j$  the colour  $\varepsilon_j$ , it will be shown that there always exists a monochromatic  $\beta$ -organisation  $Y_{\varepsilon}$ . Moreover, one can compute the  $\beta$ -organisation  $Y_{\varepsilon}$  from  $\varepsilon$  by means of a simple greedy algorithm. In this way, a KC-string has two aspects: (i) as a random partition of the copies of  $\beta$  in R, and (ii) as a generator of a  $\beta$ -organisation in R which is monochromatic under this partition. The symmetric structure R is reflected (or preserved) by each KC-string in two distinct ways.

Similar results will be established for many other homogeneous structures. The main result is formulated in Section 2 and proved in Section 3. In Section 4 we apply this theory to the Fraïssé limits of what we shall call *ranked diagrams*. It is also shown how a KC-string can be used to generate the Fraïssé limit in this case.

### 2 Preliminaries

The composition of two functions f and g, denoted by fg, is defined by fg(x) = f(g(x)). The set of non-negative integers is denoted by  $\omega$ . We view the elements of  $\omega$  as finite ordinals, so that  $n < \omega$  denotes the set  $\{0, 1, \ldots, n-1\}$ . The cardinality of a finite set A is denoted by |A|. We write  $\mathcal{N}$  for the product space  $\{0, 1\}^{\omega}$ . The set of words over the alphabet  $\{0, 1\}$  is denoted by  $\{0, 1\}^*$ . If  $\alpha = \alpha_0 \alpha_1 \alpha_2 \ldots$  is in  $\mathcal{N}$  and  $n < \omega$ , we write  $\overline{\alpha}(n)$  for the binary word  $\prod_{j < n} \alpha_j$ . We use the usual recursion-theoretic terminology  $\Sigma_r^0, \prod_r^0$  and  $\Delta_r^0$  for the description of the arithmetical subsets of  $\omega^k \times \mathcal{N}^\ell$  – see [10], for example. We write  $\lambda$  for the Lebesgue measure on  $\mathcal{N}$ . This is the unique probability measure that assigns the value  $\frac{1}{2}$  to each of the events  $A_i = \{\alpha \in \mathcal{N} | \alpha_i = 1\}$ and under which the events  $A_i$  are statistically independent.

A prefix algorithm is a partial recursive function f from  $\{0,1\}^*$  to  $\{0,1\}^*$  whose domain is prefix-free, i.e. if  $u, v \in \text{dom } f$  then neither is an initial segment of the other. It is well-known (and easy to prove) that there is an effective enumeration of prefix algorithms and, therefore, that there is some universal prefix algorithm U. For  $s \in \{0,1\}^*$  let H(s), the Kolmogorov-complexity of s, be the length of a shortest "program"  $p \in \{0,1\}^*$ , such that U(p) = s. (For the history and underlying intuition of these notions, the reader is referred to [20]. See also [15], [2], [9] or [8].) An infinite binary string  $\varepsilon$  is said to be Kolmogorov-Chaitin complex (KC-complex) if and only if

$$\exists m \forall n H\left(\overline{\varepsilon}(n)\right) \ge n - m \; .$$

The set of KC-complex strings does not depend on the choice of the universal prefix algorithm U and has  $\lambda$ -measure one. We denote this set by KC and refer to the elements as KC-strings. We shall make frequent use of the following result.

**Theorem 1.** [6] If X is a  $\Pi_2^0$ -subset of  $\mathcal{N}$  and  $\lambda(X) = 1$ , then X contains every KC-string  $\varepsilon$ .

The proof of this result is based on Martin-Löf's description [16] of the set KC.

In the sequel,  $\mathcal{L}$  will stand for the signature of a relational structure. Moreover,  $\mathcal{L}$  will always be finite and the arities of the relational symbols will all be  $\geq 1$ . This has the implication that the empty set carries a unique  $\mathcal{L}$ -structure. The definitions that follow were introduced by Fraïssé [7] in 1954. (For a general discussion of the results to be summarised, the reader is also referred to Hodges [11], Chapter 7).

The age of an  $\mathcal{L}$ -structure X, written Age(X), is the class of all finite  $\mathcal{L}$ -structures (defined on finite ordinals) which can be embedded as  $\mathcal{L}$ -structures into X. The structure X is homogeneous (some authors say ultrahomogeneous) if, given any isomorphism  $f : A \to B$  between finite substructures of X, there is an automorphism g of X whose restriction to A is f. The following result is due to Fraïssé. (See [11], Chapter 7, for a proof.)

**Proposition 1.** The countable  $\mathcal{L}$ -structure X is homogeneous if and only if, for  $A, B \in \operatorname{Age}(X)$ and embeddings  $f : A \to B$ ,  $h : A \to X$ , there is an embedding  $g : B \to X$  such that h = gf. It suffices to require this when |B| = |A| + 1.

A class **K** of finite  $\mathcal{L}$ -structures has the *amalgamation property* if, for structures  $A, B_1, B_2$  in **K** and embeddings  $f_i : A \to B_i$  (i = 1, 2) there is a structure C in **K** and there are embeddings  $g_i : B_i \to C$  (i = 1, 2), such that  $g_1 f_1 = g_2 f_2$ .

