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A class of non-contractive, trial-dependent update rules for Iterative Learning Control

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***Abstract*— In this paper a family of trial-dependent update laws is studied and contrasted with a class of fixed update laws. In particular, it is investigated whether the principle of equivalent feedback extends to trial-dependent update laws. It turns out that this is not the case. Nonetheless, it is shown that a well-known performance bound arising in feedback control architectures, Bode's Sensitivity Integral, also applies here.**

I. Introduction

In two decades time, the field of Iterative Learning Control (ILC) has evolved from a simple idea into an advanced control methodology [14], [10], [11], [3], [12]. Given its distinctive nature and specific application field, there is little reason to expect that ILC has much in common with mainstream control methods. However, looks can be deceiving. Recent publications show that at least causal ILC is in a very precise sense equivalent with conventional feedback control [7], [8], [18], [19].

Motivated by these results, the aim of this paper is to investigate some structural properties of certain types of ILC algorithms. The discussion includes both trial-dependent and trial-independent update schemes. A recurring theme is that of the need for complexity. That is, given a certain update law, the key question is whether there exists a concurrent scheme of lower complexity generating the same control action.

The outline of the paper is as follows. Section 2 serves as a general introduction to the problem of ILC. Section 3 contains the main results. Conclusions and recommendations are presented in Section 4, which is followed by an Appendix containing all the required background material as well as some new results that are not fit for inclusion in the main text.

II. Iterative Learning Control

A. Problem Statement

Given a plant $P : U \to Y$, $y = Pu$, together with some desired output $y_d \in Y$. The objective of ILC is to define an iteration [20] on the space of control inputs U such that the corresponding sequence of outputs $\{y_k\}_{k\in\mathbb{N}}$ converges to a limit value $\bar{y} := \lim_{k\to\infty} y_k$ that is close to y_d in some sense.

We consider two families of iterations: trial-dependent- and trial-independent. The first is given in the update equation below.

$$u_{k+1} = Qu_k + Le_k \tag{1}$$

Here $e_k := y_d - y_k$ denotes the current tracking error. $Q : U \to U$ and $L : Y \to U$ are taken to be bounded linear operators acting on the current input, and current tracking error, respectively.

The affine transition map $F : U \to U$ associated with the update law (1) is given by

$$\begin{aligned} u_{k+1} &= F(u_k) \\ &= (Q - LP)\,u_k + Ly_d \end{aligned} \tag{2}$$

Over the years, the update law (1) has received considerable attention [9], [10], [16], [2], [5]. Issues such as convergence, robustness and performance have been studied in detail. A well-known convergence result states that the sequence of inputs $\{u_0, u_1, \ldots\}$ induced by the recurrence relation (2) converges to some $\bar{u} \in U$ if F is contractive. The limit point $\bar{u}$ being, of course, a fixed point of F.

The second class of update laws is given as

$$u_{k+1} = Q_{k+1}u_k + L_{k+1}e_k \tag{3}$$

where Q_k and L_k are as above, except for the index "k" (indicating the trial number). For each k we define a transition map $F_k : U \to U$

$$\begin{aligned} u_{k+1} &= F_{k+1}(u_k) \\ &= (Q_{k+1} - L_{k+1}P)\,u_k + L_{k+1}y_d \end{aligned} \tag{4}$$

Hence, associated with every element in the class of update laws (3) there is an ordered set, or sequence of transition maps $\{F_1, F_2, \ldots\}$.

Under the heading of *adaptive* (iterative) learning control, trial-dependent update laws have received some attention [15], [13], [6], but not quite as much as their trial-independent counterparts. In order to establish a result on convergence, in the present paper the sequence $\{F_1, F_2, \ldots\}$ is assumed to be (strongly) convergent. From an analysis perspective, this is a natural assumption, since there is little chance the input sequence would converge otherwise. From a synthesis point of view however, there is no particular reason why the F_k's should even be treated as design parameters. In fact, it makes more sense to allow these "parameters" to be affected by the actual behaviour of the controlled system. That is to say, in general convergence cannot just be "assumed"; it needs to be proven.

The analysis of trial-dependent update laws is subtle. Since F_k is not constant over trials, it does not immediately make sense to reason about fixed points. It turns out, however, that the notion of a fixed point and the corresponding fixed point theory can be extended to strongly converging sequences of operators. Details can be found in Appendix C.

