H_∞ Control of Differential Linear Repetitive Processes

Wojciech Paszke, Krzysztof Gałkowski, Eric Rogers, and David H. Owens

***Abstract*—Repetitive processes are a distinct class of two-dimensional (2-D) systems (i.e., information propagation in two independent directions) of both systems theoretic and applications interest. They cannot be controlled by direct extension of existing techniques from either standard [termed one-dimensional (1-D) here] or 2-D systems theory. Here, we give new results on the relatively open problem of the design of control laws using an H_∞ setting. These results are for the sub-class of so-called differential linear repetitive processes which arise in applications.**

***Index Terms*—Differential repetitive processes, H_∞ control, linear matrix inequalities.**

I. Introduction

THE ESSENTIAL unique characteristic of a repetitive process is a series of sweeps, termed passes, through a set of dynamics defined over a fixed finite duration known as the pass length. On each pass, an output, termed the pass profile, is produced which acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile. This, in turn, leads to the unique control problem for these processes in that the output sequence of pass profiles generated can contain oscillations that increase in amplitude in the pass-to-pass direction.

Physical examples of repetitive processes include long-wall coal cutting and metal rolling operations (see, for example, [10]). Also in recent years applications have arisen where adopting a repetitive process setting for analysis has distinct advantages over alternatives. Examples of these so-called algorithmic applications include classes of iterative learning control (ILC) schemes [1] and iterative algorithms for solving nonlinear dynamic optimal control problems based on the maximum principle [9].

The H_∞ setting for the control related analysis of one-dimensional (1-D) linear systems is now a very mature area and it is natural question to ask if such an approach can be extended to two-dimensional (2-D) linear systems/linear repetitive processes. In the case of 2-D discrete linear systems, some work on an H_∞ approach to analysis has been reported — see, for example, [3]. The same approach to differential linear repetitive processes has not yet been considered, but it is clear that work in this area should be very profitable with (possible) onward translation to, for example, the ILC area where the problem of what is meant by robustness of such schemes is still a largely open question.

In this paper, we first give new results on the control of differential linear repetitive processes which formulate and solve the fundamental problem of finding an admissible control law, or controller, such that stability holds together with a prescribed bound on disturbance attenuation in an H_∞ setting. Also it is shown that the control problem here can, in computational terms, be solved using linear matrix inequalities (LMIs) [2]. Finally, significant new results on the robust control of these processes are developed from this setting.

Throughout this paper, the null matrix and the identity matrix with the required dimensions are denoted by 0 and I, respectively. Moreover, $M > 0$ (≥ 0) denotes a real symmetric positive definite (respectively positive semi-definite) matrix, and $M < 0$ denotes a real symmetric negative definite matrix. We also use $(\star)$ to denote the transpose of matrix blocks in some of the LMIs employed (which are required to be symmetric).

The following results are required in the proofs of some of the results developed here, as is the well known Schur's complement formula.

Lemma 1: [7] Let Σ_1, Σ_2 be real matrices of appropriate dimensions. Then for any matrix $\mathcal{F}$ satisfying $\mathcal{F}^T\mathcal{F} \leq I$ and a scalar $\epsilon > 0$ the following inequality holds:

$$\Sigma_1\mathcal{F}\Sigma_2 + \Sigma_2^T\mathcal{F}^T\Sigma_1^T \leq \epsilon^{-1}\Sigma_1\Sigma_1^T + \epsilon\Sigma_2^T\Sigma_2. \qquad (1)$$

Lemma 2: [5] Let F be a $q \times q$ symmetric matrix and let P and Q be real matrices of dimensions $s \times q$ and $h \times q$, respectively. Then, there exists an $h \times s$ matrix G such that

$$F + P^TG^TQ + Q^TGP < 0 \qquad (2)$$

if, and only if, the inequalities

$$\mathcal{N}_p^TF\mathcal{N}_p < 0 \text{ and } \mathcal{N}_q^TF\mathcal{N}_q < 0 \qquad (3)$$

both hold, where $\mathcal{N}_p \in \ker(P)$ and $\mathcal{N}_q \in \ker(Q)$.