Suppose  ${\bf K}$  is a countable class of finite  ${\mathcal L}\text{-structures},$  the domains of which are finite ordinals such that

- 1. if A is a finite  $\mathcal{L}$ -structure defined on some finite ordinal, if  $B \in \mathbf{K}$  and if there is an embedding of A into B, then  $A \in \mathbf{K}$ ;
- 2. the class **K** has the amalgamation property.

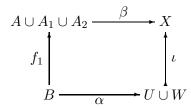
Then, Fraïssé showed that there is a countable homogeneous structure X such that Age(X) = K. Moreover, X is unique up to isomorphism. The unique X is called the *Fraïssé limit* of **K**. We also recall that, conversely, the age **K** of a countable homogeneous structure has properties (i) and (ii).

In our study of partitions of a homogeneous structure X we shall require its age to be *dense* in X in the following sense: If  $A, B \in \operatorname{Age}(X)$  and  $i : A \to B$  is an embedding, then there exist  $C \in \operatorname{Age}(X)$  and embeddings  $f_1, f_2 : B \to C$  such that  $f_1 = f_2 i$  and  $\operatorname{Im} f_1 \cap \operatorname{Im} f_2 = \operatorname{Im} f_1 i = \operatorname{Im} f_2 i$ . The Fraïssé limit of finite graphs (the random graph of Rado [17]) and the Fraïssé limit of ranked diagrams (see Section 4) are examples of homogeneous structures with dense ages. For any n, a disjoint union of countably many copies of the finite complete graph  $K_n$  is an example of a homogeneous structure whose age is not dense. (The complement of this structure does have a dense age.) The following combinatorial lemma plays a central role in the proof of Theorem 2.

**Lemma 1.** Suppose X is a countable homogeneous structure with a dense age. If U, V are disjoint subsets of X, then there is a sequence  $(V_i|i < \omega)$  of pairwise disjoint subsets of X such that  $U \cap V_i = \emptyset$  and  $U \cup V_i$  and  $U \cup V$  inherit isomorphic  $\mathcal{L}$ -structures from X, for all  $i < \omega$ .

Proof. Set  $V_0 = V$  and suppose pairwise disjoint  $V_0, \ldots, V_{k-1}$  have been constructed with  $U \cap V_i = \emptyset$ and  $U \cup V_i$  isomorphic to  $U \cup V$  for all i < k. Set  $W = \bigcup_{i < k} V_i$ . Choose  $A, B \in \operatorname{Age}(X)$  with  $A \subset B$ so that A is isomorphic to  $U \subset X$  via an isomorphism which extends to an isomorphism of B to  $U \cup W \subset X$ . Since  $\operatorname{Age}(X)$  is dense in X, there exist  $C \in \operatorname{Age}(X)$  and embeddings  $f_1, f_2 : B \to C$ such that  $A \subset C$  and  $f_1, f_2$  are both the identity on A, while  $\operatorname{Im} f_1$  and  $\operatorname{Im} f_2$  will have exactly the elements of A in common.

For  $i \in \{1,2\}$ , let  $A_i$  be the complement of A in  $\operatorname{Im} f_1$ . Then  $A_1 \cap A_2 = \emptyset$  but  $A \cup A_i$  is isomorphic to B and hence also to  $U \cup W \subset X$ . Moreover,  $A \cap A_i = \emptyset$ . Let  $\alpha$  be an isomorphism (e.g. the one from the construction of A and B above) from B onto  $U \cup W \subset X$  that maps A onto U. By Proposition 1 there is an embedding  $\beta$  such that the following diagram commutes.



We can write  $\operatorname{Im} \beta = U \cup W \cup W'$  with  $U \cup W'$  isomorphic to  $U \cup W$  and  $(U \cup W) \cap W' = \emptyset$ . Let  $V_k$  be any subset of W' such that  $U \cup V_k$  is isomorphic to  $U \cup V$ . (Such as exists by the isomorphism of  $U \cup W$  with  $U \cup W'$ .) The sequence  $(V_i | i < \omega)$  constructed in this way has the required properties.

A recursive representation of a countable  $\mathcal{L}$ -structure X is a bijection  $\phi : X \to \omega$  such that, for each  $R \in \mathcal{L}$ , if the arity of R is n, then the relation  $R^{\phi}$  defined on  $\omega^n$  by

$$R^{\phi}(x_1, x_2, \dots, x_n) \leftrightarrow R\left(\phi^{-1}(x_1), \dots, \phi^{-1}(x_n)\right)$$

is recursive. If we identify the underlying set of X with  $\omega$  via  $\phi$  and each R with  $R^{\phi}$  we call the resulting structure a *recursive*  $\mathcal{L}$ -structure.

If X is countable and homogeneous and if Age(X) has an enumeration  $A_0, A_1, A_2, \ldots$ , possibly with repetition, with the property that there is a recursive procedure that yields, for each  $i < \omega$ , and  $R \in \mathcal{L}$ , the underlying set n(i) of  $A_i$  together with the interpretation of R in n(i), then we call  $(A_i|i < \omega)$  a recursive enumeration of Age(X). It follows from the construction of Fraissé limits from their ages, as discussed in [11] (p329) that one can construct a recursive representation of Xfrom a recursive enumeration of its ages. (Conversely, it is trivial to derive a recursive enumeration of Age(X) from a recursive representation of X.) It is therefore not difficult to find recursive representations for Fraissé limits of classes **K** from recursive enumerations of their members.

