

# Machines, Logic and Quantum Physics

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19 November 1999

## Abstract

Though the truths of logic and pure mathematics are objective and independent of any contingent facts or laws of nature, our *knowledge* of these truths depends entirely on our knowledge of the laws of physics. Recent progress in the quantum theory of computation has provided practical instances of this, and forces us to abandon the classical view that computation, and hence mathematical proof, are purely logical notions independent of that of computation as a physical process. Henceforward, a proof must be regarded not as an abstract object or process but as a physical process, a species of computation, whose scope and reliability depend on our knowledge of the physics of the computer concerned.

## 1 Mathematics and the physical world

Genuine scientific knowledge cannot be certain, nor can it be justified a priori. Instead, it must be conjectured, and then tested by experiment, and this requires it to be expressed in a language appropriate for making precise, empirically testable predictions. That language is mathematics.

This in turn constitutes a statement about what the physical world must be like if science, thus conceived, is to be possible. As Galileo put it, “the universe is written in the language of mathematics” [5]. Galileo’s introduction of mathematically formulated, testable theories into physics marked the transition from the Aristotelian conception of physics, resting on supposedly necessary a priori principles, to its modern status as a theoretical, conjectural and empirical science. Instead of seeking an infallible universal mathematical design, Galilean science uses mathematics to express quantitative descriptions of an objective physical reality. Thus mathematics has become the language in which we express our knowledge of the physical world. This language is not only extraordinarily powerful and precise, but also effective in practice. Eugene Wigner referred to “the unreasonable effectiveness of mathematics in the physical sciences” [12]. But is this effectiveness really unreasonable or miraculous?

Consider how we learn about mathematics. Do we – that is, do our brains – have direct access to the world of abstract concepts and the relations between them (as Plato believed, and as Roger Penrose now advocates [8]), or do we learn mathematics by experience, that is by interacting with physical objects? We believe the latter. This is not to say that the subject-matter of mathematical theories is in any sense part of, or emerges out of, the physical world. We do not deny that numbers, sets, groups and algebras have an autonomous reality quite independent of what the laws of physics decree, and the properties of these mathematical structures are just as objective as Plato believed they were. But they are revealed to us only through the physical world. It is only physical objects, such as computers or human brains, that ever give us glimpses of the abstract world of mathematics. But how?

It is a familiar fact, and has been since the prehistoric beginnings of mathematics, that simple physical systems like fingers, tally sticks and the abacus can be used to represent some mathematical entities and operations. Historically the operations of elementary arithmetic were also the first to be delegated to more complex machines. As soon as it became clear that additions and multiplications can be performed by a sequence of basic procedures and that these procedures are implementable as physical operations, mechanical devices designed by Blaise Pascal, Gottfried Wilhelm Leibniz and others began to relieve humans from tedious tasks such as multiplying two large integers [6]. In the twentieth century, following this conquest of arithmetic, the logical concept of computability was the next to be delegated to

machines. Turing machines were invented in order to formalise the notion of “effectiveness” inherent in the intuitive idea of calculability. Alan Turing conjectured that the theoretical machines in terms of which he defined computation are capable of performing any *finite, effective* procedure (algorithm). It is worth noting that Turing machines were intended to reproduce every definite operation that a *human* computer could perform, following preassigned instructions. Turing’s method was to think in terms of physical operations, and imagine that every operation performed by the computer “consists of some change of the physical system consisting of the computer and his tape” [11]. The key point is that since the outcome is not affected by constructing “a machine to do the work of this computer”, the effectiveness of a human computer can be mimicked by a logical machine.

The Turing machine was an abstract construct, but thanks to subsequent developments in the theory of computation, algorithms can now be performed by real automatic computing machines. The natural question now arises: what, precisely, is the set of logical procedures that can be performed by a physical device? The theory of Turing machines cannot, even in principle, answer this question, nor can any approach based on formalising traditional notions of effective procedures. What we need instead is to extend Turing’s idea of *mechanising* procedures, in particular, the procedures involved in the notion of derivability. This would define mathematical proofs as being mechanically reproducible and to that extent effectively verifiable. The universality and reliability of logical procedures would be guaranteed by the mechanical procedures that effectively perform logical operations – but by no more than that. But what does it mean to involve real, physical machines in the definition of a logical notion? and what might this imply *in return* about the ‘reasonableness’ or otherwise of the effectiveness of physics in the mathematical sciences?