Lemma 3: [4] Suppose that the $n_b \times n_b$ matrices $\Sigma > 0$ and $\Gamma > 0$ are given and n_c is a positive integer. Then, there exists $n_b \times n_c$ matrices Σ_2, Γ_2 and $n_c \times n_c$ symmetric matrices Σ_3, and Γ_3, such that

$$\begin{bmatrix} \Sigma & \Sigma_2 \\ \Sigma_2^T & \Sigma_3 \end{bmatrix} > 0 \text{ and } \begin{bmatrix} \Sigma & \Sigma_2 \\ \Sigma_2^T & \Sigma_3 \end{bmatrix}^{-1} = \begin{bmatrix} \Gamma & \Gamma_2 \\ \Gamma_2^T & \Gamma_3 \end{bmatrix} \qquad (4)$$

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if, and only if

$$\begin{bmatrix} \Sigma & I \\ I & \Gamma \end{bmatrix} \geq 0. \tag{5}$$

The L_2 norm of the $r\times 1$ vector sequence $\{w_k(t)\}_k$ defined over $[0,\infty]$, $[0,\infty]$ is given by

$$\|w\|_2 = \sqrt{\sum_{k=0}^{\infty}\int_0^{\infty} w_k(t)^T w_k(t)\, dt} \tag{6}$$

and $\{w_k(t)\}_k$ is said to be a member of $L_2^r\{[0,\infty],[0,\infty]\}$, or L_2^r for short, if $\|w\|_2 < \infty$.

II. Background

The differential linear repetitive processes considered here are described by a state space model of the following form over $0 \leq t \leq \alpha$, $k \geq 0$

$$\begin{aligned} \dot{x}_{k+1}(t) &= Ax_{k+1}(t) + Bu_{k+1}(t) + B_0 y_k(t) + B_{11} w_{k+1}(t) \\ y_{k+1}(t) &= Cx_{k+1}(t) + Du_{k+1}(t) + D_0 y_k(t) + D_{11} w_{k+1}(t). \end{aligned} \tag{7}$$

Here on pass k, $x_k(t)$ is the $n\times 1$ state vector, $y_k(t)$ is the $m\times 1$ pass profile vector, $u_k(t)$ is the $l\times 1$ vector of control inputs and $w_k(t)$ is an $r\times 1$ disturbance vector which belongs to L_2^r.

To complete the process description, it is necessary to specify the boundary conditions i.e., the state initial vector on each pass and the initial pass profile (i.e., on pass 0). The simplest possible choice for these is

$$\begin{aligned} x_{k+1}(0) &= d_{k+1}, \qquad k \geq 0 \\ y_0(t) &= f(t) \end{aligned} \tag{8}$$

where the $n\times 1$ vector d_{k+1} has known constant entries and $f(t)$ is an $m\times 1$ vector whose entries are known functions of t over $[0,\alpha]$. (For ease of presentation, we will make no further explicit reference to the boundary conditions in this paper and assume that in all cases $d_{k+1} = 0$ and $f(t) = 0$).

The stability theory [10] for linear repetitive processes consists of two distinct concepts but here it is the stronger of these which is required. This is termed stability along the pass and (recall the unique control problems for these processes) is a form of bounded-input bounded-output stability independent of the pass length. Moreover, several equivalent sets of necessary and sufficient conditions for processes described by (7) with no disturbance terms present to have this property are known [10]. All of these, however, have not proved to be a suitable basis for control law design to ensure stability along the pass or this property plus a guaranteed level of performance (under some appropriate measure). This has recently led to the development of sufficient but not necessary design algorithms based on the use of LMIs, see, for example, [6] where an LMI based sufficient condition for stability along the pass of processes described by (7) with no disturbance terms present has been developed.

Since the dynamics along the pass of the processes considered here are defined by a matrix differential equation, an H_∞ based approach to the control of these processes cannot be obtained by any existing theory for 2-D discrete linear systems, such as in [3]. Moreover, it is routine to argue that the signals involved in the study of these processes can be extended from $[0,\alpha]$ to the infinite interval in such a way that projection of the infinite interval solution is possible. This has been exploited in the stability along the pass theory and here we also invoke this property (where required).

