Infinite Horizon Optimal Control: Transversality Conditions and Sensitivity Relations

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Piermarco Cannarsa[1] and Helene Frankowska[2]

***Abstract*— We investigate the infinite horizon problem in optimal control with a general—not necessarily discounted—running cost. The dynamic programming approach associates to it the value function V that may be discontinuous and may take infinite values. In general, the first order necessary optimality conditions have the form of a maximum principle, that may be abnormal and does not include any transversality condition. We show that a normal maximum principle and a sensitivity relation involving the Fréchet subdifferential are valid for a dense subset of initial conditions. In particular, a transversality condition holds true on this subset. Furthermore, if V is continuous, then for every initial state a, possibly abnormal, maximum principle can be written with a transversality condition at the initial state involving a limiting (in the normal case) or horizontal (in the abnormal case) superdifferentials of V. Finally, when V is locally Lipschitz with respect to the state variable, we prove a normal maximum principle together with sensitivity relations and a transversality condition involving generalized gradients of V. These relations simplify the investigation of the limiting behaviour of the costate.**

I. INTRODUCTION

Infinite horizon optimization attracted attention of researchers in diverse fields including electrical engineering, economics/finance and operations research/management science. In economics, the first appearance of model of maximizing levels of consumption over successive generations is due Ramsey [15]. The classical infinite horizon problem is as follows : for some $\lambda > 0$

$$W(x_0) = \inf \int_0^\infty e^{-\lambda t}\ell(x(t), u(t))\, dt, \qquad (1)$$

where $e^{-\lambda t}$ is a discount factor, (x, u) are trajectory-control pairs of the following control system

$$\begin{cases} x'(t) = f(x(t), u(t)), \quad u(t) \in U \quad \text{for a.e. } t \geq 0 \\ x(0) = x_0 \end{cases} \qquad (2)$$

and controls $u(\cdot)$ are Lebesgue measurable. Despite a very good understanding of necessary optimality conditions for finite horizon problems, this topic is still challenging in the infinite horizon context, due to the fact that on one hand the ”natural” transversality conditions do not hold in many situations, on the other hand, because, even in the absence of constraints, the maximum principle may be abnormal. We refer to [1] for an extended overview of the literature devoted to transversality conditions and normality of the maximum principle for the infinite horizon problem, examples and counterexamples, and for important bibliographical comments and also to [2], [3], [14] for a further discussion.

Given an optimal trajectory-control pair $(\bar{x}, \bar{u})$ and $T > t_0$, consider the finite horizon Bolza problem

$$\text{minimize} \int_0^T e^{-\lambda t}\ell(x(t), u(t))\, dt \qquad (3)$$

over all trajectory-control pairs (x, u) of the system

$$\begin{cases} x'(t) = f(x(t), u(t)), \quad u(t) \in U \quad \text{for a.e. } t \in [0, T] \\ x(0) = x_0. \end{cases}$$

Then the necessary optimality condition takes the form of the maximum principle: if $(\bar{x}, \bar{u})$ restricted to $[0, T]$ is an optimal trajectory-control pair for the above Bolza problem, then the solution $p_T := p$ of the adjoint system

$$-p'(t) = D_x f(\bar{x}(t), \bar{u}(t))^* p(t) - e^{-\lambda t}\ell_x(\bar{x}(t), \bar{u}(t)) \qquad (4)$$

with $p(T) = 0$, satisfies on $[0, T]$ the maximality condition

$$\begin{array}{l} \langle p(t), f(\bar{x}(t), \bar{u}(t))\rangle - e^{-\lambda t}\ell(\bar{x}(t), \bar{u}(t)) = \\ \max_{u\in U}(\langle p(t), f(\bar{x}(t), u)\rangle - e^{-\lambda t}\ell(\bar{x}(t), u)) \quad \text{a.e.} \end{array} \qquad (5)$$

The transversality condition $p(T) = 0$ is due the fact that there is no cost term depending on $x(T)$ in (3).

If for any $i \geq 1$, the restriction of $(\bar{x}, \bar{u})$ to $[0, i]$, is optimal for the Bolza problem with $T = i$, then taking a converging subsequence of $\{p_i\}$ one gets a solution of (4) satisfying the maximality condition (5) a.e. in $[0, \infty[$. Though the transversality condition does disappear in this approach, some additional assumption on f, ℓ, λ alow to conclude that $\lim_{t\to+\infty} p(t) = 0$. In a way, zero may appear to be a natural candidate for the transversality condition at infinity.

However, in general, the restriction of $(\bar{x}, \bar{u})$ to the time interval $[0, T]$ is not optimal. A remedy to this difficulty could be including the constraint $x(T) = \bar{x}(T)$. This results, however, in possibly abnormal maximum principles for finite horizon problems, and, *in fine*, leads to necessary optimality conditions without the cost function ℓ in (4), (5). Also in this approach the transversality condition at time T does disappear, becoming $-p(T) \in N_{\{\bar{x}(T)\}}(\bar{x}(T)) = \mathbb{R}^n$.

