# Decentralized Control of Stochastically Switched Linear System with Unreliable Communication 

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#### Abstract

We consider a networked control system (NCS) consisting of two plants, a global plant and a local plant, and two controllers, a global controller and a local controller. The global (resp. local) plant follows discrete-time stochastically switched linear dynamics with a continuous global (resp. local) state and a discrete global (resp. local) mode. We assume that the state and mode of the global plant are observed by both controllers while the state and mode of the local plant are only observed by the local controller. The local controller can inform the global controller of the local plant's state and mode through an unreliable TCP-like communication channel where successful transmissions are acknowledged. The objective of the controllers is to cooperatively minimize a modes-dependent quadratic cost over a finite time horizon. Following the method developed in [1] and [2], we construct a dynamic program based on common information and a decomposition of strategies, and use it to obtain explicit optimal strategies for the controllers. In the optimal strategies, both controllers compute a common estimate of the local plant's state. The global controller's action is linear in the state of the global plant and the common estimated state, and the local controller's action is linear in the actual states of both plants and the common estimated state. Furthermore, the gain matrices for the global controller depend on the global mode and its observation about the local mode, while the gain matrices for the local controller depend on the actual modes of both plants and the global controller's observation about the local mode.


## I. Introduction

The widespread applications of decentralized control in networked control systems (NCSs), power systems, smart buildings, autonomous vehicles and economic models have made it a topic of interest in the recent years [3-5]. Despite increased efforts in advancing this field, decentralized control still remains challenging with a broad range of problems to be investigated. One class of such problems is the control of switched systems where switching is governed by stochastic parameters that are partially observed by the individual decision makers. The stochastic parameters can represent local and global system conditions like the state of communication links among the controllers, mission objectives, physical conditions, and changes in the system environment.

While centralized control of switched systems has been extensively addressed by several studies [6-11], the decentralized counterpart has been the focus of relatively sparse studies [12-14]. This is due to the fact that decentralized
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control problems are generally difficult to solve (see [1517]). Not only are linear strategies suboptimal for such problems, but the problem of finding the best linear strategies may not be convex [18]. Existing methods for solving decentralized control problems require either specific information structures, such as static [19], partially nested [20-25], stochastically nested [26], switched partially nested [27] or other specific properties, such as quadratic invariance [28] or substitutability [29], [30] which make the decentralized control problems "simpler" than the general problems.
In this paper, we consider a NCS with discrete-time stochastically switched linear dynamics. The NCS includes two plants, called "global" plant and "local" plant, and two controllers, namely global controller $C^{0}$ and local controller $C^{1}$ as shown in Fig. 1 Associated with the global (resp. local) plant, there is a continuous global (resp. local) state and a discrete global (resp. local) mode. The discrete global (resp. local) mode allows us to model non-linear dynamics such as abrupt environmental disturbances, component failures or repairs, changes in subsystems interconnections, and abrupt changes in the operation point [31].

We assume that the control action of the global controller $C^{0}$ can affect both plants while the control action of the local controller $C^{1}$ can only affect the local plant as its name suggests. We assume that the state and mode of global plant are observed by both controllers while the state and mode of local plant is only observed by the local controller $C^{1}$. In addition to the observations, controller $C^{1}$ can inform controller $C^{0}$ of the local plant's state and mode through a communication channel with random packet drops. We assume a TCP-like protocol [32] where successful transmissions of packets are acknowledged by controller $C^{0}$. The objective of the controllers is to cooperatively minimize a modes-dependent quadratic cost over a finite time horizon. The dependence of cost on the global and local modes allows us to model mode-dependent changes in the control objective.

This stochastically switched system model can arise in various NCS applications. For example, the operation of a service robot in a smart home can be modeled by the NCS where the robot is the local plant and the smart home is the global plant. The global controller can be the HVAC (heating, ventilation, and air conditioning) system that controls the global state and mode including temperature and indoor air quality, while the local controller directly controls the robot with switched linear dynamics. Depending on the local situation, the robot can transmit information through wireless communication to the global controller to
request adjustments on certain comfort parameters of the smart home.

The decentralized control problem we consider in this paper does not belong to the simpler classes mentioned earlier due to either the unreliable communication or the switching dynamics and cost function. A closely related problem has been studied in [14]; however; the information structure there is partially nested. We developed a method to obtain optimal controllers for a decentralized control problem with unreliable communication in [1], [2], [33] using ideas from the common information approach [34]. More specifically, we proposed a modified dynamic program based on a decomposition of strategies which can be explicitly solved to find the optimal controllers for the problems of [1], [2]. Although this method fails to provide any structure for the dynamic program of the switched system considered in this paper, we show that it can be generalized to capture the switching nature of system dynamics of the plants. Using this method, we obtain explicit optimal strategies for the controllers. In the optimal strategies, both controllers compute a common estimate of the local plant's state. Global controller $C^{0}$,s action is linear in the state of global plant and the common estimated state, and the local controller $C^{1}$ 's action is linear in both the actual states of plants and the common estimated state. Furthermore, the gain matrix for controller $C^{0}$ depends on the global mode and its observation about the local mode, while the gain matrices for controller $C^{1}$ depend on the actual global and local modes of both plants and controller $C^{0}$ 's observation about the local mode.