While the abstract model of a machine, as used in the classical theory of computation, is a pure-mathematical construct to which we can attribute any consistent properties we may find convenient or pleasing, a consideration of actual computing machines as physical objects must take account of their actual physical properties, and therefore, in particular, of the laws of physics. Turing’s machines (with arbitrarily long tapes) can be built, but no one would ever do so except for fun, as they would be extremely slow and cumbersome. We find the computers now available much faster and more reliable. Where does this reliability come from? How do we know that the computer generates the same outputs as the appropriate abstract Tur-

ing machine, that the machinery of cog-wheels must finally display the right answer? After all, nobody has tested the machine by following all possible logical steps, or by performing all the arithmetic it can perform. If they were able and willing to do that, there would be no need to build the computer in the first place. The reason we trust the machine cannot be based entirely on logic; it must also involve our knowledge of the physics of the machine. We take for granted the laws of physics that govern the computation, i.e. the physical process that takes the machine from an initial state (input) to a final state (output). Moreover, our understanding is informed by physical theories which, though formulated in mathematical terms in the tradition of Galileo, evolved by conjectures and empirical refutations. In this perspective, what Turing really asserted was that it is possible to build a universal computer, a machine that can be programmed to perform any computation that any other physical object can perform. That is to say, a single buildable physical object, given only proper maintenance and a supply of energy and additional memory when needed, can mimic all the behavior and responses of any other physically possible object or process. In this form Turing's conjecture (which Deutsch has called in this context the *Church-Turing principle* [3]) can be viewed as a statement about the physical world.

Are there any limits to computations performed by computing machines? Obviously there are both logical and physical limits. Logic tells us that, for example, no machine can find more than one even prime, whilst physics tells us that, for example, no computations can violate the laws of thermodynamics. Moreover, logical and physical limitations can be intimately linked, as illustrated by the "halting problem". According to logic, the halting problem says that there is no algorithm for deciding whether any given machine, when started from any given initial situation, eventually stops. Therefore some computational problems, such as determining whether a specified universal Turing machine, given a specified input, will halt, cannot be solved by any Turing machine. In physical terms, this statement says that machines with certain properties cannot be physically built, and as such can be viewed as a statement about physical reality or equivalently about the laws of physics.

So where does mathematical effectiveness come from? It is not simply a miracle, "a wonderful gift which we neither understand nor deserve" [12] – at least, no more so than our ability to discover *empirical* knowledge, for our knowledge of mathematics and logic is inextricably entangled with our knowledge of physical reality: every mathematical proof depends for its ac-

ceptance upon our agreement about the rules that govern the behavior of physical objects such as computers or our brains. Hence when we improve our knowledge about physical reality, we may also gain new means of improving our knowledge of logic, mathematics and formal notions. It seems that we have no choice but to recognize the dependence of our mathematical *knowledge* (though not, we stress, of mathematical truth itself) on physics, and that being so, it is time to abandon the classical view of computation as a purely logical notion independent of that of computation as a physical process. In the following we discuss how the discovery of quantum mechanics in particular has changed our understanding of the nature of computation.

## 2 Quantum interference

To explain what makes quantum computers so different from their classical counterparts, we begin with the phenomenon of quantum interference. Consider the following computing machine whose input can be prepared in one of two states representing, 0 and 1.

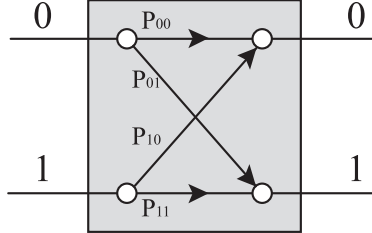


Figure 1: Schematic representation of the most general machine that performs a computation mapping  $\{0, 1\}$  to itself. Here  $p_{ij}$  is the probability for the machine to produce the output  $j$  when presented with the input  $i$ . (The action of the machine depends on no other input or stored information.)

The machine has the property that if we prepare its input with the value  $a$  ( $a = 0$  or  $1$ ) and then measure the output, we obtain, with probability  $p_{ab}$ , the value  $b$  ( $b = 0$  or  $1$ ). It may seem obvious that if the  $p_{ab}$  are arbitrary apart from satisfying the standard probability conditions  $\sum_b p_{ab} = 1$ , Fig. (1) represents *the most general* machine whose action depends on no other input or stored information and which performs a computation mapping  $\{0, 1\}$  to itself. The two possible deterministic limits are obtained by setting  $p_{01} =$

$p_{10} = 0$ ,  $p_{00} = p_{11} = 1$  (which gives a logical identity machine) or  $p_{01} = p_{10} = 1$ ,  $p_{00} = p_{11} = 0$  (which gives a negation (**not**) machine). Otherwise we have a randomising device. Let us assume, for the sake of illustration, that  $p_{01} = p_{10} = p_{00} = p_{11} = 0.5$ . Again, we may be tempted to think of such a machine as a random switch which, with equal probability, transforms any input into one of the two possible outputs. However, that is not necessarily the case. When the particular machine we are thinking of is followed by another, identical, machine the output is always the negation of the input.