Many authors contributed to proofs of normal maximum principles for the infinite horizon problem involving some transversality conditions, see [12], [13], [10], [20], [16], [1]-[3] and the bibliographies contained therein for variational approaches and [4], [14], [17] which use duality theory for weighted Sobolev spaces $L^p(0, +\infty; \mathbb{R}^n)$ with respect to the measure $e^{-\lambda t}dt$ (or more general measures).

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Recently, because of some economic applications, also the nonautonomous infinite horizon problem without the discount factor attracted attention of several researchers. In such case, W may become discontinuous and may take infinite values. When variational methods are used, some authors modify the notion of optimal solution or, alternatively, introduce new variables and cost to obtain relevant necessary optimality conditions for the finite horizon problems and then, by passing to a limit, to get a maximum principle for the infinite horizon problem.

Let us underline that in the finite horizon setting, the transversality condition at final time is one of ingredients of the maximum principle. For the infinite horizon problem the transversality condition at infinity follows from assumptions on data and the adjoint equation: one first gets a solution of the adjoint equation satisfying the maximality condition and then investigates how this solution behaves at infinity. For this reason, it seems being more relevant to impose transversality conditions at the initial state, as it was done in [4], where a discounted problem with linear dynamics was studied using the abstract Green formula.

We consider the nonautonomous optimal control problem

$$V(t_0, x_0) = \inf \int_{t_0}^{\infty} L(t, x(t), u(t))\, dt \tag{6}$$

over all trajectory-control pairs (x, u), subject to the system

$$\begin{cases} x'(t) = f(t, x(t), u(t)), \quad u(t) \in U(t) \quad \text{a.e.} \\ x(t_0) = x_0, \end{cases} \tag{7}$$

where $\mathbb{R}_+ = [0, +\infty[$, $t_0 \in \mathbb{R}_+$, $x_0 \in \mathbb{R}^n$, $L : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$, $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ are given mappings, $U(\cdot)$ is a set-valued map with nonempty images in $\mathbb{R}^m$. Selections $u(t) \in U(t)$ are supposed to be measurable.

The above setting subsumes the classical infinite horizon optimal control problem when f and U are time independent, $L(t, x, u) = e^{-\lambda t}\ell(x, u)$ for some mapping $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$ and $t_0 = 0$. Furthermore, in this case, for all $t \geq 0$,

$$V(t, x_0) = e^{-\lambda t} W(x_0).$$

To derive necessary optimality conditions, we rely solely on the dynamic programming principle and do not modify the notion of solution. Unlike the approach in [13], [16], [20], where dynamic programming was used in the presence of a discount factor, we neither introduce new variables nor modify the cost function. Instead, for a lower semicontinuous value function and for a dense set of initial states we get a normal maximum principle by exploiting the "forward sensitivity relations" derived in [8], [9]. If $V(t_0, \cdot)$ is continuous, then for *every* $x_0 \in \mathbb{R}^n$ we obtain a, possibly abnormal, maximum principle with a transversality condition involving the limiting or horizontal limiting superdifferentials of $V(t_0, \cdot)$ at x_0. Our results imply, in particular, that if for an initial condition x_0 the set of limiting horizontal supergradients of $V(t_0, \cdot)$ at x_0 is empty, then the derived maximum principle is normal. If the value function is locally Lipschitz, then the maximum principle is normal and the sensitivity relations can also be deduced from [6] to investigate the behavior of the costate and the Hamiltonian (along the trajectory-costate pair) at infinity.

We would like to underline that introducing the value function into necessary optimality conditions is an additional property of the costate, usually absent in the formulation of maximum principles. Adding such sensitivity relations may even give necessary and sufficient optimality conditions for the finite horizon problems, cf. [7]. On the other hand, as we show below, they also allow to deduce the behavior of the costate at infinity in a quite straightforward way. For this reason our approach simplifies the earlier investigations of the infinite horizon maximum principles. At the same time it *does not* apply to local minimizers. The previous works, [4], [16], [20], involving the value function in transversality conditions have addressed problems with a discount factor and locally Lipschitz value functions, while we are able to state some results also when W is merely lower semicontinuous or continuous and there is no discount factor.

The outline of the paper is as follows. Section II concerns some preliminaries and notations. In Section III we provide sufficient conditions for V to be lower semicontinuous, continuous or locally Lipschitz and give a relaxation theorem. Section IV discusses maximum principles and sensitivity relations. In subsection A we state a normal maximum principle and a sensitivity relation involving the Fréchet subdifferential of the lower semicontinuous value function. Subsection B deals with continuous (around x_0) mapping $V(t_0, \cdot)$, where we obtain a, possibly abnormal, maximum principle with a transversality condition at x_0. Finally Subsection C is devoted to maximum principles, sensitivity relations and transversality conditions at infinity for locally Lipschitz value function. Because of the lack of space, we do not include detailed proofs of our results, but only indicate by what arguments they are obtained. Complete proofs will be published elsewhere.