## A. Notation

Random variables/vectors are denoted by upper case letters, their realization by the corresponding lower case letters. For a sequence of column vectors $X, Y, Z, \ldots$, the notation $\operatorname{vec}(X, Y, Z, \ldots)$ denotes vector $\left[X^{\top}, Y^{\top}, Z^{\top}, \ldots\right]^{\top}$. The transpose and trace of matrix $A$ are denoted by $A^{\top}$ and $\operatorname{tr}(A)$, respectively. Uppercase letters are also used to denote matrices. Matrix dimensions are not specified if they can be inferred from the context. The notation $\mathbf{I}_{n}$ and $\mathbf{0}_{n \times m}$ is used to denote a $n \times n$ identity matrix and a $n \times m$ zero matrix, respectively. For block matrix $B$ and $r=0,1,[B]_{r} \bullet$ denotes the $r$-th block row of $B$. Note that the top block row corresponds to $r=0$ and the bottom corresponds to $r=1$. For example, for $B=\left[\begin{array}{cc}\mathbf{I}_{m} & \mathbf{0}_{m \times n} \\ \mathbf{0}_{n \times m} & \mathbf{I}_{n}\end{array}\right],[B]_{0 \bullet}=$ $\left[\begin{array}{ll}\mathbf{I}_{m} & \mathbf{0}_{m \times n}\end{array}\right]$ and $[B]_{1 \bullet}=\left[\begin{array}{ll}\mathbf{0}_{n \times m} & \mathbf{I}_{n}\end{array}\right]$.

In general, subscripts are used as time index while superscripts are used to index controllers. For time indices $t_{1} \leq t_{2}, X_{t_{1}: t_{2}}$ (resp. $g_{t_{1}: t_{2}}(\cdot)$ ) is the short hand notation for the variables $\left(X_{t_{1}}, X_{t_{1}+1}, \ldots, X_{t_{2}}\right)$ (resp. functions $\left.\left(g_{t_{1}}(\cdot), \ldots, g_{t_{1}}(\cdot)\right)\right)$. Similarly, $X^{0: 1}$ is the shorthand notation for the variables $X^{0}, X^{1} . \mathbb{P}(\cdot), \mathbb{E}[\cdot]$, and $\boldsymbol{\operatorname { c o v }}(\cdot)$ denote the probability of an event, the expectation of a random variable/vector, and the covariance matrix of a random vector, respectively. For random variables/vectors $X$ and $Y$, $\mathbb{E}[X \mid y]:=\mathbb{E}[X \mid Y=y]$. For a strategy $g$, we use $\mathbb{P}^{g}(\cdot)$ (resp. $\left.\mathbb{E}^{g}[\cdot]\right)$ to indicate that the probability (resp. expectation)


Fig. 1. Two-controller system model. The binary random variable $\Gamma_{t}$ indicates whether packets are transmitted successfully. Solid lines indicate communication links, dashed lines indicate control links, and the dash-dot line indicates that the global state and mode can affect the local state.
depends on the choice of $g$. Let $\Delta\left(\mathbb{R}^{n}\right)$ denote the set of all probability measures on $\mathbb{R}^{n}$ with finite second moment. For any $\theta \in \Delta\left(\mathbb{R}^{n}\right), \theta(E)=\int_{\mathbb{R}^{n}} \mathbb{1}_{E}(x) \theta(d x)$ denotes the probability of event $E$ under $\theta$. The mean and the covariance of a distribution $\theta \in \Delta\left(\mathbb{R}^{n}\right)$ are denoted by $\mu(\theta)$ and $\operatorname{cov}(\theta)$, respectively, and are defined as $\mu(\theta)=\int_{R^{n}} x \theta(d x)$ and $\operatorname{cov}(\theta)=\int_{R^{n}}(x-\mu(\theta))(x-\mu(\theta))^{\top} \theta(d x)$.

## B. Organization

The rest of the paper is organized as follows. We introduce the system model and formulate the two-controller optimal control problem in Section [II In Section III following the common information approach, we provide a dynamic program for solving this problem. Section IV introduces a decomposition for the control strategies and provides a modified dynamic program based on this decomposition. We solve the modified dynamic program in Section $\nabla$ and provide the optimal strategies for the controllers. Section VI concludes the paper. The proofs of all the technical results of the paper appear in the Appendices.

## II. System Model and Problem Formulation

Consider the discrete-time switched linear system with two plants and two associated controllers shown in Fig. 1 The two-plant system follows the switched linear dynamics described by

$$
\left[\begin{array}{l}
X_{t+1}^{0}  \tag{1}\\
X_{t+1}^{1}
\end{array}\right]=A\left(M_{t}^{0: 1}\right)\left[\begin{array}{c}
X_{t}^{0} \\
X_{t}^{1}
\end{array}\right]+B\left(M_{t}^{0: 1}\right)\left[\begin{array}{c}
U_{t}^{0} \\
U_{t}^{1}
\end{array}\right]+\left[\begin{array}{l}
W_{t}^{0} \\
W_{t}^{1}
\end{array}\right]
$$

for $t=0, \ldots, T$ where $X_{t}^{n} \in \mathbb{R}^{d_{X}^{n}}$ is the continuous state and $M_{t}^{n} \in \mathcal{M}^{n}=\left\{1, \ldots, \kappa^{n}\right\}$ with $\kappa^{n}<\infty$ is the discrete mode of plant $n$ for $n=0,1 . U_{t}^{n} \in \mathbb{R}^{d_{U}^{n}}$ is the control action of controller $C^{n}$ and $W_{t}^{n}$ is a zero mean noise vector. We assume that the collection of random variables $X_{0}^{n}, W_{t}^{n}, M_{t}^{n}$ for $n=0,1, t=0,1, \ldots, T$, are independent and with distributions $\pi_{X_{0}^{n}}$ and $\pi_{W_{t}^{n}}, \pi_{M^{n}}$, respectively. We further assume that $X_{0}^{0: 1}, W_{0: T}^{0: 1}$ have finite second moments.