III. H_∞ Norm Bound

It is easy to see that stability along the pass of a process described by (7) is independent of the disturbance terms. We will also require a Lyapunov function interpretation of this property, where the candidate function is taken to be

$$\begin{aligned} V(k,t) &= V_1(k,t) + V_2(k,t) \\ &= x_{k+1}^T(t)P_1 x_{k+1}(t) + y_k^T(t)P_2 y_k(t) \end{aligned} \tag{9}$$

where $P_1 > 0$ and $P_2 > 0$. The associated increment is

$$\Delta V(k,t) = \dot{V}_1(k,t) + \Delta V_2(k,t) \tag{10}$$

where

$$\begin{aligned} \dot{V}_1(k,t) &= \dot{x}_{k+1}^T(t)P_1 x_{k+1}(t) + x_{k+1}^T(t)P_1\dot{x}_{k+1}(t) \\ \Delta V_2(k,t) &= y_{k+1}^T(t)P_2 y_{k+1}(t) - y_k^T(t)P_2 y_k(t). \end{aligned}$$

Hence, (by substitution from (7) with $w_{k+1}(t) = 0$) we can write

$$\begin{aligned} \Delta V(k,t) &= \zeta_k^T(t)\left(\widehat{A}_1^T P + P\widehat{A}_1 + \widehat{A}_2^T P_2 \widehat{A}_2 - R\right)\zeta_k(t) \\ &=: \zeta_k^T(t) S_{2\mathrm{D}} \zeta_k(t) \end{aligned} \tag{11}$$

where $P = \mathrm{diag}\{P_1, 0\}$, $R = \mathrm{diag}\{0, P_2\}$, $\zeta_k(t) = [x_{k+1}^T(t) \quad y_k^T(t)]^T$ and

$$\widehat{A}_1 = \begin{bmatrix} A & B_0 \\ 0 & 0 \end{bmatrix}, \quad \widehat{A}_2 = \begin{bmatrix} 0 & 0 \\ C & D_0 \end{bmatrix}.$$

It is now routine to conclude (see [6]) that stability along the pass holds if $\Delta V(k,t) < 0$. (This is based on the fact that the matrix $S_{2\mathrm{D}}$ in (11) is the so-called 2-D Lyapunov equation for these processes and stability along the pass holds if $S_{2\mathrm{D}} < 0$.)

Definition 1: A differential linear repetitive process described by (7) is said to have H_∞ disturbance attenuation (or norm) bound $\gamma > 0$ if it is stable along the pass and the induced norm between w and y is bounded by γ i.e.,

$$\sup_{0\neq w\in L_2^r} \frac{\|y\|_2}{\|w\|_2} < \gamma. \tag{12}$$

Theorem 1: A differential linear repetitive process described by (7) is stable along the pass and has H_∞ disturbance attenuation bound γ if $\exists$ matrices $P_1 > 0$ and $P_2 > 0$ such that the following LMI holds:

$$\begin{bmatrix} -P_2 & P_2 C & P_2 D_0 & P_2 D_{11} \\ C^T P_2 & A^T P_1 + P_1 A & P_1 B_0 & P_1 B_{11} \\ D_0^T P_2 & B_0^T P_1 & -P_2 + I & 0 \\ D_{11}^T P_2 & B_{11}^T P_1 & 0 & -\gamma^2 I \end{bmatrix} < 0. \tag{13}$$

Proof: Introduce the associated Hamiltonian as

$$H(k,t) = \Delta V(k,t) + y_{k+1}^T(t) y_{k+1}(t) - \gamma^2 w_{k+1}^T(t) w_{k+1}(t) \tag{14}$$

and it is easily shown that H_∞ disturbance attenuation is equivalent to

$$H(k,t) < 0. \tag{15}$$

Hence, we require that $\Delta V(k,t) < 0$ and therefore stability along the pass must hold. Also we can write

$$H(k,t) = [\,\zeta_k^T(t) \quad w_{k+1}^T(t)\,]\,\Theta\begin{bmatrix}\zeta_k(t)\\ w_{k+1}(t)\end{bmatrix} \tag{16}$$

where

$$\Theta = \begin{bmatrix}\widehat{A}_1^T P + P\widehat{A}_1 + \widehat{A}_2^T R\widehat{A}_2 + L^T L - R & (*)\\ \widehat{B}_{11}^T P + \widehat{D}_{11}^T R\widehat{A}_2 & \widehat{D}_{11}^T R\widehat{D}_{11} - \gamma^2 I\end{bmatrix} \tag{17}$$

and

$$\widehat{B}_{11} = \begin{bmatrix}B_{11}\\ 0\end{bmatrix}, \quad \widehat{D}_{11} = \begin{bmatrix}0\\ D_{11}\end{bmatrix}, \quad L = [\,0 \quad I\,]. \tag{18}$$