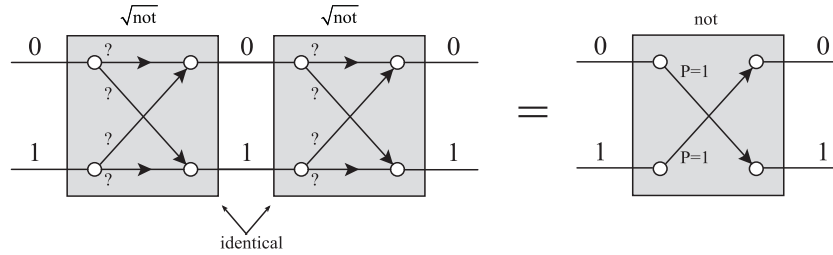


Figure 2: Concatenation of the two identical machines mapping  $\{0, 1\}$  to itself. Each machine, when tested separately, behaves as a random switch, however, when the two machines are concatenated the randomness disappears - the net effect is the logical operation **not**. This is in clear contradiction with the axiom of additivity in probability theory!

This is a very counter-intuitive claim - the machine alone outputs 0 or 1 with equal probability and independently of the input, but the two machines, one after another, acting independently, implement the logical operation **not**. That is why we call this machine  $\sqrt{\text{not}}$ . It may seem reasonable to argue that since there is no such operation in logic, the  $\sqrt{\text{not}}$  machine cannot exist. But it does exist! Physicists studying single-particle interference routinely construct them, and some of them are as simple as a half-silvered mirror i.e. a mirror which with probability 50% reflects a photon which impinges upon it and with probability 50% allows it to pass through. Thus the two concatenated machines are realised as a sequence of two half-silvered mirrors with a photon in each denoting 0 if it is on one of the two possible paths and 1 if it is on the other.

The reader may be wondering what has happened to the axiom of additivity in probability theory, which says that if  $E_1$  and  $E_2$  are mutually exclusive events then the probability of the event ( $E_1$  **or**  $E_2$ ) is the sum of the proba-

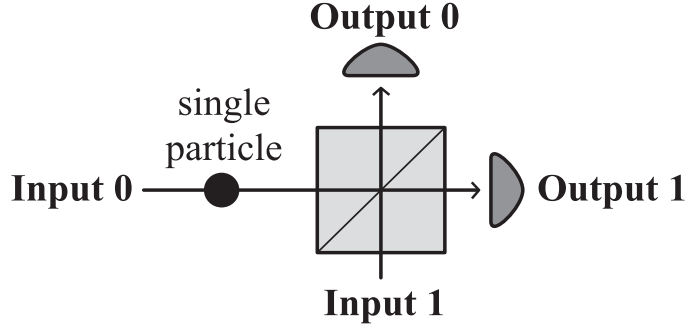


Figure 3: The experimental realisation of the  $\sqrt{\text{not}}$  gate. A half-silvered mirror reflects half the light that impinges upon it. But a single photon doesn't split: when we send a photon at such a mirror it is detected, with equal probability, either at Output 0 or 1. This does not, however, mean that the photon leaves the mirror in either of the two outputs at random. In fact the photon takes both paths at once! This can be demonstrated by concatenating two half-silvered mirrors as shown in the next figure.

bilities of the constituent events,  $E_1$ ,  $E_2$ . We may argue that the transition  $0 \rightarrow 0$  in the composite machine can happen in the two mutually exclusive ways, namely,  $0 \rightarrow 0 \rightarrow 0$  or  $0 \rightarrow 1 \rightarrow 0$ . The probabilities of the two are  $p_{00}p_{00}$  and  $p_{01}p_{10}$  respectively. Thus the sum  $p_{00}p_{00} + p_{01}p_{10}$  represents the probability of the  $0 \rightarrow 0$  transition in the new machine. Provided that  $p_{00}$  or  $p_{01}p_{10}$  are different from zero, this probability should also be different from zero. Yet we can build machines in which  $p_{00}$  and  $p_{01}p_{10}$  are different from zero, but the probability of the  $0 \rightarrow 0$  transition in the composite machine is equal to zero. So what is wrong with the above argument?

One thing that is wrong is the assumption that the processes  $0 \rightarrow 0 \rightarrow 0$  and  $0 \rightarrow 1 \rightarrow 0$  are mutually exclusive. In reality, the two transitions both occur, simultaneously. We cannot learn about this fact from probability theory or any other a priori mathematical construct. We learn it from the best physical theory available at present, namely quantum mechanics. Quantum theory explains the behavior of  $\sqrt{\text{not}}$  and correctly predicts the probabilities of all the possible outputs no matter how we concatenate the machines. This knowledge was created as the result of conjectures, experimentation, and refutations. Hence, reassured by the physical experiments that corroborate this theory, logicians are now entitled to propose a new logical operation