II. Preliminaries

Denote by $B(0, R)$ the closed ball in a finite dimensional space centered at zero with radius $R > 0$. Let $K \subset \mathbb{R}^n$ and $x \in K$. The contingent cone to K at x consists of all vectors $v \in \mathbb{R}^n$ such that there exist sequences $h_i \to 0+$ and vectors $v_i \to v$ satisfying $x + h_i v_i \in K$. The limiting normal cone to K at $x \in K$ is defined using the Peano-Kuratowski upper limit, see for instance [5],

$$N_K^L(x) = \mathrm{Limsup}_{y \to_K x} T_K(y)^-,$$

where $\to_K$ denotes the convergence in K and $T_K(y)^-$ is the negative polar of $T_K(y)$. It is well known that if x lies on the boundary of K, then $N_K^L(x)$ is not reduced to zero.

For $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ the domain of φ is the set $dom(\varphi)$ of all x such that $\varphi(x) \neq \pm\infty$. Denote by $epi(\varphi)$ and $hyp(\varphi)$ resp. the epigraph and the hypograph of φ.

The Fréchet subdifferential of φ at $x \in dom(\varphi)$, denoted by $\partial^-\varphi(x)$, consists of all vectors p satisfying $(p, -1) \in T_{epi(\varphi)}(x, \varphi(x))^-$. If $p \in \mathbb{R}^n$ is so that $(p, -1) \in N^L_{epi(\varphi)}(x, \varphi(x))$, then it is called a limiting subgradient of φ at x, while if $(p, 0) \in N^L_{epi(\varphi)}(x, \varphi(x))$, then p is called a limiting horizontal subgradient of φ at x. The sets of all limiting and limiting horizontal subgradients of φ at x are denoted by $\partial^{L,-}\varphi(x)$ and $\partial^{\infty,-}\varphi(x)$ respectively.

The limiting supergradients are defined in a similar way: every $p \in \mathbb{R}^n$ satisfying $(-p,+1) \in N^L_{hyp(\varphi)}(x,\varphi(x))$ is called a limiting supergradient of φ at x. If $(-p,0) \in N^L_{hyp(\varphi)}(x,\varphi(x))$, then p is called a limiting horizontal supergradient of φ at x. The sets of all limiting and limiting horizontal supergradients of φ at x are denoted by $\partial^{L,+}\varphi(x)$ and $\partial^{\infty,+}\varphi(x)$ respectively.

If φ is locally Lipschitz at x, then the sets of limiting horizontal subgradients and supergradients at x are empty and the set $co\,\partial^{L,-}\varphi(x)$ is the generalized gradient of φ at x, denoted by $\partial\varphi(x)$, where co stands for the convex hull.

For $a,\, b \in \mathbb{R}$ we shall use the following notations $a \wedge b = \min\{a,b\},\;\; a \vee b = \max\{a,b\}$. For a matrix A, denote by A^* its transpose.

III. Value Function

Consider the nonautonomous infinite horizon optimal control problem (6)-(7) with data as described in the introduction. A Lebesgue measurable selection $u(t) \in U(t)$ a.e. in $\mathbb{R}_+$ is called a control and the set of all controls is denoted by $\mathcal{U}$. Note that to state (6)-(7) we need only controls defined on $[t_0,+\infty[$. However, without any loss of generality, we may assume that controls are defined on $[0,+\infty[$.

Assumptions **(H1)***:*

i) *There exist locally integrable functions* $c,\theta : \mathbb{R}_+ \to \mathbb{R}_+$ *such that for a.e.* $t \in \mathbb{R}_+$,
$$|f(t,x,u)| \le c(t)|x| + \theta(t), \quad \forall\, x \in \mathbb{R}^n,\; u \in U(t);$$

ii) *For every* $R > 0$*, there exists a locally integrable function* $c_R : \mathbb{R}_+ \to \mathbb{R}_+$ *and a modulus of continuity* $\omega_R : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *such that for a.e.* $t \in \mathbb{R}_+$, $\omega_R(t,\cdot)$ *is nondecreasing,* $\lim_{r\to 0+}\omega_R(t,r) = 0$ *and for all* $x, y \in B(0,R),\; u \in U(t)$,
$$|f(t,x,u) - f(t,y,u)| \le c_R(t)|x-y|,$$
$$|L(t,x,u) - L(t,y,u)| \le \omega_R(t,|x-y|);$$

iii) *For all* $x \in \mathbb{R}^n$*, the mappings* $f(\cdot,x,\cdot)$, $L(\cdot,x,\cdot)$ *are Lebesgue-Borel measurable ;*

iv) *There exists a locally integrable function* $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ *and a locally bounded nondecreasing function* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that for a.e.* $t \in \mathbb{R}_+$,
$$L(t,x,u) \le \beta(t)\phi(|x|), \quad \forall\, x \in \mathbb{R}^n,\; u \in U(t);$$

v) $U(\cdot)$ *is Lebesgue measurable and has closed nonempty images;*

vi) *For a.e.* $t \in \mathbb{R}_+$*, and for all* $x \in \mathbb{R}^n$ *the set*
$$\big\{\big(f(t,x,u), L(t,x,u) + r\big) : u \in U(t) \text{ and } r \ge 0\big\}$$
is closed and convex.