B. On complexity and equivalent feedback

This section deals with complexity in ILC update rules. Now what are the compexity issues in ILC? To give an example, consider once more the class of trial-independent update laws that was previously introduced in equation (1). Suppose the iteration converges to some fixed point $(\bar{u}, \bar{e}) \in U \times Y$. It is not hard to see that the pair $(\bar{u}, \bar{e})$ is necessarily a solution of the "equilibrium equation"

$$(I - Q)\, u \;=\; Le \tag{5}$$

In fact, under some well-posedness conditions, it is the *only* solution. Note that the solution contains information about both the asymptotic performance and the control effort required to obtain it. Clearly, a solution cannot be determined *a priori*, since the equation contains terms involving the unknown plant. Now there are several ways to determine a solution. One is to just run the given scheme until it converges. Another way is to implement a feedback law

$$u \;=\; (I - Q)^{-1}\, Le \tag{6}$$

which will yield the exact same solution without the need to engage in a process of iteration. In other words, the update law (1) allows for a direct feedback implementation—provided all operators are causal. On top of that, it shows that whether the operators are causal or not, different operator pairs (Q, L) can induce the same solution pair (u, e). Taking the feedback control point of view, these correspond to different left-coprime factorizations of the same controller. These, and other structural properties are discussed in [19].

The bottomline is that the feedback implementation is much more efficient in terms of computational complexity. It is not hard to see that this conclusion generalizes to the class of trial-dependent update laws. In this case the result is even more striking. Having in fact an infinite number of design parameters, all the relevant information can still be condensed into essentially the same simple equation. The implication is clear: there is no point in exploiting update laws as exotic as (3) or even (1) unless it is shown that the complexity one resorts to is strictly necessary. That is to say, if the resulting control effort cannot be generated in another, more simple way. The moral of the story is twofold. First of all, it is interesting to note that despite the apparent differences in evolution and appearance, ILC and conventional feedback control have more in common than one used to think. Second, if ILC is to be recognized as a serious candidate for specific control applications, future research should emphasize and exploit its distinctive properties. It is hard to think of anything more lethal to ILC's subsistence than to prove that the same performance can be obtained through simpler means that obviously do not make use of the problem's intrinsic repetitiveness.

As a first step towards the goal outlined above, the next section introduces a class of update rules whose complexity appears to be irreducible.

III. A CLASS OF NON-CONTRACTIVE, TRIAL-DEPENDENT UPDATE LAWS

This section is concerned with the analysis of a subset of the class of trial-dependent update rules previously introduced in Section II-A. The particular class of interest is given as

$$u_{k+1} \;=\; u_k + L_{k+1} e_k \tag{7}$$

The operators $L_k : Y \to U$, $k = 1, 2, \ldots$ are assumed to be causal, bounded and linear. It is furthermore assumed that L_k strongly converges to zero ($0_{U \to Y}$).

A. Motivation

Why would this specific class be potentially interesting? Mainly because the update rule does not suggest a direct feedback implementation. The question whether or not such a feedback implementation exists needs futher investigation, but in any case it is not immediate from the update rule itself.

Why does this subclass need special treatment? Can it not be included in a general discussion on the bigger class? It could be, but then it would loose all its interesting properties. For in order to establish a result on convergence for the general class, it seems there is little to resort to other than Banach's fixed point theorem, which requires the transition map, or rather: the sequence of transition maps, to be contractive. Fact is that this condition is not likely to be met in case $Q_k = I$ for all k, or even if Q_k is merely assumed to converge to I. To illustrate this, rewrite (7) to get

$$\begin{aligned} u_{k+1} &= F_{k+1}(u_k) \\ &= (I - L_{k+1} P)\, u_k + L_{k+1} y_d \end{aligned} \tag{8}$$

Now $\{F_k, k \geq 1\}$ is contractive if and only if the condition

$$\|I - L_{k+1} P\| \;<\; 1 \tag{9}$$

is satisfied for (almost) all k. That is to say, only if LP is invertible over U, where L is defined as the limit $L := \lim_{k \to \infty} L_k$. There is no reason to assume that this holds for general L or that there even exists L for which it does. Now why should one care about this seemingly special case? The answer is that this is really the only case of interest because it is exactly in this case that the equivalent controller is not defined. This is readily observed from (6).