Hence, (15) can be replaced by $\Theta < 0$, and an obvious application of the Schur's complement formula to this last condition gives (13) and the proof is complete. ■

IV. Static H_∞ Control

For differential linear repetitive processes of the form considered here, one possible control law has the structure [6]

$$u_{k+1}(t) = [\,K_1 \quad K_2\,]\begin{bmatrix}x_{k+1}(t)\\ y_k(t)\end{bmatrix} \tag{19}$$

where K_1 and K_2 are appropriately dimensioned matrices to be designed. In effect, this control law uses feedback of the current state vector (which is assumed to be available for use) and 'feedforward' of the previous pass profile vector. Note that in repetitive processes the term 'feedforward' is used to describe the case where state or pass profile information from the previous pass (or passes) is used as (part of) the input to a control law applied on the current pass, i.e., to information which is propagated in the pass-to-pass (k) direction.

The following result shows that the LMI setting extends to allow the design of a control law of the form (19) to result in stability along the pass with a prescribed H_∞ disturbance attenuation bound.

Theorem 2: Suppose that a control law of the form (19) is applied to a differential linear repetitive process described by (7). Then the resulting process is stable along the pass and has prescribed H_∞ disturbance attenuation bound $\gamma > 0$ if $\exists$ matrices $W_1 > 0$, $W_2 > 0$, N_1 and N_2 such that the following LMI holds:

$$\begin{bmatrix}-W_2 & (\star) & (\star) & (\star) & (\star)\\ W_1C^T + N_1^T D^T & \Omega_1 & (\star) & (\star) & (\star)\\ W_2D_0^T + N_2^T D^T & W_2B_0^T + N_2^T B^T & -W_2 & (\star) & (\star)\\ D_{11}^T & B_{11}^T & 0 & -\gamma^2 I & (\star)\\ 0 & 0 & W_2 & 0 & -I\end{bmatrix} < 0 \tag{20}$$

where

$$\Omega_1 = W_1A^T + N_1^T B^T + AW_1 + BN_1.$$

Also if this condition holds, the control law matrices K_1 and K_2 are given by $N_1W_1^{-1}$ and $N_2W_2^{-1}$ respectively.

Proof: Interpreting Theorem 1 in terms of the state space model resulting from applying (19) to (7) gives that it is stable along the pass with prescribed H_∞ disturbance attenuation bound γ if

$$\begin{bmatrix}-S & S\overline{A}_2 & S\widehat{D}_{11}\\ \overline{A}_2^T S & \overline{A}_1^T P + P\overline{A}_1 + L^T L - R & P\widehat{B}_{11}\\ \widehat{D}_{11}^T S & \widehat{B}_{11}^T P & -\gamma^2 I\end{bmatrix} < 0 \tag{21}$$

where

$$\overline{A}_1 = \begin{bmatrix}A + BK_1 & B_0 + BK_2\\ 0 & 0\end{bmatrix}$$

$$\overline{A}_2 = \begin{bmatrix}0 & 0\\ C + DK_1 & D_0 + DK_2\end{bmatrix}.$$

Here, $S = \mathrm{diag}\{P_3, P_2\}$, and $P_3 > 0$ is any given matrix with the required dimensions. Now make an obvious application of the Schur's complement formula to yield

$$\begin{bmatrix}-S & S\overline{A}_2 & S\widehat{D}_{11} & 0\\ \overline{A}_2^T S & \overline{A}_1^T P + P\overline{A}_1 - R & P\widehat{B}_{11} & L^T\\ \widehat{D}_{11}^T S & \widehat{B}_{11}^T P & -\gamma^2 I & 0\\ 0 & L & 0 & -I\end{bmatrix} < 0. \tag{22}$$

Next, substitute the formulas given previously for $\overline{A}_1$ and $\overline{A}_2$ into this last expression, pre- and post-multiply the result by $\mathrm{diag}\{P_3^{-1}, P_2^{-1}, P_1^{-1}, P_2^{-1}, I, I\}$ and then set $W_1 = P_1^{-1}$, $W_2 = P_2^{-1}$, $W_3 = P_3^{-1}$, $N_1 = K_1P_1^{-1}$, $N_2 = K_2P_2^{-1}$ Finally, noting that the result does not depend on matrix W_3, leads to (20) and the proof is complete. ■

V. H_∞ Control of Uncertain Differential Linear Repetitive Processes

In this section we extend the results given in the previous section of this paper to the case where there is uncertainty associated with the process state space model. The presence of these uncertainties can arise from a number of sources, e.g., variation of physical parameters over time and/or imperfect knowledge of the process dynamics, leading to only an approximate model. Here we aim to design the control law of the previous section to ensure stability along the pass with a prescribed H_∞ disturbance attenuation level for all admissible uncertainties.