Under the above assumptions to every control $u(\cdot)$ and $(t_0,x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ corresponds a locally absolutely continuous solution $x(\cdot)$ of the system in (7) defined on $\mathbb{R}_+$. The couple (x,u) is called a trajectory-control pair. If the integral in (6) does not converge for any $u \in \mathcal{U}$, then set $V(t_0,x_0) = +\infty$.

The extended function $V : [0,+\infty[\times\mathbb{R}^n \to \mathbb{R}_+ \cup \{+\infty\}$ defined by (6)-(7) is called the value function of the infinite horizon problem. For any $t_0 \in \mathbb{R}_+,\;\; x_0 \in \mathbb{R}^n$ such that $V(t_0,x_0) < +\infty$, a trajectory-control pair $(\bar{x},\bar{u})$ is optimal for the infinite horizon problem at (t_0,x_0) if for every trajectory-control pair (x,u) satisfying $x(t_0) = x_0$ we have
$$\int_{t_0}^{\infty} L(t,\bar{x}(t),\bar{u}(t))\,dt \le \int_{t_0}^{\infty} L(t,x(t),u(t))\,dt.$$

The proof of the following Proposition is standard.

Proposition 1: Assume (H1). Then V is lower semicontinuous and for every $(t_0,x_0) \in dom(V)$, there exists a trajectory-control pair $(\bar{x},\bar{u})$ satisfying $V(t_0,x_0) = \int_{t_0}^{\infty} L(t,\bar{x}(t),\bar{u}(t))\,dt$.

Consider the relaxed infinite horizon problem
$$V^{rel}(t_0,x_0) = \inf \int_{t_0}^{\infty} \sum_{i=0}^{n} \lambda_i(t) L(t,x(t),u_i(t))\,dt \qquad (8)$$
over all trajectory-control pairs of
$$\begin{cases} x'(t) = \sum_{i=0}^{n} \lambda_i(t) f(t,x(t),u_i(t)) \\ u_i(t) \in U(t),\; \lambda_i(t) \ge 0,\; \sum_{i=0}^{n} \lambda_i(t) = 1 \\ x(t_0) = x_0, \end{cases} \qquad (9)$$
where $u_i(\cdot),\;\lambda_i(\cdot)$ are Lebesgue measurable on $\mathbb{R}_+$ for $i = 0,...,n$.

Clearly $V^{rel} \le V$. The following result can be proved by using finite horizon relaxation theorems and careful choice of relaxed controls. The detailed proof will appear elsewhere.

Theorem 1: Assume (H1) i)-v) with $\omega_R(t,r) = \bar{c}_R(t)r$, for a locally integrable $\bar{c}_R : \mathbb{R}_+ \to \mathbb{R}_+$, and that, for a.e. $t \in \mathbb{R}_+$ and all $x \in \mathbb{R}^n$, the set
$$\big\{\big(f(t,x,u), L(t,x,u)\big) \mid u \in U(t)\big\}$$
is compact. If for every $t \ge 0$, $V^{rel}(t,\cdot) : \mathbb{R}^n \to R$ is continuous, then $V^{rel} = V$ on $\mathbb{R}_+ \times \mathbb{R}^n$. In particular, if a trajectory-control pair $(\bar{x},\bar{u})$ is optimal for (6)-(7), then it is also optimal for the relaxed problem (8)-(9).

Remark 1: a) Notice that if $U(t)$ is compact for a.e. $t \ge 0$ and f, L are continuous with respect to u, then the above compactness assumption holds true.

b) Sufficient conditions for continuity of $V^{rel}(t,\cdot)$ can be deduced from Theorem 2 below.

c) Theorem 1 allows to avoid convexity requirement in assumption (H1) vi).

We provide next three results concerning continuity and local Lipschitz continuity of the value function. Their proofs are quite technically involved and will be published elsewhere. They are based on the dynamic programming, the Gronwall lemma and carefully chosen assumptions.

Theorem 2: Let (H1) hold with time independent $c(t) \equiv c \ge 0$ and $\theta(t) \equiv \theta \ge 0$. Assume also that $c_R(t) \equiv \delta \ge 0$ for every $R > 0$ and that, for all $x,\, y \in \mathbb{R}^n$, a.e. $t \ge 0$ and all $u \in U(t)$,
$$\begin{aligned} |L(t,y,u) - L(t,x,u)| \le \bar{\omega}(t,|x-y|)\times \\ \big[L(t,x,u) \wedge L(t,y,u) + h(t,|x| \vee |y|)\big], \end{aligned} \qquad (10)$$

where $\bar{\omega} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+,\ h : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ enjoy the following properties: $\bar{\omega},\ h$ are Lebesgue-Borel measurable, nondecreasing with respect to the second variable, $\bar{\omega}$ is bounded, $\lim_{r\to 0+} \bar{\omega}(t,r) = 0$ for a.e. $t > 0$ and

$$\int_0^\infty h(t,(R+\theta t)e^{ct})dt < +\infty \quad \forall\, R \geq 0. \qquad (11)$$

If $\mathrm{dom(V)} \neq \emptyset$, then $\mathrm{dom(V)} = \mathbb{R}_+ \times \mathrm{R^n}$ and $V(t,\cdot)$ is continuous for any $t \geq 0$.