Let X be a countable, homogeneous structure with a recursive representation  $\phi$ . For  $\beta \in Age(X)$ , let  $[X, \beta]$  be the set of copies (images under embeddings) of  $\beta$  in X. Suppose, in addition, that X has a dense age. We can use  $\phi$  to find a recursive enumeration  $\beta_0, \beta_1, \ldots$ , without repetition, of the set  $[X, \beta]$ . The density of Age(X) in X ensures that  $[X, \beta]$  is infinite (see Lemma 1) and the representation  $\phi$  can be used to decide whether a given finite subset of X inherits a structure isomorphic to  $\beta$ .

If  $\alpha$  is an infinite binary string then  $\alpha$  induces a 2-colouring  $\chi_{\alpha}$  of  $[X, \beta]$  where  $\chi_{\alpha}$  assigns to the *i*-th copy  $\beta_i$  of  $\beta$  in X the colour  $\alpha_i$ , the *i*-th bit of  $\alpha$ . The main theorem of the paper can now be formulated.

**Theorem 2.** Let X be a recursive homogeneous structure with a dense age. For each  $\beta \in \text{Age}(X)$ and each KC-string  $\varepsilon$ , there exists an embedding  $\nu : X \to X$  such that  $\chi_{\varepsilon}(\beta') = 1$  for each  $\beta' \in [\nu(X), \beta]$ . In addition,  $\nu$  can be so constructed that it is recursive relative to  $\varepsilon$ .

One can think of the mappings  $\chi_{\alpha} : [X,\beta] \to 2$  as random partitions. It follows from Theorem 2 that when  $[X,\beta]$  is subjected to a random partitioning then, with probability 1, one can find copies  $X_0, X_1$  of X in X such that  $\chi_{\alpha}$  is of colour *i* on  $[X_i,\beta]$  (i = 0, 1). This is because  $\alpha$  is in KC with probability 1. Moreover, when  $\alpha$  is a KC-string, we can effectively generate, relative to  $\alpha$ , the automorphic copies  $X_0$  and  $X_1$  of X. The proof of the theorem appears in Section 3.

Recall that Ramsey's Theorem [18] says that for  $X = K_{\omega}$ , the complete graph on the natural numbers, for  $\beta = K_n$ , the complete graph on n points, and  $\varepsilon$  an arbitrary binary sequence, there exists an embedding  $\nu : X \to X$  such that  $[\nu(X), \beta]$  is monochromatic under the 2-colouring of  $[X, \beta]$  induced by  $\varepsilon$ . E. Specker [19] has observed that there exists a recursive sequence  $\varepsilon$  such that, for the colouring of  $[X, K_2]$  induced by  $\varepsilon$ , there exists no recursive copy X' of X such that  $[X', K_2]$ is monochromatic. This has been further refined by C.G. Jockusch [12] who showed that there exists a recursive sequence  $\varepsilon$  such that, for the colouring of  $[X, K_n]$  induced by  $\varepsilon$ , there is no  $\Sigma_n^0$ copy X' of X for which  $[X', \beta]$  is monochromatic. However, for any recursive  $\varepsilon$ , there always exists a  $\Pi_n^0$  copy X' of X for which  $[X', \beta]$  is monochromatic. It follows, however, from Theorem 2 that when  $\varepsilon$  is a KC-string, one can find a monochromatic X' which is *recursive* in  $\varepsilon$ . This emphasises that Jockusch's results exploit the non-random nature of recursive partitions.

### 3 Complex partitions of Fraissé limits

In the following we will denote the class of all finite subsets of a set Y by Fin Y. If  $w \in \text{Fin } \omega$  we denote the largest element of w by max w. If w is empty, then max w = -1. If  $n \in \omega$ , then by wn we mean  $w \cup \{n\}$ . We write v < w if there is a  $t \neq \emptyset$  with  $w = v \cup t$  and max  $v < \min t$ .

**Definition 1.** Let Y be a countably infinite set. An encoding of Y is a function  $\pi$ : Fin  $\omega \to$  Fin Y such that

(i)  $\pi(\emptyset) = \emptyset$  and for some  $w_0 \in \operatorname{Fin} \omega$ ,

$$\pi(w_0) \neq \emptyset \tag{1}$$

(ii) whenever  $n > m > \max w$ 

$$\pi(wn) \cap \pi(wm) = \pi(w); \tag{2}$$

(iii) for each w with  $\pi(w) \neq \emptyset$ ,

$$\sum 2^{-|\pi(wk)\setminus\pi(w)|} = \infty \tag{3}$$

where the summation is over all  $k > \max w$  such that  $\pi(wk) \neq \pi(w)$ .