The system matrices $A\left(M_{t}^{0: 1}\right)$ and $B\left(M_{t}^{0: 1}\right)$ depend on the system modes $M_{t}^{0}$ and $M_{t}^{1}$. As illustrated in Fig. 1 state and mode of plant 0 and the action of controller $C^{0}$
can affect plant 1 , but state and mode of plant 1 and the action of controller $C^{1}$ do not affect plant 0 .

$$
\begin{align*}
A\left(M_{t}^{0: 1}\right) & =\left[\begin{array}{cc}
A^{00}\left(M_{t}^{0}\right) & \mathbf{0} \\
A^{10}\left(M_{t}^{0: 1}\right) & A^{11}\left(M_{t}^{0: 1}\right)
\end{array}\right],  \tag{2}\\
B\left(M_{t}^{0: 1}\right) & =\left[\begin{array}{cc}
B^{00}\left(M_{t}^{0}\right) & \mathbf{0} \\
B^{10}\left(M_{t}^{0: 1}\right) & B^{11}\left(M_{t}^{0: 1}\right)
\end{array}\right] . \tag{3}
\end{align*}
$$

Since $M_{t}^{0}$ affects both plants and $M_{t}^{1}$ only affects plant 1 , we refer to $M_{t}^{0}$ as the global mode and to $M_{t}^{1}$ as the local mode. For notational simplicity, we denote $X_{t}=$ $\operatorname{vec}\left(X_{t}^{0: 1}\right), U_{t}=\operatorname{vec}\left(U_{t}^{0: 1}\right), W_{t}=\operatorname{vec}\left(W_{t}^{0: 1}\right)$, and $S_{t}=$ $\operatorname{vec}\left(X_{t}^{0: 1}, U_{t}^{0: 1}\right)$. Then, the system dynamics can be expressed as $X_{t+1}=D\left(M_{t}^{0: 1}\right) S_{t}+W_{t}$ where

$$
\begin{equation*}
D\left(M_{t}^{0: 1}\right)=\left[A\left(M_{t}^{0: 1}\right) \quad B\left(M_{t}^{0: 1}\right)\right] \tag{4}
\end{equation*}
$$

At each time $t$, the state $X_{t}^{0}$ and mode $M_{t}^{0}$ of the global plant are observed by both controllers $C^{0}$ and $C^{1}$ while the state $X_{t}^{1}$ and mode $M_{t}^{1}$ of the local plant is only observed by controller $C^{1}$. Controller $C^{1}$ can inform controller $C^{0}$ of the local plant's state and mode through an unreliable link with random packet drops. Let $\Gamma_{t}$ be a Bernoulli random variable describing the state of this link, that is, $\Gamma_{t}=0$ when the link is broken and otherwise, $\Gamma_{t}=1$. We denote $p(i):=\mathbb{P}\left(\Gamma_{t}=i\right)$ for $i \in\{0,1\}$. We assume that $\Gamma_{0: T}$ are independent random variables and they are independent of $X_{0}^{0: 1}, W_{0: T}^{0: 1}, M_{0: T}^{0: 1}$. Furthermore, let $\left(Z_{t}, \tilde{Z}_{t}\right)$ be the channel output, that is,

$$
\begin{align*}
& Z_{t}= \begin{cases}X_{t}^{1} & \text { when } \Gamma_{t}=1 \\
\emptyset & \text { when } \Gamma_{t}=0\end{cases}  \tag{5}\\
& \tilde{Z}_{t}= \begin{cases}M_{t}^{1} & \text { when } \Gamma_{t}=1 \\
\emptyset & \text { when } \Gamma_{t}=0\end{cases} \tag{6}
\end{align*}
$$

We assume that the channel outputs $Z_{t}$ and $\tilde{Z}_{t}$ are perfectly observed by controller $C^{0}$. The successful transmissions of packets is acknowledged by the controller $C^{0}$ (see, for example, TCP-like protocols [32]). Thus, effectively, $Z_{t}$ and $\tilde{Z}_{t}$ are perfectly observed by controller $C^{1}$ as well.
Both controllers select their control actions at time $t$ after observing $Z_{t}$ and $\tilde{Z}_{t}$. We assume that the links from controllers $C^{0}$ and $C^{1}$ to the plants are perfect.