By considering special subclasses, such as the one introduced in (7), specialized techniques can be deployed in order to establish results on convergence that do not rely on the contractivity of the transition map. However, this comes at a price. In the case of the update rule (7) with the conditions as stated, one readily observes that as L_k

tends to zero, F_k will tend to identity ($I_{U\to U}$). Since every point in the domain of an identity map is a fixed point, it is immediate that the fixed point is no longer unique and will thus depend on the initial conditions.

B. Convergence analysis

Next a theorem is presented that gives conditions under which the class of update rules (7) converges to a bounded solution. It requires the notion of summable sequences.

Definition 1: Let $\{L_k\}$ be a sequence of bounded linear operators. The sequence is said to be *summable-in-norm* if

$$\lim_{N\to\infty}\sum_{k=1}^{N}\|L_k\| \;<\; \infty \tag{10}$$

Theorem 2 (Convergence on U): Let $\{L_k\}$ be a sequence of causal bounded linear operators from Y to U. Suppose $\{L_k\}$ is summable-in-norm. Then the sequence of control inputs $\{u_k\}$ converges to some bounded input $\bar{u}\in U$.

Note that the assumption that L_k strongly converges to zero is implicit in the condition that $\{L_k\}$ is summable. In order to prove Theorem 2, we need the following lemma.

Lemma 3 (Convergence of an infinite product): If $a_i \geq 0$ for all values of i, the product $\prod_{i=1}^{k}(1+a_i)$ and the series $\sum_{i=1}^{k} a_i$ converge or diverge together.

Proof: See [17] ∎

We prove the main theorem.

Proof: [of Theorem 2] Using induction it can be shown that for $k\geq 0$

$$\begin{aligned} u_k &= \prod_{i=0}^{k-1}(I-L_{i+1}P)\,u_0+\dots \\ &\quad \dots+\sum_{j=0}^{k-1}\prod_{i=j+1}^{k-1}(I-L_{i+1}P)\,L_{j+1}y_d \end{aligned}\tag{11}$$

Boundedness is proved term by term. Taking the first term out, note that

$$\begin{aligned} \|\prod_{i=0}^{k}(I-L_{i+1}P)\,u_0\| &\leq \prod_{i=0}^{k}\|I-L_{i+1}P\|\,\|u_0\| \\ &\leq \prod_{i=0}^{k}(1+\|L_{i+1}\|\|P\|)\,\|u_0\| \end{aligned}$$

Boundedness follows from the summability assumption and Lemma 3. The same argument applies for second term,

$$\begin{aligned} &\sum_{j=0}^{k}\prod_{i=j+1}^{k}(I-L_{i+1}P)\,L_{j+1}y_d \\ &\leq \sum_{j=0}^{k}\prod_{i=j+1}^{k}(1+\|L_{i+1}\|\|P\|)\,\|L_{j+1}\|\|y_d\| \\ &\leq \prod_{i=0}^{\infty}(1+\|L_{i+1}\|\|P\|)\sum_{j=0}^{k}\|L_{j+1}\|\|y_d\| \end{aligned}\tag{12}$$

where both the sum and the product converge by the assumption of summability (and Lemma 3). ∎

The next example shows how the result of Theorem 2 can be applied. Let $P\in\mathcal{RH}_\infty$ be a stable plant. For $k\geq 1$ we define $L_k := L/k^2$. Assume $L\in\mathcal{RH}_\infty$. It is easy to see that $\{L_k\}$ is summable

$$\begin{aligned} \sum_{k=1}^{\infty}\|L_k\| &= \sum_{k=1}^{\infty}\|L\|/k^2 \\ &= (\pi^2/6)\|L\| \end{aligned}\tag{13}$$