As a first attempt at this task, we assume that the uncertainty is norm bounded in both the state and pass profile updating equations. This form corresponds to the case of processes where uncertainty is modeled as an additive perturbation to the nominal model state space matrices and can be written as

$$\begin{aligned}\begin{bmatrix}\dot{x}_{k+1}(t)\\ y_{k+1}(t)\end{bmatrix} &= \left(\begin{bmatrix}A & B_0\\ C & D_0\end{bmatrix} + \begin{bmatrix}\Delta A & \Delta B_0\\ \Delta C & \Delta D_0\end{bmatrix}\right)\begin{bmatrix}x_{k+1}(t)\\ y_k(t)\end{bmatrix}\\ &\quad + \left(\begin{bmatrix}B\\ D\end{bmatrix} + \begin{bmatrix}\Delta B\\ \Delta D\end{bmatrix}\right)u_{k+1}(t)\\ &\quad + \left(\begin{bmatrix}B_{11}\\ D_{11}\end{bmatrix} + \begin{bmatrix}\Delta B_{11}\\ \Delta D_{11}\end{bmatrix}\right)w_{k+1}(t)\end{aligned} \tag{23}$$

where the admissible uncertainties are assumed to be of the form

$$\begin{bmatrix}\Delta A & \Delta B_0 & \Delta B & \Delta B_{11}\\ \Delta C & \Delta D_0 & \Delta D & \Delta D_{11}\end{bmatrix} = \begin{bmatrix}H_1\\ H_2\end{bmatrix}\mathcal{F}\,[E_1 \quad E_2 \quad E_3 \quad E_4] \tag{24}$$

and H_1, H_2, E_1, E_2, E_3, E_4 are known constant matrices of compatible dimensions. The matrix $\mathcal{F}$ is unknown with constant entries and satisfies

$$\mathcal{F}^T\mathcal{F} \le I. \tag{25}$$

The following result gives a solution to the problem of designing the control law (19) to solve the problem considered here.

Theorem 3: Suppose that a control law of the form (19) is applied to a differential linear repetitive process described by (23), with uncertainty structure modeled by (24) and (25). Then, the resulting process is stable along the pass for all admissible uncertainties and has prescribed H_∞ disturbance attenuation bound $\gamma > 0$ if $\exists$ matrices $W_1 > 0$, $W_2 > 0$, N_1 and N_2 and a scalar $\epsilon > 0$ such that the LMI shown in (26) at the bottom of the page, holds, where

$$\Omega_2 = W_1A^T + N_1^TB^T + AW_1 + BN_1 + 3\epsilon H_1H_1^T.$$

If (26) holds, the control law matrices K_1 and K_2 are given by $N_1W_1^{-1}$ and $N_2W_2^{-1}$, respectively.

Proof: First interpret (20) in terms of the state space model resulting from application of the control law to obtain (27) shown at the bottom of the page, where

$$\begin{aligned}\Omega_3 &= W_1\Delta A^T + N_1^T\Delta B^T + \Delta AW_1 + \Delta BN_1\\ \Omega_4 &= W_1A^T + N_1^TB^T + AW_1 + BN_1.\end{aligned}$$

The first term in the above inequality can be rewritten as

$$\overline{H}\overline{\mathcal{F}}\overline{E} + \overline{E}^T\overline{\mathcal{F}}^T\overline{H}^T \tag{28}$$

where

$$\begin{aligned}\overline{H} &= \begin{bmatrix}0 & H_2 & H_2 & H_2 & 0\\ 0 & H_1 & H_1 & H_1 & 0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\end{bmatrix}\\ \overline{\mathcal{F}} &= \mathrm{diag}\{\mathcal{F},\mathcal{F},\mathcal{F},\mathcal{F},\mathcal{F}\}\\ \overline{E} &= \mathrm{diag}\{0, E_1W_1 + E_3N_1, E_2W_2 + E_3N_2, E_4, 0\}.\end{aligned}$$

An obvious application of (1) (Lemma 1) followed by application of the Schur's complement formula yields (26) and the proof is complete. ■

VI. H_∞ Control With Dynamic (or Pass Profile) Controller

In the control law used in the previous two sections, full access to the current state vector has been assumed. Here, we consider the application of a controller which is activated only by the previous pass profile vector. (Note again that the pass profile is the output vector of these processes and hence on any pass the previous pass profile, unlike the current pass state vector, is always available for use.)