To prove the above theorem it is enough to show that (10), (11) yield upper semicontinuity of $V(t,\cdot)$.

Under further restrictions, V turns out to be locally Lipschitz.

Lemma 1: Assume (H1) with time independent $c(t) \equiv c \geq 0$, $\theta(t) \equiv \theta \geq 0$, $c_R(t) \equiv \delta \geq 0$ for every $R > 0$ and suppose that, for all $x,\, y \in \mathbb{R}^n$, a.e. $t \geq 0$ and all $u \in U(t)$

$$|L(t,y,u) - L(t,x,u)| \leq k(t,|x| \vee |y|)|x-y|, \qquad (12)$$

where $k : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is Lebesgue-Borel measurable, $k(t,\cdot)$ is nondecreasing, and

$$\int_0^\infty e^{\delta t}k(t,(R+\theta t)e^{ct})dt < +\infty \quad \forall\, R \geq 0. \qquad (13)$$

If $\mathrm{dom(V)} \neq \emptyset$, then $V(t,\cdot)$ is locally Lipschitz continuous on $\mathbb{R}^n$ for all $t \geq 0$ and for all $x_1,\ x_2 \in B(0,R)$

$$|V(t,x_2) - V(t,x_1)| \leq e^{-\delta t}\, K_t(R)\, |x_2 - x_1| \qquad (14)$$

where for all $t \geq 0$

$$K_t(R) := \int_t^\infty e^{\delta\tau} k\big(\tau, M_t(\tau,R)\big)d\tau < +\infty \qquad (15)$$

and $M_t(\tau,R) := [R + \theta(\tau - t)]e^{c(\tau-t)}$.

Corollary 1: Under the assumptions of Lemma 1, fix any $(t_0,x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$ and let $(\bar{x},\bar{u})$ be any trajectory-control pair satisfying $\bar{x}(t_0) = x_0$. Then for all $t \geq t_0$ and $x_1, x_2 \in B(\bar{x}(t),1)$ we have

$$|V(t,x_2) - V(t,x_1)| \leq e^{-\delta t}\, K_{t_0}(1 + |x_0|)\, |x_2 - x_1|. \qquad (16)$$

Theorem 3: Under all the assumptions of Lemma 1, suppose that $\beta(\cdot)$ is nonincreasing. If $\mathrm{dom(V)} \neq \emptyset$, then $\mathrm{dom(V)} = \mathbb{R}_+ \times \mathrm{R^n}$, V is locally Lipschitz continuous, and for every $T \geq t \geq 0$, $R > 0$ and for all $t_1, t_2 \in [t,T]$, $x_1, x_2 \in B(0,R)$,

$$|V(t_2,x_2) - V(t_1,x_1)| \leq$$
$$e^{-\delta t}\, K_t(R)\, |x_2 - x_1| + N_t(T,R)|t_2 - t_1|,$$

where

$$N_t(T,R) = e^{-\delta t}K_t(R)\big[\,\theta + c\, M_t(T,R) + \beta(t)\phi\big(M_t(T,R)\big)\,\big].$$

Remark 2: As in Corollary 1, even the Lipschitz constant of V with respect to time and space can be estimated along any trajectory-control pair. More precisely, for any given trajectory-control pair $(\bar{x},\bar{u})$ with $\bar{x}(t_0) = x_0$ and all $(t_i,x_i) \in \mathbb{R}_+ \times \mathbb{R}^n$ $(i = 1,2)$ satisfying $|x_i - \bar{x}(t)| \leq 1$ and $t \leq t_i \leq t+1$ for some $t \geq t_0$, we have that

$$\begin{aligned}
&|V(t_2,x_2) - V(t_1,x_1)| \\
&\quad\leq \quad e^{-\delta t}\, K_{t_0}(1 + |x_0|)\{|x_2 - x_1| + \\
&\quad[\theta \quad + c\, M_t(t+1,R_t) + \beta(t)\phi(M_t(t+1,R_t))]|t_2 - t_1|\},
\end{aligned}$$

where $R_t = 1 + |\bar{x}(t)| \leq 1 + M_{t_0}(t,|x_0|)$. Since

$$\begin{aligned}
M_t(t+1,R_t) &\leq \big[\,1 + \theta + M_{t_0}(t,|x_0|)\,\big]e^c \\
&\leq \big[\,1+\theta\,\big]e^c + \big[\,|x_0| + \theta(t - t_0)\,\big]\, e^{c(t-t_0+1)},
\end{aligned}$$

it is easy to realise that if $\delta > c$, then the Lipschitz constant of V at $(t,\bar{x}(t))$ when $t \to \infty$ can be estimated from the above upon the behaviour at infinity of the function $t \mapsto e^{-\delta t}\,\beta(t)\,\phi\big(t(1+\theta)\,e^{ct}\big)$.