Now Theorem 2 says that, no matter what L or P is, the sequence of inputs $\{u_k\}$ will always converge to a bounded solution in $\mathcal{H}_2$. Thus, convergence does not depend on detailed knowledge of the plant. In fact, assuming mere boundedness is enough. The error e_k satisfies $e_k = Z_k e_0$, where Z_k is defined as $Z_k := \prod_{i=0}^{k-1}(I-L_{i+1}P)$. This tells us that the ultimate performance depends on the initial error e_0 which, in turn, depends on the initial input u_0. To remove ambiguity, it is henceforth assumed that the initial input is set to zero, i.e. $u_0 = 0$. Under this assumption e_0 equals y_d. The transformation

$$Z \;:=\; \lim_{k\to\infty} Z_k \tag{14}$$

which maps y_d onto $\bar{e}$ can be interpreted as an (output) sensitivity function. In some cases a closed form expression for Z can be found. As it turns out in our example

$$Z(s) \;=\; \frac{\sin\left(\pi\sqrt{L(s)P(s)}\right)}{\pi\sqrt{L(s)P(s)}} \tag{15}$$

A plot of $Z(j\omega)$ for $P = 1/(s\tau+1)$ and $L = 1$ is given in Figure 1.

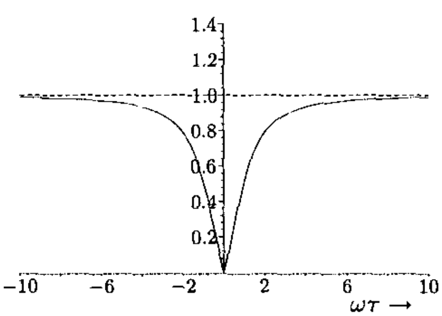

Fig. 1. The "sensitivity function" $|Z(j\omega)|$ as in (15) for $P = 1/(s\tau+1)$ and $L = 1$.

Tracking is particularly good in the low-frequency range (up till $\omega\tau\approx 1$). We also observe that $|Z(j\omega)| < 1$ for all ω. This suggests that the intial error is never amplified. One may wonder whether this property is intrinsic to the class of update rules. But this is not so as can be seen for the case $P = 1/(s\tau+1)^2$, depicted in Figure 2. More generally, the following proposition shows that $Z(s)$ is constrained by an integral relation.

Proposition 4 (Sensitivity integral): Assume that $P\in\mathcal{RH}_\infty$ has a pole-zero excess greater than one and let $\{L_k\}$

be a summable sequence in $\mathcal{RH}_\infty$. Then $Z(s)$ as defined in (14) satisfies the following integral relation

$$\int_{-\infty}^{\infty} \log|Z(j\omega)| d\omega \;\geq\; 0 \tag{16}$$

Proof: The key to the proof is Poisson's integral formula

$$F(s) \;=\; \frac{1}{\pi}\int_{-\infty}^{\infty} F(j\omega)\frac{x}{x^2+(y-\omega)^2} d\omega$$

where $s = x + jy$. This formula holds for all complex functions that are analytic in the closed right half plane and satisfy

$$\lim_{R\to\infty} \frac{|F(Re^{j\omega})|}{R} \;=\; 0$$

The idea is to apply this identity to the complex function $\log|Z(s)|$. However, since $Z(s)$ is allowed to have zeroes in the closed right half plane (CRHP), the condition on analyticity may not be satisfied. This problem is circumvented by considering instead $\tilde{Z} = B^{-1}Z$, where B is a Blaschke product containing all the CRHP zeroes $\{z_i\}$ of Z, counted according to their multiplicity

$$B(s) \;=\; \prod_i \frac{s - z_i}{s + \bar{z}_i}$$

The function $\log|\tilde{Z}|$ is analytic in the CRHP. Applying Poisson's integral formula gives

$$\int_{-\infty}^{\infty} \log|\tilde{Z}(j\omega)|\frac{x^2}{x^2+\omega^2} d\omega \;=\; \pi x \log|\tilde{Z}(x)|$$

Note that since B is inner, $|Z|$ equals $|\tilde{Z}|$ along the imaginary axis. Hence, as x tends to infinity, the left hand side tends to the sensitivity integral. The right hand side can be split into two parts

$$\pi x \log|\tilde{Z}(x)| \;=\; \pi x \log|Z(x)| + \pi x \log|B^{-1}(x)|$$

Taking the limit, the first term on the right hand side vanishes under the assumption that P has pole excess greater than one. The second term tends to

$$\begin{aligned}
\lim_{x\to\infty} \log|B^{-1}(x)| \;&=\; \lim_{x\to\infty} \log \prod_i \left|\frac{z_i + x}{z_i - x}\right| \\
&=\; \lim_{x\to\infty} \log \sum_i x \log \frac{1+\frac{\bar{z}_i}{x}}{1-\frac{z_i}{x}} \\
&=\; 2\sum_i \mathrm{Re}\, z_i \\
&\geq\; 0
\end{aligned}$$