The controller used in this section has the following state space model, were due to space limitations we do not consider the case when the process model has uncertainty in its state space model (this follows by a routine extension of the analysis below)

$$\begin{aligned}\begin{bmatrix}\dot{x}^c_{k+1}(t)\\ y^c_{k+1}(t)\end{bmatrix} &= \begin{bmatrix}A_{c11} & A_{c12}\\ A_{c21} & A_{c22}\end{bmatrix}\begin{bmatrix}x^c_{k+1}(t)\\ y^c_k(t)\end{bmatrix} + \begin{bmatrix}B_{c1}\\ B_{c2}\end{bmatrix}y_k(t)\\ u_{k+1}(t) &= [C_{c1} \quad C_{c2}]\begin{bmatrix}x^c_{k+1}(t)\\ y^c_k(t)\end{bmatrix} + D_cy_k(t)\end{aligned} \tag{29}$$

$$\begin{bmatrix}-W_2 + 3\epsilon H_2H_2^T & (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & (\star)\\ W_1C^T + N_1^TD^T + 3\epsilon H_1H_2^T & \Omega_2 & (\star) & (\star) & (\star) & (\star) & (\star) & (\star)\\ W_2D_0^T + N_2^TD^T & W_2B_0^T + N_2^TB^T & -W_2 & (\star) & (\star) & (\star) & (\star) & (\star)\\ D_{11}^T & B_{11}^T & 0 & -\gamma^2I & (\star) & (\star) & (\star) & (\star)\\ 0 & 0 & W_2 & 0 & -I & (\star) & (\star) & (\star)\\ 0 & E_1W_1 + E_3N_1 & 0 & 0 & 0 & -\epsilon I & (\star) & (\star)\\ 0 & 0 & E_2W_2 + E_3N_2 & 0 & 0 & 0 & -\epsilon I & (\star)\\ 0 & 0 & 0 & E_4 & 0 & 0 & 0 & -\epsilon I\end{bmatrix} < 0 \tag{26}$$

$$\begin{bmatrix}0 & (\star) & (\star) & (\star) & (\star)\\ W_1\Delta C^T + N_1^T\Delta D^T & \Omega_3 & (\star) & (\star) & (\star)\\ W_2\Delta D_0^T + N_2^T\Delta D^T & W_2\Delta B_0^T + N_2^T\Delta B^T & 0 & (\star) & (\star)\\ \Delta D_{11}^T & \Delta B_{11}^T & 0 & 0 & (\star)\\ 0 & 0 & 0 & 0 & 0\end{bmatrix} + \begin{bmatrix}-W_2 & (\star) & (\star) & (\star) & (\star)\\ W_1C^T + N_1^TD^T & \Omega_4 & (\star) & (\star) & (\star)\\ W_2D_0^T + N_2^TD^T & W_2B_0^T + N_2^TB^T & -W_2 & (\star) & (\star)\\ D_{11}^T & B_{11}^T & 0 & -\gamma^2I & (\star)\\ 0 & 0 & W_2 & 0 & -I\end{bmatrix} < 0 \tag{27}$$

$$\begin{bmatrix} \mathcal{N}_c^T & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} -T_{v_{11}} + D_0 T_{v_{11}} D_0^T & CU_{h_{11}} + D_0 T_{v_{11}} B_0^T & D_{11} & D_0 T_{v_{11}} \\ U_{h_{11}} C^T + B_0 T_{v_{11}} D_0^T & U_{h_{11}} A^T + AU_{h_{11}} + B_0 T_{v_{11}} B_0^T & B_{11} & B_0 T_{v_{11}} \\ D_{11}^T & B_{11}^T & -\gamma^2 I & 0 \\ T_{v_{11}} D_0^T & T_{v_{11}} B_0^T & 0 & -I + T_{v_{11}} \end{bmatrix} \begin{bmatrix} \mathcal{N}_c & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} < 0 \quad (34)$$