Let $H_{t}^{n}$ denote the information available to controller $C^{n}$, $n \in\{0,1\}$ to make decisions at time $t$. Then,

$$
\begin{align*}
& H_{t}^{0}=\left\{X_{0: t}^{0}, M_{0: t}^{0}, Z_{0: t}, \tilde{Z}_{0: t}, U_{0: t-1}^{0}\right\} \\
& H_{t}^{1}=H_{t}^{0} \cup\left\{X_{t}^{1}, M_{t}^{1}\right\} . \tag{7}
\end{align*}
$$

Let $\mathcal{H}_{t}^{n}$ be the space of all possible realizations of $H_{t}^{n}$, $n \in\{0,1\}$. Then, $C^{n}$ 's actions are selected according to

$$
\begin{equation*}
U_{t}^{n}=\lambda_{t}^{n}\left(H_{t}^{n}\right), \quad n \in\{0,1\} \tag{8}
\end{equation*}
$$

where $\lambda_{t}^{n}: \mathcal{H}_{t}^{n} \rightarrow \mathbb{R}^{d_{U}^{n}}$ is a Borel measurable mapping. The collection of mappings $\lambda_{0}^{n}, \ldots, \lambda_{T}^{n}$ is called the strategy of controller $C^{n}$ and is denoted by $\lambda^{n}$. The collection of both controllers' strategies, $\lambda^{0: 1}$, is called the strategy profile.

[^0]The instantaneous cost $c_{t}\left(X_{t}^{0: 1}, M_{t}^{0: 1}, U_{t}^{0: 1}\right)$ of the system is a quadratic function given by

$$
\begin{equation*}
c_{t}\left(X_{t}^{0: 1}, M_{t}^{0: 1}, U_{t}^{0: 1}\right)=X_{t}^{\top} Q_{t}\left(M_{t}^{0: 1}\right) X_{t}+U_{t}^{\top} R_{t}\left(M_{t}^{0: 1}\right) U_{t} \tag{9}
\end{equation*}
$$

where $Q_{t}\left(m^{0: 1}\right)$ is a symmetric positive semi-definite (PSD) matrix and $R_{t}\left(m^{0: 1}\right)$ is a symmetric positive definite (PD) matrix for all $m^{0} \in \mathcal{M}^{0}$ and $m^{1} \in \mathcal{M}^{1}$.

The performance of strategies $\lambda^{0: 1}$ is the total expected cost given by

$$
\begin{equation*}
J\left(\lambda^{0: 1}\right)=\mathbb{E}^{\lambda^{0: 1}}\left[\sum_{t=0}^{T} c_{t}\left(X_{t}^{0: 1}, M_{t}^{0: 1}, U_{t}^{0: 1}\right)\right] \tag{10}
\end{equation*}
$$

Let $\Lambda^{0}$ and $\Lambda^{1}$ denote the set of all possible control strategies of $C^{0}$ and $C^{1}$, respectively, that ensure all random variables (state and control actions) have finite second moments. The optimal control problem is formally defined below.

Problem 1. For the system described by (10), we would like to solve the following strategy optimization problem,

$$
\begin{equation*}
\inf _{\lambda^{0} \in \Lambda^{0}, \lambda^{1} \in \Lambda^{1}} J\left(\lambda^{0: 1}\right) \tag{11}
\end{equation*}
$$

Problem 1 is a two-controller decentralized optimal control problem. However, we cannot a priori assume that linear control strategies are optimal in this problem because 1) the information structure is not partially nested [20], 2) the system dynamics given in (1) are not linear due to the presence of the local and global modes $M_{t}^{0: 1}$, 3) $X_{0}^{0: 1}$ and $W_{0: T}^{0: 1}$ are not necessarily Gaussian.

## III. EQuivalent Problem and Dynamic Program

According to (7), $H_{t}^{0}$ is the common information among the controllers $C^{0}$ and $C^{1}$. Using the common information approach [34], we formulate a centralized decision-making problem which can be used to find optimal strategies in the decentralized Problem 1. In this centralized problem, $C^{0}$ is the only decision-maker and at each time $t$, it makes two decisions given the realization $h_{t}^{0}$ :

1) $C^{0}$ 's control action $u_{t}^{0}=\phi_{t}^{0}\left(h_{t}^{0}\right)$,
2) A prescription $\rho_{t}$ for $C^{1}$ which is a Borel measurable mapping from $\mathbb{R}^{d_{X}^{1}} \times \mathcal{M}^{1}$ to $\mathbb{R}^{d_{U}^{1}}$. That is, $\rho_{t}=\phi_{t}^{1}\left(h_{t}^{0}\right)$ where $\rho_{t}$ belongs to the space $\mathcal{P}=\left\{\rho: \mathbb{R}^{d_{X}^{1}} \times \mathcal{M}^{1} \rightarrow\right.$ $\mathbb{R}^{d_{U}^{1}}$ such that $\rho$ is measurable $\}$.
For each realization of $C^{1}$,s private information $\left(x_{t}^{1}, m_{t}^{1}\right)$, the mapping $\rho_{t}$ prescribes an action $u_{t}^{1}=\rho_{t}\left(x_{t}^{1}, m_{t}^{1}\right)$.