This concludes the proof. ∎

The result in Theorem 4 is sometimes referred to as the *waterbed effect*. What it says is that if the (initial) error is attenuated in one region, it is amplified in another. It also shows that under the present conditions the error can never be zero over a whole frequency range, because in order to satisfy the sensitivity integral, it would mean that the error would grow unbounded in another region.

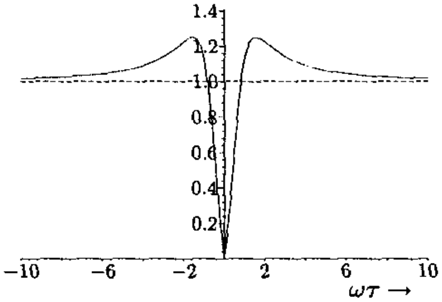

Fig. 2. The "sensitivity function" $|Z(j\omega)|$ as in (15) for $P = 1/(s\tau+1)^2$ and $L = 1$.

C. Equivalent Feedback

Does the update scheme (7) allow for a direct feedback implementation or not? And if so, what can be said about the corresponding controller structure? Suppose an equivalent feedback controller exists and call it K. The corresponding closed-loop sensitivity function is given as $S := 1/(1+PK)$. Setting $S = Z$ and solving for K gives

$$K \;=\; \frac{1-Z}{ZP} \tag{17}$$

To get an idea of what this amounts to, consider the next example. Take $L_1 = L$ for $k = 1$ and $L_k = 0$ for all other k. It is easy to verify that $Z = (1 - LP)$. The equivalent controller K is given as $K = L/1 - LP$. Using the theory of Youla parameterization, it can be verified that K is stablilizing for all $L \in \mathcal{RH}_\infty$. It is also clear that in order to implement K, exact knowledge of the plant is required. It appears that this is a general property. For suppose that the equivalent controller would be determined by the design parameters L_k only. Now let L_k be fixed and vary P over the space of all bounded linear operators. By Theorem 2 the input $\bar{u}$ and hence the error $\bar{e}$ would be bounded for all P. But this implies that the same equivalent controller would have to be stabilizing for all P. There is but one controller for which this is true and that is the zero controller. From (17) it is clear that $K = 0$ if and only if $Z = 1$. It is also not hard to see that $Z = 1$ if and only if $L_k = 0$ for all k. The conclusion is thus that the computational scheme (7) has irreducible complexity for all but one set of admissable parameter values.

IV. Conclusions

In this paper, a class of trial-dependent update laws was studied and contrasted with the more familiar class of fixed update rules. We argued that since ILC's *raison d'être* originates from the idea of exploiting repetitiveness, one should look for classes of algorithms that do just that. That is to say, we should constrain attention to those that are not (obviously) equivalent with conventional feedback or feedforward architectures. One such class of algorithms was introduced in this paper. It was shown that for this class an equivalent feedback controller could not be defined independent of any plant knowledge. Nonetheless, the corresponding sensitivity function was still shown to be subject to performance constraints similar to those arising in feedback architectures.

APPENDIX

The majority of the material in this section was taken from refereces [1] and [4].

A. Contractions and fixed point theory

Let (X,d) be a metric space. A map $F : X \to X$ is said to be *Lipschitzian* if there exists a constant $\alpha \geq 0$ such that

$$d(F(x),F(x')) \leq \alpha d(x,x') \tag{18}$$

for all $x,x' \in X$. The smallest α for which (18) holds is said to be the *Lipschitz constant* for F. If $L < 1$ we say that F is a *contraction*, whereas if $L = 1$, we say that F is *nonexpansive*.

Definition 5 (Fixed point): Given a map $F : X \to X$. A point $x \in X$ is a *fixed point* of F if it satisfies $F(x) = x$.

A map $F : X \to X$ may have more than one fixed point, or none at all. The following theorem gives a sufficient condition for F to have a *unique* fixed point in X, along with an iterative procedure to compute it.