$$\begin{bmatrix} A^T P_{h_{11}} + P_{h_{11}} A + C^T S_{v_{11}} C & (\star) \\ B_{11}^T P_{h_{11}} + D_{11}^T S_{v_{11}} C & D_1^T S_{v_{11}} D_{11} - \gamma^2 \end{bmatrix} < 0 \quad (35)$$

$$\begin{bmatrix} P_{h11} & I \\ I & U_{h11} \end{bmatrix} \geq 0, \begin{bmatrix} S_{v11} & I \\ I & T_{v11} \end{bmatrix} \geq 0 \quad (36)$$

where the controller state vectors $x_{k+1}^c(t)$ and $y_k^c(t)$ are of dimensions $n_1 \times 1$ and $m_1 \times 1$ respectively. Also introduce

$$\Xi = \begin{bmatrix} A & B_0 \\ C & D_0 \end{bmatrix}$$
$$\Omega = \begin{bmatrix} B_{11} \\ D_{11} \end{bmatrix}$$
$$B_2 = \begin{bmatrix} B \\ D \end{bmatrix}$$
$$C_2 = [0 \quad I]$$
$$A_c = \begin{bmatrix} A_{c11} & A_{c12} \\ A_{c21} & A_{c22} \end{bmatrix}$$
$$B_c = \begin{bmatrix} B_{c1} \\ B_{c2} \end{bmatrix}$$
$$C_c = [C_{c1} \quad C_{c2}] \quad (30)$$

and define the so-called augmented state and pass profile vectors for the process resulting from application of this controller to (7) as

$$\dot{\overline{x}}_{k+1}(t) = \begin{bmatrix} \dot{x}_{k+1}(t) \\ \dot{x}_{k+1}^c(t) \end{bmatrix}, \quad \overline{y}_{k+1}(t) = \begin{bmatrix} y_{k+1}(t) \\ y_{k+1}^c(t) \end{bmatrix} \quad (31)$$

together with the matrices

$$\Pi = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$
$$\Pi_1 = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\Pi_2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

to obtain

$$\begin{bmatrix} \dot{x}_{k+1}(t) \\ \overline{y}_{k+1}(t) \end{bmatrix} = \overline{A} \begin{bmatrix} \overline{x}_{k+1}(t) \\ \overline{y}_k(t) \end{bmatrix} + \overline{B} w_{k+1}(t)$$
$$y_k(t) = \overline{C} \begin{bmatrix} \overline{x}_{k+1}(t) \\ \overline{y}_k(t) \end{bmatrix} \quad (32)$$

where

$$\overline{B} = \Pi_1 \begin{bmatrix} \Omega \\ 0 \end{bmatrix} + \Pi_2 \begin{bmatrix} \Omega \\ 0 \end{bmatrix} = \overline{B}_1 + \overline{B}_2, \; \overline{C} = [C_2 \quad 0]\Pi$$
$$\overline{A} = \Pi_1 \begin{bmatrix} \Xi + B_2 D_c C_2 & B_2 C_c \\ B_c C_2 & A_c \end{bmatrix} \Pi + \Pi_2 \begin{bmatrix} \Xi + B_2 D_c C_2 & B_2 C_c \\ B_c C_2 & A_c \end{bmatrix} \Pi = \overline{A}_1 + \overline{A}_2.$$

Theorem 1 interpreted in terms of the state space model (32) now gives the following result which serves as an existence condition for the controller considered in this section.

Theorem 4: Suppose that a controller of the form (29) is applied to a differential linear differential repetitive process described by (7). Then, the resulting process is stable along the pass and has prescribed H_∞ disturbance attenuation bound $\gamma > 0$ if there exist matrices $S_v > 0$ and $P_h > 0$ such that the following inequality holds:

$$\begin{bmatrix} -S & S\overline{A}_2 & S\overline{B}_2 & 0 \\ \overline{A}_2^T S & \overline{A}_1^T P + P\overline{A}_1 - R & P\overline{B}_1 & \overline{C}^T \\ \overline{B}_2^T S & \overline{B}_1^T P & -\gamma^2 I & 0 \\ 0 & \overline{C} & 0 & -I \end{bmatrix} < 0 \quad (33)$$

where here $S = \text{diag}\{S_h, S_v\}$ $P = \text{diag}\{P_h, 0\}$, $R = \text{diag}\{0, S_v\}$.