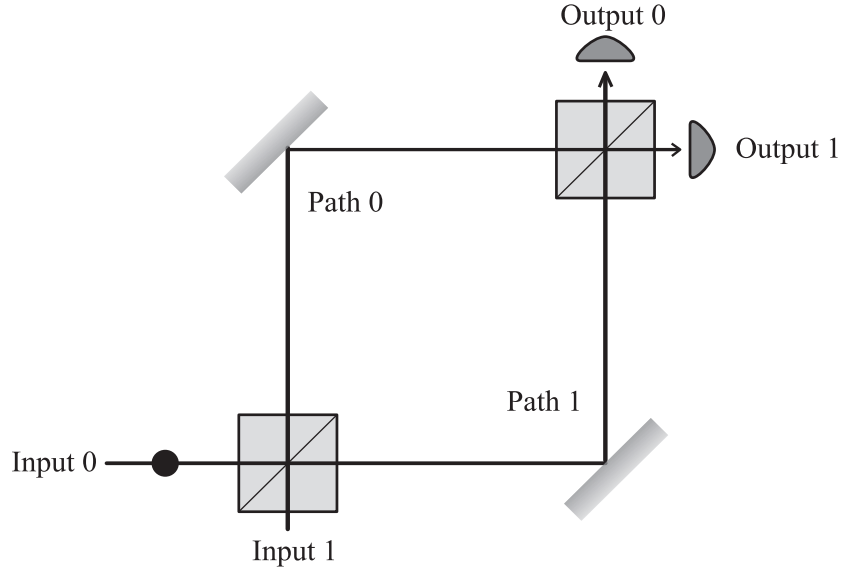


Figure 4: The experimental realisation of the two concatenated  $\sqrt{\text{not}}$  gates, known as a single-particle interference. A photon which enters the interferometer via Input 0 always strikes a detector Output 1 and never a detector at Output 0. Any explanation which assumes that the photon takes exactly one path through the interferometer leads to the conclusion that the two detectors should on average each fire on half the occasions when the experiment is performed. But experiment shows otherwise!

$\sqrt{\text{not}}$ . Why? Because a faithful physical model for it exists in nature!

Let us now introduce some of the mathematical machinery of quantum mechanics which can be used to describe quantum computing machines ranging from the simplest, such as  $\sqrt{\text{not}}$ , to the quantum generalisation of the universal Turing machine. At the level of predictions, quantum mechanics introduces the concept of *probability amplitudes* – complex numbers  $c$  such that the quantities  $|c|^2$  may under suitable circumstances be interpreted as probabilities. When a transition, such as “a machine composed of two identical sub-machines starts in state 0 and generates output 0, and nothing else happens”, can occur in several alternative ways, the overall probability amplitude for the transition is the sum, not of the probabilities, but of the probability amplitudes for each of the constituent transitions considered separately.



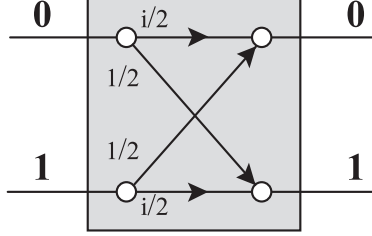


Figure 5: Transitions in quantum machines are described by probability amplitudes rather than probabilities. Probability amplitudes are complex numbers  $c$  such that the quantities  $|c|^2$  may under suitable circumstances be interpreted as probabilities. When a transition, such as “a machine composed of two identical sub-machines starts in state 0 and generates output 0, and nothing else happens”, can occur in several alternative ways, the probability amplitude for the transition is the sum of the probability amplitudes for each of the constituent transitions considered separately.

In the  $\sqrt{\text{not}}$  machine, the probability amplitudes of the  $0 \rightarrow 0$  and  $1 \rightarrow 1$  transitions are both  $i/\sqrt{2}$ , and the probability amplitudes of the  $0 \rightarrow 1$  and  $1 \rightarrow 0$  transitions are both  $1/\sqrt{2}$ . This means, for example, that the  $\sqrt{\text{not}}$  machine preserves the bit value with the probability amplitude  $c_{00} = c_{11} = i/\sqrt{2}$  and negates it with the probability amplitude  $c_{01} = c_{10} = 1/\sqrt{2}$ . In order to obtain the corresponding probabilities we have to take the modulus squared of the probability amplitudes which gives  $1/2$  both for preserving and swapping the bit value. This describes the behavior of the  $\sqrt{\text{not}}$  machine in Fig.(1). However, when we concatenate the two machines, as in Fig.(2) then, in order to calculate the probability of output 0 on input 0, we have to add the probability amplitudes of all computational paths leading from input 0 to output 0. There are only two of them -  $c_{00}c_{00}$  and  $c_{01}c_{10}$ . The first computational path has probability amplitude  $i/\sqrt{2} \times i/\sqrt{2} = -1/2$  and the second one  $1/\sqrt{2} \times 1/\sqrt{2} = +1/2$ . We add the two probability amplitudes first and then we take the modulus squared of the sum. We find that the probability of output 0 is zero. Unlike probabilities, probability amplitudes can cancel each other out!