We call $u_{t}^{p r s}=\left(u_{t}^{0}, \rho_{t}\right)$ the prescription at time $t$. We denote $\phi_{t}^{p r s}=\left(\phi_{t}^{0}, \phi_{t}^{1}\right)$ and write $u_{t}^{p r s}=\phi_{t}^{p r s}\left(h_{t}^{0}\right)$ to indicate that the prescription is a function of the common information $h_{t}^{0}$. The functions ( $\left.\phi_{t}^{p r s}, 0 \leq t \leq T\right)$ are collectively referred to as the prescription strategy and denoted by $\phi^{p r s}$. The prescription strategy is required to satisfy the following conditions: 1) $\phi^{0} \in \Lambda^{0}$, and 2) if we define $\varphi_{t}^{1}\left(X_{t}^{1}, M_{t}^{1}, H_{t}^{0}\right):=\left[\phi_{t}^{1}\left(H_{t}^{0}\right)\right]\left(X_{t}^{1}, M_{t}^{1}\right)$, then $\varphi^{1} \in \Lambda^{1}$ where the notation $\left[\phi_{t}^{1}\left(H_{t}^{0}\right)\right]\left(X_{t}^{1}, M_{t}^{1}\right)$ means that we first use $\phi_{t}^{1}\left(H_{t}^{0}\right)$ to find the mapping $\rho_{t}$ and then evaluate $\rho_{t}$ at $X_{t}^{1}, M_{t}^{1}$.

Denote by $\Phi^{\text {prs }}$ the set of all prescription strategies satisfying the above conditions. Consider the following optimization problem.
Problem 2 (Equivalent Centralized Problem). For the system described by (1)-(8), we would like to solve the following optimization problem,

$$
\begin{equation*}
\inf _{\phi^{p r s} \in \Phi{ }^{p r s}} \mathbb{E}^{\phi^{p r s}}\left[\sum_{t=0}^{T} c_{t}^{p r s}\left(X_{t}^{0: 1}, M_{t}^{0: 1}, U_{t}^{p r s}\right)\right] \tag{12}
\end{equation*}
$$

where for any $x_{t}^{0: 1}, m_{t}^{0: 1}$, and $u_{t}^{p r s}=\left(u_{t}^{0}, \rho_{t}\right)$,

$$
\begin{equation*}
c_{t}^{p r s}\left(x_{t}^{0: 1}, m_{t}^{0: 1}, u_{t}^{p r s}\right)=c_{t}\left(x_{t}^{0: 1}, m_{t}^{0: 1}, u_{t}^{0}, \rho_{t}\left(x_{t}^{1}, m_{t}^{1}\right)\right) \tag{13}
\end{equation*}
$$

Using arguments similar to [2], the solution to Problem 22 can be characterized by a Dynamic Program (DP). The information state for this DP consists of $x_{t}^{0}, m_{t}^{0}, \tilde{z}_{t}$, and the common belief on the state $X_{t}^{1}$ defined as follows: Under prescription strategies $\phi_{0: t-1}^{p r s} \in \Phi^{p r s}$ and for any measurable sets $E \subset \mathbb{R}^{d_{X}^{1}}$,

$$
\begin{equation*}
\theta_{t}(E):=\mathbb{P}^{\phi_{0: t-1}^{p r s}}\left(X_{t}^{1} \in E \mid h_{t}^{0}\right) \tag{14}
\end{equation*}
$$

This belief is sequentially updated for any realization $h_{t}^{0}$ according to

$$
\begin{equation*}
\theta_{t+1}=\psi_{t}\left(\theta_{t}, u_{t}^{p r s}, x_{t}^{0}, m_{t}^{0}, \tilde{z}_{t}, z_{t+1}\right) \tag{15}
\end{equation*}
$$

where $u_{t}^{p r s}=\left(u_{t}^{0}, \rho_{t}\right)=\phi_{t}^{p r s}\left(h_{t}^{0}\right)$ and $\psi_{t}$ is a fixed transformation which does not depend on the choice of prescription strategies ${ }^{2}$.

Let $\tilde{\mathcal{M}}^{1}=\mathcal{M}^{1} \cup\{\emptyset\}$. Then, the following theorem provides a DP characterization for the solution of Problem 2
Theorem 1. Suppose there exist functions $\left\{V_{t}: \mathbb{R}^{d_{X}^{0}} \times \mathcal{M}^{0} \times\right.$ $\Delta\left(\mathbb{R}^{d_{X}^{1}}\right) \times \tilde{\mathcal{M}}^{1} \rightarrow \mathbb{R}$ for $t=0,1, \ldots, T+1$ such that for each $x_{t}^{0} \in \mathbb{R}^{d_{X}^{0}}, m_{t}^{0} \in \mathcal{M}^{0}, \theta_{t} \in \Delta\left(\mathbb{R}^{d_{X}^{1}}\right)$, and $\tilde{z}_{t} \in \tilde{\mathcal{M}}^{1}$, the following are true:

- $V_{T+1}\left(x_{T+1}^{0}, m_{T+1}^{0}, \theta_{T+1}, \tilde{z}_{T+1}\right)=0$,
- For any $t=0,1, \ldots, T$

$$
\begin{align*}
& V_{t}\left(x_{t}^{0}, m_{t}^{0}, \theta_{t}, \tilde{z}_{t}\right)=\min \left\{\mathbb { E } \left[c_{t}^{p r s}\left(x_{t}^{0}, X_{t}^{1}, m_{t}^{0}, M_{t}^{1}, u_{t}^{p r s}\right)\right.\right. \\
& +V_{t+1}\left(X_{t+1}^{0}, M_{t+1}^{0}, \psi_{t}\left(\theta_{t}, u_{t}^{p r s}, x_{t}^{0}, m_{t}^{0}, \tilde{z}_{t}, Z_{t+1}\right), \tilde{Z}_{t+1}\right) \\
& \left.\left.\qquad x_{t}^{0}, m_{t}^{0}, \theta_{t}, \tilde{z}_{t}, u_{t}^{p r s}\right]\right\}, \tag{16}
\end{align*}
$$