Theorem 6 (Banach's Fixed Point Theorem): Let (X,d) be a complete metric space and let $F : X \to X$ be a contraction. Then F has a unique fixed point $\bar{x} \in X$. Moreover, for any $x \in X$ we have

$$\lim_{n\to\infty} F^n(x) = \bar{x} \tag{19}$$

B. Operator Theory

Henceforth it is assumed that X is a complete normed vector space. In addition, only mappings from X into itself are considered.

Definition 7 (Linear operator): An operator F is *linear* if, for all $x,x' \in X$ and all scalars α it holds that $F(x + x') = Fx + Fx'$, and $F(\alpha x) = \alpha Fx$.

Definition 8 (Affine operator): An operator F is *affine* if the associated operator $F_l(x) := F(x) - F(0)$, is linear.

Definition 9 (Bounded linear operator): F is a *bounded linear* operator if it is linear and there exists a real number c such that for all $x \in X$, $\|Fx\| \leq c\|x\|$.

Note that every bounded linear operator is Lipschitzian.

Definition 10 (Bounded affine operator): Let F be an affine operator. F is a *bounded affine* operator if the associated linear operator F_l is bounded in the sense of Definition 9.

Definition 11 (Strong convergence): Let $\{F_n, n \geq 1\}$ be a sequence of bounded affine operators. If

$$\|F_n x - Fx\| \to 0 \tag{20}$$

as $n \to \infty$ for all $x \in X$ then we say that F_n *converges strongly* to F.

In a complete vector space X, a necessary and sufficient condition for a sequence $\{F_n x, n \geq 1\}$ to converge to a limit point in X is that the sequence is Cauchy, i.e. for any $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m,n > N$

$$\|F_n x - F_m x\| < \varepsilon \tag{21}$$

From this it is clear that a necessary condition for the sequence $\{F_n, n \geq 1\}$ to converge strongly to a limit F is that the sequence $\{F_n x, n \geq 1\}$ is Cauchy for every $x \in X$. This fact will prove useful later on.

C. Fixed point theory: extensions

We present some extensions to the classical fixed point theory. For notational convenience we adopt the shorthand notation

$$\left(\prod_{n=1}^{N} F_n\right)(x) := F_N\Big(F_{N-1}\big(\cdots F_1(x)\cdots\big)\Big)$$

Definition 12 (Fixed point): Let $\{F_n, n \geq 1\}$ be a sequence of bounded affine operators that strongly converges to a limit $F := \lim_{n\to\infty} F_n$ (see Definition 11). We say that a point $x \in X$ is a fixed point of the sequence of operators $\{F_n\}$ if it is a fixed point of F, i.e. if $\lim_{n\to\infty} \|F_n(x) - x\| = 0$.

Definition 13 (Contraction): Let $\{F_n, n \geq 1\}$ be a strongly converging sequence of bounded affine operators, and let L_n denote the Lipschitz constant of the associated linear operator $F_n(x) - F_n(0)$. We say that $\{F_n\}$ is contractive if $\sup_n L_n < 1$.

We have the following result.

Theorem 14: Let X be a Banach space and let $\{F_n, n \geq 1\}$ be a strongly converging sequence of bounded affine operators on X. Define $F := \lim_{n\to\infty} F_n$. Suppose $\{F_n\}$ is contractive. Then there exists a unique fixed point $\bar{x} \in \mathcal{X}$ such that $F(\bar{x}) = \bar{x}$. Furthermore, for any $x \in X$, we have that

$$\bar{x} = \lim_{N\to\infty} \left(\prod_{n=1}^{N} F_n\right)(x) \tag{22}$$

Proof: Let us prove uniqueness first. Suppose by contradiction that F has two fixed points $x,x' \in X$ and $x \neq x'$. Let L_n denote the Lipschitz constant of the linear operator $(F_n)_l$. Observe that

$$\|F_n x - F_n x'\| \leq L_n \|x - x'\| \tag{23}$$

Hence

$$\lim_{n\to\infty} \|F_n x - F_n x'\| \leq L \|x - x'\| \tag{24}$$

where $L := \sup_n L_n < 1$. On the other hand

$$\|F_n x - F_n x' - (x - x')\| \leq \|F_n x - x\| + \|F_n x' - x'\| \tag{25}$$

implies that

$$\lim_{n\to\infty} \|F_n x - F_n x'\| = \|x - x'\| \tag{26}$$

Equations (24) and (26) cannot both hold true, unless $x = x'$. But this contradicts our starting assumption. Hence we conclude that F has only one fixed point.