### 3 Quantum algorithms

Addition of probability amplitudes, rather than probabilities, is one of the fundamental rules for prediction in quantum mechanics and applies to all physical objects, in particular quantum computing machines. If a computing machine starts in a specific initial configuration (input) then the probability that after its evolution via a sequence of intermediate configurations it ends up in a specific final configuration (output) is the squared modulus of the sum of all the probability amplitudes of the computational paths that connect the input with the output. The amplitudes are complex numbers and may cancel each other, which is referred to as *destructive interference*, or enhance each other, referred to as *constructive interference*. The basic idea of quantum computation is to use quantum interference to amplify the correct outcomes and to suppress the incorrect outcomes of computations. Let us illustrate this by describing a variant of the first quantum algorithm, proposed by David Deutsch in 1985.

Consider the Boolean functions  $f$  that map  $\{0, 1\}$  to  $\{0, 1\}$ . There are exactly four such functions: two constant functions ( $f(0) = f(1) = 0$  and  $f(0) = f(1) = 1$ ) and two “balanced” functions ( $f(0) = 0, f(1) = 1$  and  $f(0) = 1, f(1) = 0$ ). Suppose we are allowed to evaluate the function  $f$  *only once* (given, say, a lengthy algorithm for evaluating it on a given input, or a look-up table that may be consulted only once) and asked to determine whether  $f$  is constant or balanced (in other words, whether  $f(0)$  and  $f(1)$  are the same or different). Note that we are not asking for the particular values  $f(0)$  and  $f(1)$  but for a global property of the function  $f$ . Our classical intuition insists, and the classical theory of computation confirms, that to determine this global property of  $f$ , we have to evaluate both  $f(0)$  and  $f(1)$ , which involves evaluating  $f$  twice. Yet this is simply not true in physical reality, where quantum computation can solve Deutsch’s problem with a single function evaluation. The machine that solves the problem, using quantum interference, is composed of the two  $\sqrt{\text{NOT}}$ s with the function evaluation machine in between them, as in Fig.(6).

We need not go into the technicalities of the physical implementation of the evaluation of  $f(x)$ , where  $f$  is in general a Boolean function mapping  $\{0, 1\}^n \rightarrow \{0, 1\}^m$ . But generally, a machine that evaluates such a function must be capable of traversing as many computational paths as there are possible values  $x$  in the domain of  $f$  (so we can label them with  $x$ ). Its effect is that the probability amplitude on path  $x$  is multiplied by the phase factor

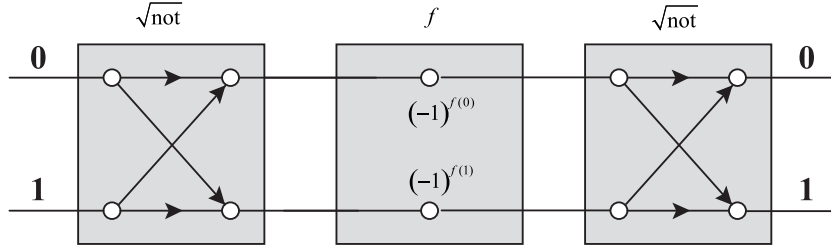


Figure 6: Schematic representation of a quantum machine that solves Deutsch's problem with a single function evaluation.

$\exp\left(\frac{2\pi i f(x)}{2^m}\right)$  [2]. In the case of Deutsch's problem the two phase factors are  $(-1)^{f(0)}$  and  $(-1)^{f(1)}$ . Now we can calculate the probability amplitude of output 0 on input 0. The probability amplitudes on the two different computational paths are  $i/\sqrt{2} \times (-1)^{f(0)} \times i/\sqrt{2} = -1/2 \times (-1)^{f(0)}$  and  $1/\sqrt{2} \times (-1)^{f(1)} \times 1/\sqrt{2} = 1/2 \times (-1)^{f(1)}$ . Their sum is

$$\frac{1}{2} \left( (-1)^{f(1)} - (-1)^{f(0)} \right), \quad (1)$$

which is 0 when  $f$  is constant and  $\pm 1$  when  $f$  is balanced. Thus the probability of output 0 on input 0 is given by the modulus squared of the expression above, which is zero when  $f$  is constant and unity when  $f$  is balanced. Deutsch's result laid the foundation for the new field of quantum computation. The hunt began for useful tasks for quantum computers to do. A sequence of steadily improved quantum algorithms led in 1994 to Peter Shor's discovery of a quantum algorithm that, in principle, could perform efficient factorisation [10]. Since the intractability of factorisation underpins the security of many of the most secure known methods of encryption, including the most popular public key cryptosystem RSA [9]<sup>1</sup>, Shor's algorithm was soon hailed as the first 'killer application' for quantum computation — something very useful that only a quantum computer could do.