**Definition 2.** An encoding  $\pi$  is called effective relative to a bijection  $\sigma: \omega \to Y$  when there exist a recursive binary relation  $R_{\sigma}$  and a recursive function f, such that, for  $i \in \omega$  and  $w \in \operatorname{Fin} \omega$ ,

(i) 
$$R_{\sigma}(i, w) \leftrightarrow \sigma(i) \in \pi(w)$$
,

and also

(*ii*)  $f(w) = |\pi(w)|$ .

These definitions have been adapted from [5]. The next theorem is a generalization of Theorem A of [5].

**Theorem 3.** If the encoding  $\pi$ : Fin $\omega \to$  FinY is effective relative to  $\sigma$  and if  $\varepsilon \in KC$ , then there exists a strictly increasing sequence

$$w_1 < w_2 < w_3 < \dots$$

in Fin  $\omega$  such that

$$\varepsilon(j) = 1$$
 whenever  $\sigma(j) \in \bigcup_{n \ge 1} \pi(w_n).$ 

There exists an oracle computation of this sequence from  $\varepsilon$ .

*Proof.* Let  $\pi$  be an encoding which is effective relative to  $\sigma$ , as defined above. Apply (1) to fix  $w_0 = v_0 k \in \operatorname{Fin} \omega$ , where  $k = \max w_0$ , such that  $\pi(v_0) = \emptyset$  but  $\pi(v_0 k) \neq \emptyset$ .

Let  $\varepsilon$  be in *KC*. We construct a *strictly increasing* sequence in Fin $\omega$  by induction so that for each *n* 

$$w_0 < w_1 < \ldots < w_n$$
 and  $\varepsilon(j) = 1$  for all  $\sigma(j) \in \bigcup_{k=1}^n \pi(w_k)$ .

The construction will be recursive in  $\varepsilon$ . This will suffice to prove the theorem.

Suppose  $n \ge 0$  and  $w_0, \ldots, w_n$  have been constructed. For every  $k > \max w_n$ , we define  $B_k \subseteq \mathcal{N}$  by:

$$\alpha \in B_k \leftrightarrow (\forall j) [\sigma(j) \in \pi(w_n k) \setminus \pi(w_n) \to \alpha_j = 1].$$

By Definition 2,  $R_{\sigma}(i, w)$  and the function  $w \mapsto |\pi(w)|$  are both recursive, so there exists a total recursive function  $\psi : \omega \to \omega$  such that  $j \leq \psi(k)$  whenever  $\sigma(j) \in \pi(w_n k)$ . The function  $\psi$  could, for example, compute the largest j so that  $\sigma(j) \in \pi(w_n k)$  when  $w_n$  and k have been given. Now,

$$\alpha \in B_k \leftrightarrow (\forall j \le \psi(k)) [R_{\sigma}(j, w_n k) \land \neg R_{\sigma}(j, w_n) \to \alpha_j = 1].$$

It now follows that the relation  $\alpha \in B_k$  is recursive in k and  $\alpha$ .

We shall define a sequence  $X_0, X_1, X_2, \ldots$  of statistically independent random variables on the probability space  $(\mathcal{N}, \Sigma, \lambda)$  where  $\Sigma$  is the collection of Borel subsets of  $\mathcal{N}$  and  $\lambda$  the Lebesgue measure, as before. Let  $X_i(\alpha) = \alpha_i$  for  $\alpha \in \mathcal{N}$  and  $i \in \omega$ . If  $k > \ell > \max w_n$  and both  $\pi(w_n k) \neq \pi(w_n)$  and  $\pi(w_n \ell) \neq \pi(w_n)$ , then the events  $B_k$  and  $B_\ell$  are statistically independent. To see this, note that  $B_k$  belongs to the  $\sigma$ -algebra generated by

$$\{X_j | \sigma(j) \in \pi(w_n k) \setminus \pi(w_n)\},\$$

and  $B_{\ell}$  belongs to the  $\sigma$ -algebra generated by

$$\{X_j | \sigma(j) \in \pi(w_n \ell) \setminus \pi(w_n)\}$$

Independence follows from the fact that  $\pi(w_n k) \cap \pi(w_n \ell) = \pi(w_n)$ .

Since the probability

$$P\left(\alpha \in B_k\right) = 2^{-|\pi(w_n k) \setminus \pi(w_n)|}$$

and we know, by (3), that the sum of the probabilities of these *independent* events diverges, it follows from the second Borel-Cantelli lemma that the event  $B_k$ , with  $\pi(w_n k) \neq \pi(w_n)$ , occurs infinitely often with probability 1. In particular, if we define B by

$$\alpha \in B \leftrightarrow \exists k (k > \max w_n \land \pi(w_n k) \neq \pi(w_n) \land \alpha \in B_k)$$

then  $\lambda(B) = 1$ . But B is a  $\Sigma_1^0$ -set and  $\Sigma_1^0 \subset \Pi_2^0$ , so it follows directly from Theorem 1 that  $\varepsilon \in B$ . Choose the smallest  $k > \max w_n$  for which  $\pi(w_n k) \neq \pi(w_n)$  and such that  $\varepsilon \in B_k$ . Set  $w_{n+1} = w_n k$ . Then  $\varepsilon(j) = 1$  for all j with  $\sigma(j) \in \bigcup_{\ell \le n+1} \pi(w_\ell)$ . Every step – including this last one – is effective relative to  $\varepsilon$ .