where $X_{t}^{1}$ is a random vector distributed according to $\theta_{t}, u_{t}^{\text {prs }}=\left(u_{t}^{0}, \rho_{t}\right)$ and the minimization is over $u_{t}^{0} \in$ $\mathbb{R}^{d_{U}^{0}}, \rho_{t} \in \mathcal{P}$.
Further, suppose there exists a feasible prescription strategy $\phi^{\text {prs* }} \in \Phi^{\text {prs }}$ such that for any realization $h_{t}^{0} \in \mathcal{H}_{t}^{0}$ and its corresponding common beliefs $\theta_{t}$ (defined in (14)-(49)), the prescription $u_{t}^{p r s *}=\left(u_{t}^{0 *}, \rho_{t}^{*}\right)=\phi^{\text {prs* }}\left(h_{t}^{0}\right)$ achieves the minimum in the definition of $V_{t}\left(x_{t}^{0}, m_{t}^{0}, \theta_{t}, \tilde{z}_{t}\right)$. Then, $\phi^{\text {prs* }}$ is an optimal prescription strategy for Problem 2$]$

Proof. See Appendix VI for a proof.

[^1]
## IV. Modified Dynamic Program based on Strategy Decomposition

In this section, we first introduce a decomposition for the control strategies of the controller $C^{1}$. Then, we modify the DP of Theorem 1 based on this decomposition. As we will show in Section $\mathbb{V}$, this decomposition helps us find a solution to the DP of Theorem 1, and provide optimal strategies for the controllers.

## A. Decomposition of controller $C^{1}$ 's strategies

Consider arbitrary prescription strategies $\phi^{p r s} \in \Phi^{\text {prs }}$ of Problem 2 Under these strategies, $U_{t}^{1}$ can be decomposed as,

$$
\begin{align*}
& U_{t}^{1}=\left[\phi_{t}^{1}\left(H_{t}^{0}\right)\right]\left(X_{t}^{1}, M_{t}^{1}\right) \\
& =\underbrace{\mathbb{E}_{0}^{p r s t-1}\left[U_{t}^{1} \mid H_{t}^{0}, M_{t}^{1}\right]}_{\left[\bar{\phi}_{t}^{1}\left(H_{t}^{0}\right)\right]\left(M_{t}^{1}\right)}+\underbrace{\left\{U_{t}^{1}-\mathbb{E}_{0, t-1}^{\phi_{0}^{p r s}}\left[U_{t}^{1} \mid H_{t}^{0}, M_{t}^{1}\right]\right\}}_{\left[\tilde{\phi}_{t}^{1}\left(H_{t}^{0}\right)\right]\left(X_{t}^{1}, M_{t}^{1}\right)} . \tag{17}
\end{align*}
$$

Note that for any realization $h_{t}^{0}$ of $H_{t}^{0}, \bar{\phi}_{t}^{1}\left(h_{t}^{0}\right)$ is a measurable mapping from $\mathcal{M}^{1}$ to $\mathbb{R}^{d_{U}^{1}}$ and $\tilde{\phi}_{t}^{1}\left(h_{t}^{0}\right)$ is a measurable mapping from $\mathbb{R}^{d_{X}^{1}} \times \mathcal{M}^{1}$ to $\mathbb{R}^{d_{U}^{1}}$. Furthermore, for any realization $h_{t}^{0}$ of $H_{t}^{0}$ and $m_{t}^{1}$ of $M_{t}^{1}$, $\left[\tilde{\phi}_{t}^{1}\left(H_{t}^{0}\right)\right]\left(X_{t}^{1}, M_{t}^{1}\right)$ is conditionally zero-mean given $h_{t}^{0}, m_{t}^{1}$, that is, $\mathbb{E}^{\phi_{0: t-1}^{p r s}}\left[\left[\tilde{\phi}_{t}^{1}\left(H_{t}^{0}\right)\right]\left(X_{t}^{1}, M_{t}^{1}\right) \mid h_{t}^{0}, m_{t}^{1}\right]=0$. Also note that $X_{t}^{1}$ and $M_{t}^{1}$ are conditionally independent given the common information $h_{t}^{0}$. Hence, we have

$$
\begin{align*}
& \mathbb{E}^{\phi_{0: t-1}^{p r s}}\left[\left[\tilde{\phi}_{t}^{1}\left(H_{t}^{0}\right)\right]\left(X_{t}^{1}, M_{t}^{1}\right) \mid h_{t}^{0}, m_{t}^{1}\right] \\
& =\mathbb{E}^{\phi_{0: t-1}^{p r s}}\left[\left[\tilde{\phi}_{t}^{1}\left(h_{t}^{0}\right)\right]\left(X_{t}^{1}, m_{t}^{1}\right) \mid h_{t}^{0}\right]=\int\left[\tilde{\phi}_{t}^{1}\left(h_{t}^{0}\right)\right]\left(x_{t}^{1}, m_{t}^{1}\right) \theta_{t}\left(d x_{t}^{1}\right) . \tag{18}
\end{align*}
$$