Next we verify that $\bar{x}$ as defined by (22) is a fixed point of F.To that end, let us assume that the right hand side of (22) converges. Then, by definition

$$\lim_{N\to\infty}\left(\prod_{n=1}^{N}F_n\right) = \lim_{N\to\infty}F_N\left(\prod_{n=1}^{N-1}F_n\right) \quad (27)$$

By assumption, the left hand side of (27), acting on x, converges to $\bar{x}$. The claim is that the right hand side, acting on x, converges to $F(\bar{x})$. This can be shown as follows. Take any $x \in X$, define $x_n := \left(\prod_{i=1}^{n}F_n\right)(x)$, and observe that $\|F_N(x_{N-1}) - F(\bar{x})\|$

$$\begin{aligned} &= \|F_N(x_{N-1}) - F_N(\bar{x}) + F_N(\bar{x}) - F(\bar{x})\| \\ &\le \|F_N(x_{N-1}) - F_N(\bar{x})\| + \|F_N(\bar{x}) - F(\bar{x})\| \\ &= \|F_N(x_{N-1} - \bar{x})\| + \|F_N(\bar{x}) - F(\bar{x})\| \\ &\le L_N\,\|x_{N-1} - \bar{x} + \|F_N(\bar{x}) - F(\bar{x})\| \qquad (28)\end{aligned}$$

Here we used the fact that F_N is bounded and affine over its domain. Recall that L_N is uniformly bounded by some $L < 1$ because $\{F_n\}$ is contractive. As N tends to infinity both terms on the right hand side of (28) vanish. The first because by assumption x_{N-1} converges to $\bar{x}$ and the second because $\{F_n\}$ is strongly converging. This shows that $F_N(x_{N-1})$ tends to $F\bar{x}$. We conclude that $\bar{x}$ is indeed a fixed point of F.

Next we show that the sequence $\left(\prod_{n=1}^{N}F_N\right)(x)$ converges for all $x \in X$. To that end, take any $x \in X$ and define x_n as before. Decompose the affine operator F_n into a an affine part $(F_n)_a$, and a linear part $(F_n)_l$ as follows:

$$F_n(x) := (F_n)_l x + (F_n)_a \quad (29)$$

We arrive at the following expression for x_n

$$x_n = \left(\prod_{i=1}^{n}(F_i)_l\right)x_0 + \sum_{j=1}^{n-1}\left(\prod_{i=j}^{n-1}(F_{i+1})_l\right)(F_j)_a + (F_n)_a$$

Define $A := \sup_n \|(F_n)_a\|$ and recall that $\|(F_n)_l\| = L_n$ is uniformly bounded by some $L < 1$. Thus

$$\begin{aligned}\|x_n\| &\le L^n\|x_0\| + \left(\sum_{j=0}^{n-1}L^j\right)A \\ &= L^n\|x_0\| + \left(\frac{1-L^n}{1-L}\right)A \qquad (30)\end{aligned}$$

Clearly the right hand side of (30) is bounded and hence $\{x_n\}$ is bounded. What remains is to show that $\|x_n - x_{n-1}\|$ will tend to zero as $n \to \infty$. We have that $\|x_n - x_{n-1}\|$

$$\begin{aligned} &= \|F_n(x_{n-1}) - F_n(x_{n-2}) + F_n(x_{n-2}) - F_{n-1}x_{n-2}\| \\ &\le \|F_n(x_{n-1}) - F_n(x_{n-2})\| + \|F_n(x_{n-2}) - F_{n-1}(x_{n-2})\| \\ &\le L\,\|x_{n-1} - x_{n-2}\| + \|F_n(x_{n-2}) - F_{n-1}(x_{n-2})\|\end{aligned}$$

The second term on the right hand side vanishes as n tends to infinity. This is because $\{F_n x, n \ge 1\}$ is Cauchy for all $x \in X$. Consequently, since $L < 1$, $\|x_n - x_{n-1}\|$ will converge zero. This completes the proof. ∎

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