We now proceed to prove the main theorem of the paper (Theorem 2).

*Proof.* Let X be a recursive homogeneous structure with a dense age. There is a universal procedure that yields, for finite subsets U, V of X with  $U \cap V = \emptyset$  and each  $k < \omega$  a set  $V_k$  such that the sequence  $(V_k|k < \omega)$  is as in the conclusion of Lemma 1. This is evident from the proof of Lemma 1 since the inductive constructions of the  $V_k$  can be done recursively for a given recursive structure X.

Since X is recursive we can identify its domain with  $\omega$ . Our aim is to construct a function  $\mu : \operatorname{Fin} \omega \to \operatorname{Fin} \omega$  such that, for  $w \in \operatorname{Fin} \omega$ , there is an embedding  $\nu(\omega)$  from the  $\mathcal{L}$ -structure on  $|w| \subset X$  to an  $\mathcal{L}$ -structure  $\mu(w) \subset X$  such that, for  $k > \max w$ , the embedding  $\nu(wk)$  will be an extension of  $\nu(w)$ .

The construction will be such that if  $n > m > \max w$  then

$$\mu(wn) \cap \mu(wm) = \mu(w)$$

and  $\mu(wm)$  will always contain a copy of  $\beta$  which is not in  $\mu(w)$ . Finally, we shall ensure that that the embeddings  $\nu(w)$  will depend recursively on w. The construction is as follows:

- (1) Set  $\mu(\emptyset) = \emptyset$  and  $\nu(\emptyset) = \emptyset$ .
- (2) Assume μ(w), ν(w) and k > max w are given. Construct V (which will be a finite set) such that V ∩ μ(w) = Ø and if we set Z = μ(w) ∪ V then Z contains a copy of |w| + 1, extending the copy of |w| in μ(w), and Z contains a copy of β not in μ(w). (Proposition 1 shows that we can extend |w| and Lemma 1 implies that there are infinitely many copies of β.) Next, construct a pairwise disjoint sequence V<sub>0</sub>, V<sub>1</sub>, V<sub>2</sub>, ... (again using Lemma 1) which are all also disjoint from μ(w), such that if we set Z<sub>j</sub> = μ(w) ∪ V<sub>j</sub> then Z<sub>j</sub> is isomorphic to Z. Finally, set μ(wk) = Z<sub>k</sub> and let ν(wk) be an embedding of |w|+1 into Z<sub>k</sub> which extends ν(w).

Set  $\pi(w) = [\mu(w), \beta]$ . We now show that  $\pi$  is an encoding of  $Y = [X, \beta]$  in the sense of Definition 1. By the construction we see immediately that  $\pi$  satisfies conditions (1) and (2) of Definition 1. In order to verify the condition (3), we note that if  $n > \max w$  then  $\pi(wn) \setminus \pi(w)$  is non-empty and its size is independent of n (again by Step 2 of the construction). The divergence of the series follows.

Let  $\beta_0, \beta_1, \ldots$  be an effective enumeration without repetition of Y. For  $i < \omega$ , set  $\sigma(i) = \beta_i$ . Note that, since we have an effective representation of X, the straight-forward (greedy!) algorithm for giving  $\mu$  and  $\pi$ , respectively, shows that both are recursive. Since  $\pi$  is recursive, so is the mapping  $w \mapsto |\pi(w)|$ . Also, whether  $[\sigma(i) \in \pi(w)]$  holds, can be determined by listing and comparing the elements of  $\sigma(i)$  and  $\mu(w)$ , where  $\mu$  is as above. Therefore,  $\pi$ , as defined, is effective relative to  $\sigma$ (in the sense of Definition 2).

Theorem 3 now gives an oracle computation of a strictly increasing sequence  $w_1 < w_2 < w_3 < \ldots$ from  $\varepsilon$  such that  $\varepsilon(j) = 1$  whenever  $\sigma(j) \in \bigcup \pi(w_n)$ . In other words, since  $\mu(w_n)$  is increasing in n, if  $\sigma(j) \subset \bigcup \mu(w_n)$  then  $\varepsilon(j) = 1$ .

The embeddings  $\nu(w_n) : |w_n| \to \mu(w_n)$  are mutually compatible and thus define an embedding  $\nu : X \to X$  such that  $\operatorname{Im} \nu \subset \bigcup_n \mu(w_n)$ . This embedding  $\nu$  is the required embedding, which is indeed recursive relative to  $\varepsilon$  since  $w \mapsto \nu(w)$  is recursive and the sequence  $w_1 < w_2 < w_3 < \ldots$  is recursive relative to  $\varepsilon$ .

## 4 Ranked diagrams

In [5] it was shown that partitioning the edges of the complete countable graph  $K_{\omega}$  into two classes  $E_0, E_1$  by means of a KC-string  $\varepsilon$  yields two graphs  $(\omega, E_0)$  and  $(\omega, E_1)$  both of which are isomorphic to the Fraïssé limit of finite graphs. In this section we want to do the same for so-called ranked diagrams. These structures can be viewed as the Hasse diagrams of posets.