Few if any mathematicians doubt that the factoring problem is in the

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<sup>1</sup>In December 1997 the British Government officially confirmed that public-key cryptography was originally invented at the Government Communications Headquarters (GCHQ) in Cheltenham. By 1975, James Ellis, Clifford Cocks, and Malcolm Williamson from GCHQ had discovered what was later re-discovered in academia and became known as RSA and Diffie-Hellman key exchange.

“**BPP**” class (where **BPP** stands for “bounded error probabilistic polynomial time”), but interestingly, this has never been proved. In computational complexity theory it is customary to view problems in **BPP** as being “tractable” or “solvable in practice” and problems not in **BPP** as “intractable” or “unsolvable in practice on a computer” (see, for example, [7]). A ‘**BPP** class algorithm’ for solving a problem is an efficient algorithm which, for any input, provides an answer that is correct with a probability greater than some constant  $\delta > 1/2$ . In general we cannot check easily if the answer is correct or not but we may repeat the computation some fixed number  $k$  times and then take a majority vote of all the  $k$  answers. For sufficiently large  $k$  the majority answer will be correct with probability as close to 1 as desired. Now, Shor’s result proves that factoring is not in reality an intractable task – and we learned this by studying quantum mechanics!

As a matter of fact, Richard Feynman, in his talk during the First Conference on the Physics of Computation held at MIT in 1981, observed that it appears to be impossible to simulate a general quantum evolution on a classical probabilistic computer in an efficient way [4]. That is to say, any classical simulation of quantum evolution involves an exponential slowdown in time compared with the natural evolution, since the amount of information required to describe the evolving quantum state in classical terms generally grows exponentially with time. However, instead of viewing this fact as an obstacle, Feynman regarded it as an opportunity. If it requires so much computation to work out what will happen in a complicated multiparticle interference experiment then, he argued, the very act of setting up such an experiment and measuring the outcome is tantamount to performing a complex computation. Thus, Feynman suggested that it might be possible to simulate a quantum evolution efficiently after all, provided that the simulator itself is a quantum mechanical device. Furthermore, he conjectured that if one wanted to simulate a different quantum evolution, it would not be necessary to construct a new simulator from scratch. It should be possible to choose the simulator so that minor systematic modifications of it would suffice to give it any desired interference properties. He called such a device a universal quantum simulator. In 1985 Deutsch proved that such a universal simulator or a universal quantum computer does exist and that it could perform any computation that any other quantum computer (or any Turing-type computer) could perform [3]. Moreover, it has since been shown that the time and other resources that it would need to do these things would not increase exponentially with the size or detail of the physical system being simulated,

so the relevant computations would be tractable by the standards of complexity theory [1]. This illustrates the fact that the more we know about physics, the more we know about computation and mathematics. Quantum mechanics proves that factoring is tractable: without quantum mechanics we do not yet know how to settle the issue either way.

## 4 Deterministic, Probabilistic, and Quantum Computers

Any quantum computer, including the universal one, can be described in a fashion similar to the special-purpose machines we have described above, essentially by replacing probabilities by probability amplitudes. Let us start with a classical Turing machine. This is defined by a finite set of quintuples of the form

$$(q, s, q', s', d), \tag{2}$$

where the first two characters describe the initial condition at the beginning of a computational step and the remaining three characters describe the effect of the instruction to be executed in that condition ( $q$  is the current configuration,  $s$  is the symbol currently scanned,  $q'$  is the configuration to enter next,  $s'$  is the symbol to replace  $s$ , and  $d$  indicates motion of one square to the right, or one square to the left, or stay fixed, relative to the tape). In this language a computation consists of presenting the machine with an input which is a finite string of symbols from the alphabet  $\Sigma$  written in the tape cells, then allowing the machine to start in the initial state  $q_0$  with the head scanning the leftmost symbol of the input and to proceed with its basic operations until it stops in its final (halting) state  $q_h$ . (In some cases the computation might not terminate.) The output of the computation is defined as the contents of some chosen part of the tape when (and if) the machine reaches its halting state.

During a computation the machine goes through a sequence of configurations; each configuration provides a global description of the machine and is determined by the string written on the entire tape, the state of the head and the position of the head. For example, the initial configuration is given by the input string, state  $q_0$ , and the head scanning the leftmost symbol from the input. There are infinitely many possible configurations of the machine but in all successful computations the machine goes through only a finite

sequence of them. The transitions between configurations are completely determined by the quintuples (2).



Figure 7: A three step deterministic computation.