Example 1: Given $\lambda > 0$, a closed nonempty set $U \subset \mathbb{R}^m$, $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $\ell : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}_+$, consider the classical infinite horizon problem: (1)-(2) Assume that f satisfies assumptions **(H1)** with time independent $c(t) \equiv c$, $\theta(t) = \theta$ and $c_R(t) \equiv \delta$ for all $R > 0$. Suppose in addition that for all $u \in U$, $\ell(0,u) \leq M$ for some $M \geq 0$ and

$$|\ell(x,u) - \ell(y,u)| \leq C[1 + (|x| \vee |y|)^r]|x-y|$$

for some constants $C, r \geq 0$. Then, taking

$$\beta(t) = e^{-\lambda t} \quad \text{and} \quad \phi(s) = M + C\big[1 + s^r\big] \ \ \forall\, x,y \in \mathbb{R}^n,$$

it is easy to check that Theorem 3 can be applied provided that

$$\lambda > \delta + c\, r. \qquad (17)$$

Indeed, (12) holds true with $k(t,s) = Ce^{-\lambda t}(1 + s^r)$, which in turn satisfies (13) owing to (17). The above assumptions were used in [4] to study analogous problems for linear f and convex compact U. In particular, if

$$\theta := \sup_{u\in U} \ell(0,u) + \sup_{x\in\mathbb{R}^n,\, u\in U} |f(x,u)| < \infty$$

and there exist $0 \leq \delta < \lambda$, $\delta_1 \geq 0$ such that $f(\cdot,u)$ and $\ell(\cdot,u)$ are resp. $\delta-$Lipschitz and δ_1-Lipschitz for every $u \in U$, then, taking $k(t,s) = e^{-\lambda t}\delta_1 s,\ c = 0,\ \phi(s) = \theta + \delta_1 s$, we deduce from Theorem 3 that the value function V satisfies

$$|V(t_2,x_2) - V(t_1,x_1)| \leq e^{-\delta t}\, K_t(R)\, |x_2 - x_1| +$$
$$e^{-\delta t}\, K_t(R)[\theta + e^{-\lambda t}(\theta + \delta_1(R + \theta(T-t)))]|t_2 - t_1|$$

for all $x_1,\ x_2 \in B(0,R)$ and $t_1,\ t_2 \in [t,T]$, where

$$K_t(R) = \delta_1 \int_t^\infty e^{(\delta-\lambda)\tau}(R + \theta(\tau - t))d\tau \leq K_0(R).$$

Notice that we can always suppose that $\delta > 0$. Remark 2 implies then that for any trajectory $\bar{x}$ of the control system, Lipschitz constant of V at $(t,\bar{x}(t))$ decreases exponentially to zero when $t \to \infty$.

IV. Maximum Principles

We shall need the following assumption :

(H2) For every $R > 0$, there exists a locally integrable function $\alpha_R : \mathbb{R}_+ \to \mathbb{R}_+$ such that for a.e. $t \in \mathbb{R}_+$ and all $x, y \in B(0,R)$,

$$|L(t,x,u) - L(t,y,u)| \leq \alpha_R(t)|x-y|, \quad \forall\, u \in U(t).$$

The Hamiltonian $H : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \to R$ is defined by

$$H(t,x,p) := \sup_{u\in U(t)} (\langle p, f(t,x,u)\rangle - L(t,x,u)).$$

Then $H(t, x, \cdot)$ is convex and under assumptions (H1), (H2) for a.e. $t \geq 0$, the supremum is attained for all $(x, p) \in \mathbb{R}^n \times \mathbb{R}^n$. Moreover $H(t, \cdot, p)$ is Lipschitz on $B(0, R)$ with Lipschitz constant $c_R(t)|p| + \alpha_R(t)$.

A. Sensitivity Relation for LSC Value Function

If $f(t, \cdot, u)$ and $L(t, \cdot, u)$, are differentiable, denote $D_x f$ and L_x their (partial) Jacobian and gradient with respect to x. Recall that if $V(t, \cdot)$ is lower semicontinuous, then $\partial_x^- V(t, x_0) \neq \emptyset$ on a dense subset of points x_0 of $dom(V(t, \cdot))$, where $\partial_x^- V(t, x_0)$ denotes the partial Fréchet subdifferential of $V(t, \cdot)$ at x_0.

Theorem 4: Assume (H1), (H2) and let $(\bar{x}, \bar{u})$ be optimal at $(t_0, x_0) \in dom(V)$ with $\partial_x^- V(t_0, x_0) \neq \emptyset$. If $f(t, \cdot, u)$ and $L(t, \cdot, u)$ are differentiable for a.e. $t \in \mathbb{R}_+$ and all $u \in U(t)$, then for every $p_0 \in \partial_x^- V(t_0, x_0)$ the solution $p(\cdot)$ of the adjoint system

$$\begin{cases} -p'(t) = D_x f(t, \bar{x}(t), \bar{u}(t))^* p(t) - L_x(t, \bar{x}(t), \bar{u}(t)) \\ p(t_0) = -p_0 \end{cases}$$

satisfies the maximality condition

$$\langle p(t), f(t, \bar{x}(t), \bar{u}(t))\rangle - L(t, \bar{x}(t), \bar{u}(t)) = H(t, \bar{x}(t), p(t)) \tag{18}$$

a.e. in $[t_0, +\infty[$ and the sensitivity relation

$$-p(t) \in \partial_x^- V(t, \bar{x}(t)) \quad \forall\, t \geq t_0.$$

Furthermore, if assumptions of Lemma 1 hold true with $\delta > 0$, then $p(t)$ converges exponentially to zero when $t \to \infty$.