The following result extends this last theorem to give a controller design algorithm.

Theorem 5: A differential linear repetitive process described by (32) is stable along the pass and has prescribed H_∞ disturbance attenuation bound $\gamma > 0$ if there exist matrices $P_{h_{11}} > 0$, $U_{h_{11}} > 0$, $S_{v_{11}} > 0$, $T_{v_{11}} > 0$ such that the LMIs defined by (34)–(36) at the top of the page hold, where $\mathcal{N}_c$ is a full column rank matrix whose image satisfies

$$\text{Im}\mathcal{N}_c = \ker \begin{bmatrix} D^T \\ B^T \end{bmatrix}. \quad (37)$$

Proof: Omitted due to space limitations, the details can be found in [8]. In summary, use is made of the Schur's complement formula, congruence transforms and the results of all the Lemmas given in the background section of this paper. ■

Suppose now that this last result holds. Then the following is a systematic procedure for obtaining the corresponding controller state space matrices.

1) Compute the matrices P_{h12}, P_{v12} using the following formulas:

$$S_{v_{11}} - T_{v_{11}}^{-1} = S_{v_{12}} S_{v_{12}}^T$$
$$P_{h_{11}} - U_{h_{11}}^{-1} = P_{h_{12}} P_{h_{12}}^T$$

where $P_{h22} = I$ and $S_{v22} = I$.

2) Construct the matrices $P_h > 0$ and $S_v > 0$ as

$$P_h = \begin{bmatrix} P_{h_{11}} & P_{h_{12}}^T \\ P_{h_{12}} & I \end{bmatrix}, \quad S_v = \begin{bmatrix} S_{v_{11}} & S_{v_{12}}^T \\ S_{v_{12}} & I \end{bmatrix}$$

and then we have $S = \text{diag}\{I, S_v\}$ $P = \text{diag}\{P_h, I\}$, $R = \text{diag}\{0, S_v\}$.

3) Compute the matrices M, N and Ψ defined as

$$\Psi = \begin{bmatrix} -S & S\mathcal{A}_2 & S\overline{B}_2 & 0 \\ \mathcal{A}_2^T S & \mathcal{A}_1^T P + P\mathcal{A}_1 - R & P\overline{B}_1 & \overline{C}^T \\ \overline{B}_2^T S & \overline{B}_1^T P & -\gamma^2 I & 0 \\ 0 & \overline{C} & 0 & -I \end{bmatrix}$$

$$M = [\Gamma_2^T S \quad \Gamma_1^T P \quad 0 \quad 0], N = [0 \quad \mathcal{C}_2 \quad 0 \quad 0]$$

where

$$\mathcal{A}_1 = \Pi_1 \begin{bmatrix} \Xi & 0 \\ 0 & 0 \end{bmatrix} \Pi$$

$$\mathcal{A}_2 = \Pi_2 \begin{bmatrix} \Xi & 0 \\ 0 & 0 \end{bmatrix} \Pi$$

$$\Gamma_1 = \Pi_1 \begin{bmatrix} B_2 & 0 \\ 0 & I \end{bmatrix}$$

$$\Gamma_2 = \Pi_2 \begin{bmatrix} B_2 & 0 \\ 0 & I \end{bmatrix}$$

$$\mathcal{C}_2 = \begin{bmatrix} C_2 & 0 \\ 0 & I \end{bmatrix}.$$

4) Solve the following LMI:

$$\Psi + M^T \Theta_c N + N^T \Theta_c^T M < 0$$

to obtain

$$\Theta_c = \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}$$

i.e., the matrices which define the controller state space model **(29)**.

VII. Conclusion

This paper has developed substantial new results on the relatively open problem of the control of differential linear repetitive processes which are a distinct class of 2-D linear systems of both systems theoretic and applications interest. The result is physically based control laws in an H_∞ setting where the required computations are LMI based. Also it has been shown that these results can be extended to the case of uncertainty in the model where here this is assumed to be norm bounded in both the state and pass profile updating equations of the defining state space model. Extensions to other uncertainty representations are also possible and will be reported elsewhere.

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