Computations do not, in principle, have to be deterministic. Indeed, we can augment a Turing machine by allowing it “to toss an unbiased coin” and to choose its steps randomly. Such a probabilistic computation can be viewed as a directed, tree-like graph where each node corresponds to a configuration of the machine, and each edge represents one step of the computation. The computation starts from the root node representing the initial configuration and it subsequently branches into other nodes representing configurations reachable with non-zero probability from the initial configuration. The action of the machine is completely specified by a finite description of the form:

$$\delta : Q \times \Sigma \times Q \times \Sigma \times \{\text{Left, Right, Nothing}\} \mapsto [0, 1], \quad (3)$$

where  $\delta(q, s, q', s', d)$  gives the probability that if the machine is in state  $q$  reading symbol  $s$  it will enter state  $q'$ , write  $s'$  and move in direction  $d$ . This description must conform to the laws of probability as applied to the computation tree. If we associate with each edge of the graph the probability that the computation follows that edge then we must require that the sum of the probabilities on edges leaving any single node is always equal to 1. Probability of a particular path being followed from the root to a given node is the product of the probabilities along the path’s edges, and the probability of a particular configuration being reached after  $n$  steps is equal to the sum of the probabilities along all paths which in  $n$  steps connect the initial configuration with that particular configuration. Such randomized algorithms can solve some problems (with arbitrarily high probability less than 1) much faster than any known deterministic algorithms.

The classical model described above suggests a natural quantum generalisation. A quantum computation can be represented by a graph similar to that of a probabilistic computation. Following the rules of quantum dynamics we associate with each edge in the graph the probability *amplitude*

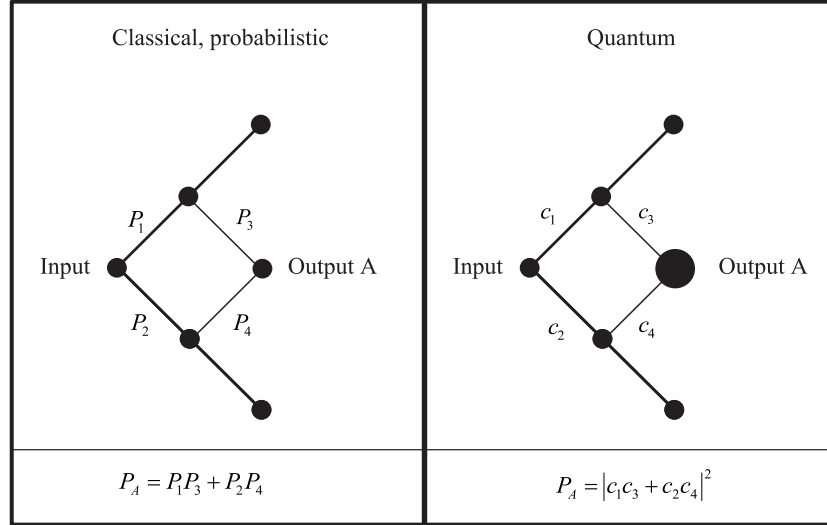


Figure 8: The probabilistic Turing machine (left) - the probability of output A is the sum of the probabilities of all computations leading to output A. In the quantum Turing machine (on the right) the probability of output A is obtained by adding all probability amplitudes leading from the initial state to output A and then taking the squared modulus of the sum. In the quantum case probabilities of some outcomes can be enhanced (constructive interference) or suppressed (destructive interference) compared with what classical probability theory would permit.

that the computation follows that edge. As before, the probability amplitude of a particular path being followed is the product of the probability amplitudes along the path's edges and the probability amplitude of a particular configuration is the sum of the amplitudes along all possible paths leading from the root to that configuration. If a particular final configuration can be reached via two different paths with amplitudes  $c$  and  $-c$  then the probability of reaching that configuration is  $|c - c|^2 = 0$  despite the fact that the probability for the computation to follow either of the two paths separately is  $|c|^2$  in both cases. Furthermore a single quantum computer can follow many distinct computational paths simultaneously and produce a final output depending on the interference of all of them. This is in contrast to a classical probabilistic Turing machine which follows only some *single* (randomly chosen) path. The action of any such quantum machine is completely specified

by a finite description of the form

$$\delta : Q \times \Sigma \times Q \times \Sigma \times \{\text{Left, Right, Nothing}\} \longrightarrow \mathbf{C} \quad (4)$$

where  $\delta(q, s, q', s', d)$  gives the probability *amplitude* that if the machine is in state  $q$  reading symbol  $s$  it will enter state  $q'$ , write  $s'$  and move in direction  $d$ .

## 5 Deeper implications

When the physics of computation was first investigated, starting in the 1960s, one of the main motivations was a fear that quantum-mechanical effects might place fundamental bounds on the accuracy with which physical objects could render the properties of the abstract entities, such as logical variables and operations, that appear in the theory of computation. Thus it was feared that the power and elegance of that theory, its most significant concepts such as computational universality, its fundamental principles such as the Church-Turing thesis and its powerful results such as the more modern theory of complexity, might all be mere figments of pure mathematics, not really relevant to anything in nature.