Now, if we define $\bar{q}_{t}=\bar{\phi}_{t}^{1}\left(h_{t}^{0}\right), \tilde{q}_{t}=\tilde{\phi}_{t}^{1}\left(h_{t}^{0}\right)$, and

$$
\begin{align*}
\overline{\mathcal{Q}}= & \left\{\bar{q}_{t}: \mathcal{M}^{1} \rightarrow \mathbb{R}^{d_{U}^{1}} \text { measurable }\right\}, \\
\tilde{\mathcal{Q}}\left(\theta_{t}\right)= & \left\{\tilde{q}_{t}: \mathbb{R}^{d_{X}^{1}} \times \mathcal{M}^{1} \rightarrow \mathbb{R}^{d_{U}^{1}} \text { measurable },\right. \\
& \left.\int \tilde{q}_{t}\left(x_{t}^{1}, m_{t}^{1}\right) \theta_{t}\left(d x_{t}^{1}\right)=0 \quad \text { for all } m_{t}^{1} \in \mathcal{M}^{1}\right\}, \tag{19}
\end{align*}
$$

then we have $\bar{q}_{t} \in \overline{\mathcal{Q}}$ and $\tilde{q}_{t} \in \tilde{\mathcal{Q}}\left(\theta_{t}\right)$. This together with the representation $U_{t}^{1}$ of (17) suggests that at each time $t$, for any $h_{t}^{0} \in \mathcal{H}_{t}^{0}$ and its corresponding common beliefs $\theta_{t}$, finding a function $\rho_{t} \in \mathcal{P}$ is equivalent to finding two functions $\bar{q}_{t} \in \overline{\mathcal{Q}}$ and $\tilde{q}_{t} \in \tilde{\mathcal{Q}}\left(\theta_{t}\right)$. The following lemma states this point formally.
Lemma 1. In Problem 2 for any $h_{t}^{0} \in \mathcal{H}_{t}^{0}$, let $\theta_{t}$ be its corresponding common belief defined by (14)-(49). Then,

$$
\begin{equation*}
\mathcal{P}=\left\{\bar{q}_{t} \circ h_{2}+\tilde{q}_{t}: \bar{q}_{t} \in \overline{\mathcal{Q}}, \tilde{q}_{t} \in \tilde{\mathcal{Q}}\left(\theta_{t}\right)\right\} \tag{20}
\end{equation*}
$$

where $h_{2}: \mathbb{R}^{d_{X}^{1}} \times \mathcal{M}^{1} \rightarrow \mathcal{M}^{1}$ is a projection function defined as $h_{2}\left(x^{1}, m^{1}\right)=m^{1}$.
Proof. See Appendix VII for a proof.
According to Lemma 1 any prescription strategy $\phi^{p r s}=$ $\left(\phi^{0}, \phi^{1}\right)_{\sim_{1}} \in \Phi^{p r s}$ can be equivalently represented by $\left(\phi^{0}, \bar{\phi}^{1}, \tilde{\phi}^{1}\right)$. In the new representation, the control action $u_{t}^{1}$ of $C^{1}$ applied to the system described by (1)-(6) is $u_{t}^{1}=\bar{q}_{t}\left(M_{t}^{1}\right)+\tilde{q}_{t}\left(X_{t}^{1}, M_{t}^{1}\right)$ where $\bar{q}_{t}=\bar{\phi}_{t}^{1}\left(h_{t}^{0}\right)$ and $\tilde{q}_{t}=$
$\tilde{\phi}_{t}^{1}\left(h_{t}^{0}\right)$. In the following, we use the equivalent representation $\left(\phi^{0}, \bar{\phi}^{1}, \tilde{\phi}^{1}\right)$ for any prescription strategy $\phi^{\text {prs }} \in \Phi^{\text {prs }}$.

## B. Modified Dynamic Program for Problem 2

The following theorem provides a modified dynamic program for optimal prescription strategies of Problem 2 based on the new representation of strategies described above.
Theorem 2. Theorem $\square$ holds if $u_{t}^{\text {prs }}=\left(u_{t}^{0}, \bar{q}_{t} \circ h_{2}+\tilde{q}_{t}\right)$ and the minimization in (16) is over $u_{t}^{0} \in \mathbb{R}^{d_{U}^{0}}, \bar{q}_{t} \in \overline{\mathcal{Q}}, \tilde{q}_{t} \in$ $\tilde{\mathcal{Q}}\left(\theta_{t}\right)$.
Proof. The proof is obtained by applying Lemma $\square$ to Theorem 1

Note that although each step of the DP of Theorem 2 involves a functional optimization similar to Theorem in the next section, we show that the functions satisfying (16) exist and it is possible to find a solution to the DP of Theorem[2] This solution can then be used to provide optimal strategies for the controllers.

Remark 1. The decomposition of the controller $C^{1}$ 's strategies proposed in (17) is different from the one in [1], [2] where $U_{t}^{1}$ is decomposed into two terms: the conditional mean of $U_{t}^{1}$ given the common information $H_{t}^{0}$ and the deviation of $U_{t}^{1}$ from the mean. One can follow the decomposition of [1], [2] here and observe that it fails to provide any structure for solving the DP of Theorem [2]