### 4.1 An $\aleph_0$ -categorical first-order theory of ranked diagrams.

In the sequel,  $\ell \geq 2$  is fixed. Let  $\mathcal{L}$  be the signature having  $\ell$  unary relations,  $L_0 \dots L_{\ell-1}$  (denoting the *levels* of the ranked diagram), and one binary relation, S (succession). The theory,  $RD_{\ell}$ , of ranked diagrams on  $\ell$  levels ( $\ell$ -diagrams), has the following three axioms :

(i) For all  $x: L_0(x) \vee \ldots \vee L_{\ell-1}(x)$ 

(ii) For all x:

$$\bigwedge_{0 \le i < j < \ell} \neg [L_i(x) \land L_j(x)]$$

(iii) For all x and y:

$$S(x,y) \to \bigwedge_{i=0}^{\ell-2} [L_i(x) \to L_{i+1}(y)]$$

The preceding axioms imply that there exists, for each x, a unique  $L_i$  such that  $L_i(x)$  holds (or – in different notation –  $x \in L_i$ ) and also that S(x, y) can hold only if x and y are on adjacent levels, y being "above" x. A model of the theory  $RD_\ell$  is an  $\ell$ -diagram. (A special class of these diagrams, namely the *k*-layered posets, has been investigated in [1].)

We shall identify a class of countable  $\ell$ -diagrams, having the property that each one of them also contains a copy of *every* other countable  $\ell$ -diagram. This class is defined by an  $\aleph_0$ -categorical first-order theory consisting of the axioms of  $RD_\ell$  as well as a collection of extension axioms similar to the extension axioms used by Compton [4] in his proof of the fact that the class of partial orders has a (labelled) first order 0-1 law. In view of the result of Kleitman and Rothschild

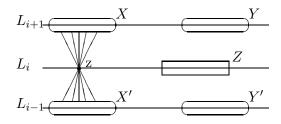


Figure 1: The extension axioms assert the existence of such a z for any X, Y, X', Y', Z.

[13], showing that a finite partial order will be ranked and of height 3 with labelled asymptotic probability 1, it makes sense to investigate random partial orders via  $\ell$ -diagrams.

We now single out those  $\ell$ -diagrams that are not only models of  $RD_{\ell}$  but also satisfy the following countable collection of axioms (indexed by the cardinalities of X, Y, X', Y', Z, for example):

(iv) (Extension Axioms) For each  $i < \ell$  and configuration of non-negative integers,  $(n_1, n_2, n_3, n_4, n_5)$ , an axiom stating that when X, Y are disjoint subsets of  $L_{i+1}, Z$  is a subset of  $L_i$  and X', Y'disjoint subsets of  $L_{i-1}$  such that  $(|X|, |Y|, |Z|, |X'|, |Y'|) = (n_1, n_2, n_3, n_4, n_5)$  then, for some  $z \in L_i$  such that  $z \notin Z$ , we have

$$\begin{array}{ccc} S(z,x) \ , & S(x',z) \ , & \neg S(z,y) & \text{ and } & \neg S(y',z) \\ \text{for all} & x \in X \ , & x' \in X' \ , & y \in Y & \text{ and } & y' \in Y' \end{array}$$

respectively. (See Figure 1.) We think of  $L_{i-1}$ , respectively  $L_{i+1}$ , as a name for the empty set when i = 0, respectively  $i = \ell - 1$ .

These extension axioms guarantee that we can extend a given arbitrary finite configuration on levels i-1, i, i+1 in the required way ( to a new  $\ell$ -diagram ) by just finding an appropriate z on level i. Axioms (i)-(iv) all together give a countable collection of first-order sentences in our language  $\mathcal{L}$ . These make up a theory  $T_{\ell}$ . We shall call its countable models the generic  $\ell$ -diagrams. Instances of the form  $X = X' = Y = Y' = \emptyset$  of (iv) guarantee that in any model of  $T_{\ell}$ , the unary relations  $L_i$ are modelled by *infinite* sets, so that any countably infinite model necessarily has infinitely many elements on each level.

#### 4.2 Explicit construction of an generic $\ell$ -diagram.

We now give an example of how to construct a recursive object that represents a generic  $\ell$ -diagram. A similar construction can be given for Rado's random graph [17]. Let  $A = \ell \times \omega$  be our underlying set and fix a collection

$$p(i,n)$$
  $i \in \ell, n < \omega$ 

of distinct odd primes. Now define the binary relation  $P_{\ell}$  on  $\ell \times \omega$  by

$$(i,n)P_{\ell}(i+1,m) \quad \leftrightarrow \quad \left( \begin{array}{cc} m \neq 0 \text{ and } p(i,n) \mid m \\ \text{OR} \\ n \neq 0 \text{ and } p(i+1,m) \mid n \end{array} \right).$$

$$(4)$$

In order to verify that  $(A, P_{\ell})$  is generic, we need to check the extension property (iv). We first assume  $0 < i < \ell - 1$ . Take any finite subsets

$$X = \{(i+1, x_0), \dots, (i+1, x_p)\}$$

$$Y = \{(i+1, y_0), \dots, (i+1, y_q)\}$$
  

$$Z = \{(i, z_0), \dots, (i, z_r)\}$$
  

$$X' = \{(i-1, x'_0), \dots, (i-1, x'_s)\}$$
  

$$Y' = \{(i-1, y'_0), \dots, (i-1, y'_t)\}$$

of  $\ell \times \omega$  such that  $X \cap Y = \emptyset = X' \cap Y'$ . It is sufficient to show that there exists  $z \notin \{0, z_0, \dots, z_r\}$  so that