Proof: We use the approach developed in [9] for the finite horizon Bolza problem and the uniqueness of solution of the adjoint system. Details will appear elsewhere. ■

The above necessary optimality condition holds true only when the subdifferential $\partial_x^- V(t_0, x_0)$ is nonempty and involves elements of this subdifferential. An alternative way is discussed in the next section, but before, for the sake of completeness, we state one more result that follows directly from nonsmooth maximum principles for the finite horizon Bolza problem, but does not include any transversality condition and may be abnormal.

Below, let $\partial_x f$ denote the partial generalized Jacobian of f with respect to x and $\partial_x L$ the partial generalized gradient of L with respect to x.

Theorem 5: Assume (H1), (H2) and let $(\bar{x}, \bar{u})$ be optimal at some $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$. Then

i) either there exists a solution $p(\cdot)$ of the adjoint inclusion

$$-p'(t) \in \partial_x f(t, \bar{x}(t), \bar{u}(t))^* p(t) - \partial_x L(t, \bar{x}(t), \bar{u}(t)) \tag{19}$$

satisfying the maximality condition (18) a.e. in $[t_0, +\infty[$;

ii) or there exists a nonvanishing solution $p(\cdot)$ of the adjoint inclusion

$$-p'(t) \in \partial_x f(t, \bar{x}(t), \bar{u}(t))^* p(t) \tag{20}$$

satisfying a.e. in $[t_0, +\infty[$ the abnormal maximum principle

$$\langle p(t), f(t, \bar{x}(t), \bar{u}(t))\rangle = \max_{u \in U(t)} \langle p(t), f(t, \bar{x}(t), u)\rangle. \tag{21}$$

Moreover, if for all large $t > t_0$, $V(t, \cdot)$ is locally Lipschitz at $\bar{x}(t)$ with a Lipschitz constant independent from t, then the normal maximum principle $i)$ holds true.

Furthermore, if there exist integrable mappings $C_f : \mathbb{R}_+ \to \mathbb{R}_+$ and $C_L : \mathbb{R}_+ \to \mathbb{R}_+$ such that for a.e. $t \geq 0$ and all $u \in U(t)$, $f(t, \cdot, u)$ is $C_f(t)$-Lipschitz and $L(t, \cdot, u)$ is $C_L(t)$-Lipschitz, then $\lim_{t\to\infty} p(t)$ does exist.

Proof: The proof is a bit lengthly so we only indicate the main idea. The complete proof will appear elsewhere. For every integer $i \geq t_0$ we introduce a new finite horizon Bolza problem

$$\text{minimize } \{V(i, x(i)) + \int_{t_0}^{i} L(t, x(t), u(t))\, dt \mid \ u \in \mathcal{U}\}$$

over all trajectory-control pairs (x, u) of (7). Observe that the restriction of $(\bar{x}, \bar{u})$ to the time interval $[t_0, i]$ is optimal for this new Bolza problem and $V(i, \cdot)$ is lower semicontinuous. So we can write a, possibly abnormal, nonsmooth maximum principle with a costate p_i satisfying on $[t_0, i]$ the conclusions of theorem. For this aim we adapt [19, Theorem 6.2.1] to our framework. If $\{p_i(t_0)\}_i$ contains a bounded subsequence, then it is enough to extract a subsequence from $\{p_i\}_i$ converging almost uniformly to some p satisfying $i)$ or $ii)$. Otherwise, replacing p_i by $p_i/|p_i(t_0)|$, we take a subsequence converging almost uniformly to some p as in $ii)$. The last statement results from the Filippov theorem, [5]. ■

The above theorem lacks the transversality condition and also it does not exclude the abnormality of the maximum principle. For this reason its conclusion is less informative than the one of Theorem 4.

B. Transversality Condition for Continuous Value Function

Actually, if $V(t_0, \cdot)$ is continuous, then a maximum principle holds true at all points $x_0 \in \mathbb{R}^n$.