Those fears turned out to be groundless. Quantum mechanics, far from placing limits on which Turing computations can be performed in nature, permits them all, and in addition provides new modes of computation such as those we have described. As far as the elegance of the theory goes, it turns out that the quantum theory of computation hangs together better, and fits in far more naturally with fundamental theories in other fields, than its classical approximation was ever expected to. The very word ‘quantum’ means the same as the word ‘bit’ — an elementary chunk — and this reflects the fact that classical physical systems, being subject to the generic instability known as ‘chaos’, would not support digital computation at all (so even Turing machines, the theoretical prototype of all classical computers, were secretly quantum-mechanical all along!). The Church-Turing thesis in the classical theory (that all ‘natural’ models of computation are essentially equivalent to each other), was never proved. Its analogue in the quantum theory of computation (the Church-Turing Principle, that the universal quantum computer can simulate the behavior of any finite physical system) was straightforwardly proved in Deutsch’s 1985 paper [3]. A stronger result (also



conjectured but never proved in the classical case), namely that such simulations can always be performed in a time that is at most a polynomial function of the time taken for the physical evolution, has since been proved in the quantum case [1].

Among the many ramifications of quantum computation for apparently distant fields of study are its implications for the notion of mathematical proof. Performing any computation that provides a definite output is tantamount to *proving* that the observed output is one of the possible results of the given computation. Since we can describe the computer's operations mathematically, we can always translate such a computation into the proof of some mathematical theorem. This was the case classically too, but in the absence of interference effects it is always possible to keep a record of the steps of the computation, and thereby produce (and check the correctness of) a proof that satisfies the classical definition - as "a sequence of propositions each of which is either an axiom or follows from earlier propositions in the sequence by the given rules of inference". Now we are forced to leave that definition behind. Henceforward, a proof must be regarded as a process — the computation itself — for we must accept that in future, quantum computers will prove theorems by methods that neither a human brain nor any other arbiter will ever be able to check step-by-step, since if the 'sequence of propositions' corresponding to such a proof were printed out, the paper would fill the observable universe many times over.

## 6 Concluding remarks

This brief discussion has merely scratched the surface of the rapidly developing field of quantum computation. We have concentrated on the fundamental issues and have avoided discussing physical details and technological practicalities. However, it should be mentioned that quantum computing is a serious possibility for future generations of computing devices. This is one reason why the field is now attracting increasing attention from both academic researchers and industry worldwide. At present it is not clear when, how and even whether fully-fledged quantum computers will eventually be built; but notwithstanding this, the quantum theory of computation already plays a much more fundamental role in the scheme of things than its classical predecessor did. We believe that anyone who seeks a fundamental understanding of either physics, computation or logic must incorporate its new

insights into their world view.

## References

- [1] E. BERNSTEIN and U. VAZIRANI, *Quantum complexity theory*, *Proceedings of the 25th Annual Symposium on the Theory of Computing*, ACM, New York, 1993, pp. 11-20.
- [2] R. CLEVE, A. EKERT, C. MACCHIAVELLO, and M. MOSCA, *Quantum Algorithms Revisited*, *Proceedings of the Royal Society*, A vol. 454, (1998), pp. 339-354.
- [3] D. DEUTSCH, *Quantum theory, the Church-Turing principle and the universal quantum computer*, *Proceedings of the Royal Society*, A vol. 400 (1985), pp. 97-117.
- [4] R.P. FEYNMAN, *Simulating physics with computers*, *International Journal of Theoretical Physics*, vol. 21 (1982), pp. 467-488.
- [5] G. GALILEI, *Saggiatore*, [1623], *Opere*, A Favaro (ed.), vol. 6, Edizione Nazionale, Firenze, 1896.
- [6] H.H. GOLDSTINE, *The Computer from Pascal to von Neumann*, Princeton University Press, Princeton, 1972.
- [7] C.H. PAPADIMITRIOU, *Computational Complexity*, Addison-Wesley, Reading, 1994.
- [8] R. PENROSE, *Shadows of the mind*, Oxford University Press, Oxford, 1994.
- [9] R. RIVEST, A. SHAMIR, and L. ADLEMAN, *On Digital Signatures and Public-Key Cryptosystems*, Technical Report MIT/LCS/TR-212, MIT Laboratory for Computer Science, January 1979.
- [10] P. SHOR, *Algorithms for quantum computation: discrete log and factoring*, *Proceedings of the 35th Annual Symposium on the Foundations of Computer Science*, S. Goldwasser (editor), IEEE Computer Society Press, Los Alamitos, 1994, pp. 124-134.

- [11] A. TURING, *On computable numbers with an application to the Entscheidungsproblem*, *Proceedings of the London Mathematical Society*, 2 vol.42 (1936-37), pp. 230-265.
- [12] E. P. WIGNER, *The unreasonable effectiveness of mathematics in the natural sciences*, *Communications on Pure and Applied Mathematics*, vol. 13,(1960), pp. 1-14.