 $\begin{array}{ll} p(i+1,x_k) \mid z &, k \leq p \\ p(i-1,x_k') \mid z &, k \leq s \\ p(i+1,y_k) \not\mid z &, k \leq q \\ p(i-1,y_k') \not\mid z &, k \leq t \end{array}$ 

and also

$$\begin{array}{ll} p(i,z) & \not\mid y_k \quad \text{or} \quad y_k = 0 \\ p(i,z) & \not\mid y'_k \quad \text{or} \quad y'_k = 0 \\ \end{array} , k \le t .$$

This can be achieved by setting

$$z = \left(\prod_{k \le p} p(i+1, x_k)\right) \cdot \left(\prod_{k \le s} p(i-1, x'_k)\right) \cdot 2^w$$

where w has been chosen sufficiently *large* to make  $z \neq z_0, \ldots, z_r$  and for p(i, z) not to divide any of the non-zero second components of elements of  $Y \cup Y'$ . This determines a z with the required properties. The cases i = 0 and  $i = \ell - 1$  are similarly dealt with.

#### 4.3 Application of Theorem 2 to ranked diagrams.

Let X be a generic  $\ell$ -diagram. If A is a finite  $\ell$ -diagram and  $f : A \to X$  any embedding, and if B is a  $\ell$ -diagram with |B| = |A| + 1 and  $B \supset A$ , then it follows directly from the extension axioms (iv) that f can be extended to an embedding of B into X. Since each singleton  $\ell$ -diagram can be embedded into X, it thus follows upon induction that any finite  $\ell$ -diagram can be embedded into X. Finally, it follows from Proposition 1 that X is homogeneous. We conclude that X is the Fraïssé limit of finite  $\ell$ -diagrams. We note that Age(X) is dense in X so that Theorem 2 also applies to generic  $\ell$ -diagrams.

#### 4.4 Binary sequences that generate generic $\ell$ -diagrams.

Fix some canonical recursive bijection

$$\psi: (\ell - 1) \times \omega \times \omega \to \omega.$$

Given  $\alpha \in \mathcal{N}$  we generate a ranked diagram  $S_{\alpha}$  on the underlying set  $A = \ell \times \omega$  by putting

$$(i,n)S_{\alpha}(i+1,m)$$
 whenever  $\alpha_{\psi(i,n,m)} = 1.$  (5)

We would now like to know for which  $\alpha \in \mathcal{N}$ , the ranked diagram  $(A, S_{\alpha})$  generated by the binary sequence  $\alpha$  is  $\ell$ -generic, where  $A = \ell \times \omega$ , as before. Let

$$G = \{ \alpha \in \mathcal{N} | \langle A, S_{\alpha}, \{0\} \times \omega, \dots, \{\ell - 1\} \times \omega \rangle \text{ is a model for } T_{\ell} \}.$$

The construction of  $S_{\alpha}$ , as in equation (5), is already such that the axioms (i)-(iii) of  $T_{\ell}$  are automatically satisfied for all  $\alpha$ .

Let  $P(\alpha, X, Y, Z, X', Y', z)$  stand for the predicate over  $\mathcal{N} \times (\operatorname{Fin} A)^5 \times A$  which states that  $z \notin Z$  and

$$S_{\alpha}(z,x)$$
,  $S_{\alpha}(x',z)$ ,  $\neg S_{\alpha}(z,y)$  and  $\neg S_{\alpha}(y',z)$ 

holds, for all  $x \in X$ ,  $x' \in X'$ ,  $y \in Y$  and  $y' \in Y'$  respectively. If we identify Fin A with  $\omega$  via a recursive bijection, then it is clear that P is a recursive predicate. Set  $K_i = \{i\} \times \omega$  for  $i < \ell$  and  $K_{-1} = K_{\ell-1} = \emptyset$ . Let  $Q(\alpha)$  be the predicate

$$(\forall 0 \le i < l)(\forall X \in \operatorname{Fin} K_{i+1})(\forall Y \in \operatorname{Fin} K_{i+1})(\forall Z \in \operatorname{Fin} K_i)(\forall X' \in \operatorname{Fin} K_{i-1}) \\ (\forall Y' \in \operatorname{Fin} K_{i-1})(\exists z \in K_i)(X \cap Y = X' \cap Y' = \emptyset \rightarrow P(\alpha, X, Y, Z, X', Y', z))$$

which is to say that  $Q(\alpha)$  holds if and only if  $\alpha$  codes a generic  $\ell$ -diagram. It is clear that Q is a  $\Pi_2^0$ -predicate. We have thus shown that

### Lemma 2. G is a $\Pi_2^0$ -set.

Let us now consider the *probability* that a uniformly randomly generated  $\alpha$  will give an  $\ell$ -generic RD on A, where our probability measure is the Lebesgue measure  $\lambda$ , as before.