Theorem 6: Let (H1), (H2) hold and $(t_0, x_0) \in dom(V)$. Assume that $V(t_0, \cdot)$ is continuous on a neighborhood of x_0 and let $(\bar{x}, \bar{u})$ be optimal at (t_0, x_0). Then

i) either there exists a solution $p(\cdot)$ of (19) satisfying the transversality condition

$$-p(t_0) \in \partial_x^{L,+} V(t_0, x_0)$$

and the maximality condition (18) a.e. in $[t_0, +\infty[$;

ii) or there exists a nonvanishing solution $p(\cdot)$ of (20) satisfying the abnormal maximum principle (21) a.e. in $[t_0, +\infty[$ and the transversality condition

$$-p(t_0) \in \partial_x^{\infty,+} V(t_0, x_0).$$

Proof: For any integer $i \geq t_0$, consider the problem

$$\text{minimize}\{\int_{t_0}^{i} L(t, x(t), u(t))\, dt - V(t_0, x(t_0))\}$$

over all trajectory-control pairs (x, u) of

$$\begin{cases} x'(t) = f(t, x(t), u(t)) & \text{for a.e. } t \in [t_0, i] \\ x(i) = \bar{x}(i). \end{cases}$$

Since $-V(t_0, \cdot)$ is continuous near x_0 we can write a, possibly abnormal, nonsmooth maximum principle on $[t_0, i]$

to get a costate p_i. The proof ends by applying the same arguments as those of the proof of Theorem 5. ■

C. Sensitivity Relations for Locally Lipschitz Value Function

We show here that if the value function is locally Lipschitz with respect to the second variable, then *every* optimal trajectory-control pair of the infinite horizon problem satisfies the maximum principle and a sensitivity relation involving the partial generalized gradient of the value function.

Theorem 7: Assume (H1) $i) - v)$, (H2) and that for all large $T > 0,$ the mapping $V(T, \cdot)$ is locally Lipschitz. Then for every $t \geq 0,\ V(t, \cdot)$ is locally Lipschitz with the local Lipschitz constants depending only on the magnitude of t.

Moreover, if $(\bar{x}, \bar{u})$ is optimal at some $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$, then there exists a solution $p(\cdot)$ of the adjoint inclusion (19), satisfying the maximality condition (18) a.e. in $[t_0, +\infty[$ and the sensitivity relations

$$-p(t_0) \in \partial_x V(t_0, x_0),\ -p(t) \in \partial_x V(t, \bar{x}(t)) \text{ for a.e. } t > t_0.$$

Furthermore, if assumptions of Lemma 1 hold true with $\delta > 0$, then $p(t)$ converges exponentially to zero when $t \to \infty$.

Proof: Consider the finite horizon problems as in the proof of Theorem 5. It is enough to apply [11] to get costates p_i satisfying the maximum principle and sensitivity relations as above on $[t_0, i]$. Taking the limits for an appropriately chosen subsequence of $\{p_i\}_i$ ends the proof. ■

The next result, whose proof is similar to the above, but uses [6] instead of [11], involves a different adjoint inclusion and provides one more sensitivity relation. It helps to study the limit at infinity of $H(t, \bar{x}(t), p(t))$, whenever $\bar{x}$ is an optimal trajectory and p is a corresponding costate.

Theorem 8: Assume (H1) i)-v), (H2) with bounded $c(\cdot),\ \theta(\cdot),\ \beta(\cdot)$ and that for every $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$ the set $\{(f(t, x, u), L(t, x, u)) : u \in U(t)\}$ is closed. If for all large $T > 0,\ V(T, \cdot)$ is locally Lipschitz, then V is locally Lipschitz on $[0, \infty[\times \mathbb{R}^n$.

Let $(\bar{x}, \bar{u})$ be optimal at some $(t_0, x_0) \in \mathbb{R}_+ \times \mathbb{R}^n$. Then there exists a locally absolutely continuous function $p : [t_0, +\infty[\to \mathbb{R}^n$ satisfying the Hamiltonian inclusion

$$(-p, \bar{x})'(t) \in \partial_{(x,p)} H(t, \bar{x}(t), p(t)), \quad \text{for a.e. } t \geq t_0, \tag{22}$$

the transversality condition

$$-p(t_0) \in \partial_x^{L,+} V(t_0, x_0), \tag{23}$$

and the two sensitivity relations

$$-p(t) \in \partial_x V(t, \bar{x}(t)) \quad \text{a.e. } t > t_0\ ; \tag{24}$$

$$(H(t, \bar{x}(t), p(t)), -p(t)) \in \partial V(t, \bar{x}(t)) \quad \text{a.e. } t > t_0. \tag{25}$$

Furthermore, if assumptions of Lemma 1 hold true with $\delta > 0$, then $p(t)$ converges exponentially to zero when $t \to \infty$. Moreover if assumptions of Theorem 3 are satisfied, $\delta > c$ and

$$ess - lim_{t \to \infty} e^{-\delta t} \beta(t) \phi(t(1 + \theta) e^{ct}) = 0, \quad \forall\ R > 0,$$

then also ess-lim$_{t \to \infty} H(t, \bar{x}(t), p(t)) = 0$, where ess-lim denotes the essential limit.

V. CONCLUSIONS AND FUTURE WORK

We have studied the value function of the infinite horizon problem without a discount factor and provided sufficient conditions for its lower semicontinuity, continuity and local Lipschitz continuity. In this very general framework we derived maximum principle and sensitivity relations that greatly simplify the investigation of normality of the maximum principle and of the behaviour at infinity of the costate. The direct approach we have developed allows to generalize several earlier results and links maximum principle to dynamic programming in a very general framework. Our future work will concern the infinite horizon optimal control problem under state constraints, useful in many applied models.

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