**Lemma 3.** With probability 1, a sequence  $\alpha \in \mathcal{N}$  defines a generic  $\ell$ -diagram.

*Proof.* We have to show that  $\lambda(G) = 1$ . Note that

$$G = \bigcap_{z \in K_i} \{ \alpha | P(\alpha, X, Y, Z, X', Y', z) \}$$

where the intersection runs over all i, X, Y, Z, X', Y' such that  $0 \le i < \ell; X, Y \in \operatorname{Fin} K_{i+1}; Z \in \operatorname{Fin} K_i; X', Y' \in \operatorname{Fin} K_{i-1}$  such that  $X \cap Y = X' \cap Y' = \emptyset$ .

Since this is a countable intersection, we can henceforth regard all parameters, save z, as fixed, and need only prove that

$$\lambda(\bigcup_{z \in K_i \setminus Z} \{\alpha | P(\alpha, X, Y, Z, X', Y', z)\}) = 1$$

when X, Y, X', Y' are as above.

Now, if z' and z'' are distinct elements of  $K_i \setminus Z$ , then  $P(\alpha, X, Y, Z, X', Y', z')$  holding for  $\alpha$  and  $P(\alpha, X, Y, Z, X', Y', z'')$  holding for  $\alpha$  are independent events. For, the evaluation of these two instances of the predicate reference disjoint (finite) sets of digits in the sequence  $\alpha$  ( $\psi$  above being one-to-one). In each case, the probability that P holds is  $2^{-n}$  where n = |X| + |Y| + |X'| + |Y'|. We may therefore apply the second Borel-Cantelli lemma to conclude that the union,

$$\bigcup_{z \in K_i \setminus Z} \{ \alpha | P(X, Y, Z, X', Y', z) \}$$

does indeed have measure 1, which proves the lemma.

Theorem 1 together with Lemmas 2 and 3 now immediately give the following theorem.

**Theorem 4.** If  $\alpha$  is a KC-string, then the ranked diagram  $(A, S_{\alpha})$  is  $\ell$ -generic.

### References

- G. Brightwell H.J. Prömel A. Steger, 'The Average Number of Linear Extensions of a Partial Order', J. Comb. Th. Series A 73 (1996) 193–206.
- [2] G.J. Chaitin, Algorithmic Information Theory (Cambridge University Press, 1987).
- [3] C.C. Chang H.J. Keisler, *Model Theory* (North Holland, Amsterdam, 1973).
- [4] K.J. Compton, 'Laws in Logic and Combinatorics', in I. Rival (ed), Algorithms and Order (Kluwer Acad. Publ., Dordrecht, 1989) 353–383.
- [5] W.L. Fouché, 'Descriptive Complexity and Reflective Properties of Combinatorial Configurations', Journal of the London Mathematical Society (2) 54 (1996) 199–208.
- [6] W.L. Fouché, 'Identifying randomness given by high descriptive complexity', Acta Applicandae Mathematicae 34 (1994) 313–328.
- [7] R. Fraïssé, 'Sur l'extension aux relations de quelques propriétés des ordres', Ann. Sci. École Norm. Sup. 71 (1954) 363–388.
- [8] P. Gács, 'Randomness and Probability Complexity of Description', Encyclopedia of Statistical Sciences (John Wiley & Sons, 1986) 551–555.
- [9] P. Gács, A review of G. Chaitin's Algorithmic Information Theory, Journal of Symbolic Logic 54 (1989) 624–627.
- [10] P.G. Hinman, Recursion-theoretic Hierarchies (Springer, New York, 1978).
- [11] W. Hodges, *Model Theory*, (Cambridge University Press, Cambridge, 1993).
- [12] C.G. Jockusch jr., 'Ramsey's Theorem and Recursion Theory', Journal of Symbolic Logic 37 (1972) 268–280.
- [13] D.J. Kleitman B.L. Rothschild, 'Asymptotic Enumeration of Partial orders on a Finite Set', Trans. Am. Math. Soc. 205 (1975) 205–220.
- [14] A.N. Kolmogorov, 'Three approaches to the quantitative definition of information', Probl. Inform. Transmission 1 (1965) 1–7.
- [15] A.N. Kolmogorov V.A. Uspenskii, 'Algorithms and randomness', Theory Probab. Appl. 32 (1987) 389–412.
- [16] P. Martin-Löf, The Definition of Random Sequences, Information and Control 9 (1966) 602– 619.
- [17] R. Rado, 'Universal graphs and universal functions', Acta Arith. 9 (1964) 393–407.
- [18] F.P. Ramsey, 'On a problem of formal logic', Proc. London Math. Soc. 30 (1930) 264–286.
- [19] E. Specker, 'Ramsey's Theorem does not hold in recursive set theory', Studies in logic and the foundations of mathematics (North-Holland, Amsterdam, 1971).
- [20] P. Vitányi M. Li, An Introduction to Kolmogorov Complexity and Its Applications (Springer-Verlag, 1993).