## V. Optimal Control Strategies

In this section, we identify the structure of the value function in the modified dynamic program described in Section IV] Using the structure, we explicitly solve the dynamic program and obtain the optimal strategies for Problem (1)
Theorem 3. For any $x_{t}^{0} \in \mathbb{R}^{d_{x}^{0}}, m_{t}^{0} \in \mathcal{M}^{0}, \theta_{t} \in \Delta\left(\mathbb{R}^{d_{x}^{1}}\right)$, and $\tilde{z}_{t} \in \tilde{\mathcal{M}}^{1}$ at time $t$, there exist positive semi-definite matrices $P_{t}\left(m_{t}^{0}, \tilde{z}_{t}\right)$ and $\tilde{P}_{t}\left(m_{t}^{0}, \tilde{z}_{t}\right)$ such that the value function of the dynamic program (16) is given by

$$
\begin{align*}
V_{t}\left(x_{t}^{0}, m_{t}^{0}, \theta_{t}, \tilde{z}_{t}\right) & =Q F\left(P_{t}\left(m_{t}^{0}, \tilde{z}_{t}\right), \operatorname{vec}\left(x_{t}^{0}, \mu\left(\theta_{t}\right)\right)\right) \\
& +\operatorname{tr}\left(\tilde{P}_{t}\left(m_{t}^{0}, \tilde{z}_{t}\right) \operatorname{cov}\left(\theta_{t}\right)\right)+e_{t}, \tag{21}
\end{align*}
$$

where $e_{t}$ is a function of statistics of $W_{t+1: T}^{0: 1}$ and matrices $P_{t+1: T}, \tilde{P}_{t+1: T}$.

Furthermore, for any $x_{t}^{0} \in \mathbb{R}^{d_{x}^{0}}, m_{t}^{0} \in \mathcal{M}^{0}, \theta_{t} \in$ $\Delta\left(\mathbb{R}^{\mathbb{R}_{x}^{1}}\right)$, and $\tilde{z}_{t} \in \tilde{\mathcal{M}}^{1}$ at time $t$, there exist matrices $K_{t}\left(m_{t}^{0}, \tilde{z}_{t}\right)$ and $\tilde{K}_{t}\left(m_{t}^{0}, \tilde{z}_{t}\right)$ such that the minimizing $u_{t}^{0}, \bar{q}_{t}, \tilde{q}_{t}$ are

- If $\tilde{z}_{t}=\emptyset$ :

$$
\begin{align*}
& {\left[\begin{array}{c}
u_{*}^{0 *} \\
\bar{q}_{t}^{*}(1) \\
\vdots \\
\bar{q}_{t}^{*} \\
\left.\bar{q}_{t}^{1}\right)
\end{array}\right]=K_{t}\left(m_{t}^{0}, \emptyset\right)\left[\begin{array}{c}
x_{t}^{0} \\
\mu\left(\theta_{t}\right)
\end{array}\right],}  \tag{22}\\
& \tilde{q}_{t}^{*}\left(x_{t}^{1}, m_{t}^{1}\right)=\tilde{K}_{t}\left(m_{t}^{0}, m_{t}^{1}\right)\left(x_{t}^{1}-\mu\left(\theta_{t}\right)\right) \\
& \forall x_{t}^{1} \in \mathbb{R}^{d_{X}^{1}}, \forall m_{t}^{1} \in \mathcal{M}^{1} . \tag{23}
\end{align*}
$$

- If $\tilde{z}_{t}=l$ for any $l \in \mathcal{M}^{1}$ :

$$
\begin{align*}
& {\left[\begin{array}{c}
u_{t}^{0 *} \\
\bar{q}_{t}^{*}(l)
\end{array}\right]=K_{t}\left(m_{t}^{0}, l\right)\left[\begin{array}{c}
x_{t}^{0} \\
\mu\left(\theta_{t}\right)
\end{array}\right],}  \tag{24}\\
& \bar{q}_{t}^{*}\left(m_{t}^{1}\right)=0, \quad \forall m_{t}^{1} \in \mathcal{M}^{1} \backslash\{l\},  \tag{25}\\
& \tilde{q}_{t}^{*}\left(x_{t}^{1}, m_{t}^{1}\right)=0, \quad \forall x_{t}^{1} \in \mathbb{R}^{d_{X}^{1}}, \forall m_{t}^{1} \in \mathcal{M}^{1} \tag{26}
\end{align*}
$$

Proof. See Appendix VIII for a proof.
The matrices $P_{t}$ and $\tilde{P}_{t}$ can be explicitly computed using coupled "Riccati-like" backward recursions, and the gain matrices $K_{t}$ and $\tilde{K}_{t}$ are simple functions of $P_{t}$ and $\tilde{P}_{t}$. The computation of these matrices can be done offline with computational complexity similar to that of an optimal centralized LQR controller. See Appendix $\Pi$ for the recursions for $P_{t}$ and $\tilde{P}_{t}$ and the equations for $K_{t}$ and $\tilde{K}_{t}$
From Theorem 3, we can explicitly compute the optimal strategies for Problem The optimal strategies of controllers $C^{0}$ and $C^{1}$ are given in the following theorem.
Theorem 4. For any $x_{t}^{0} \in \mathbb{R}^{d_{X}^{0}}$ and $m_{t}^{0} \in \mathcal{M}^{0}$, the optimal strategies of Problem $\square$ are given by $u_{t}^{0 *}$ and $u_{t}^{1 *}=$ $\rho_{t}^{*}\left(x_{t}^{1}, m_{t}^{1}\right)=\bar{q}_{t}^{*}\left(m_{t}^{1}\right)+\tilde{q}_{t}^{*}\left(x_{t}^{1}, m_{t}^{1}\right)$ where (i) $u_{t}^{0 *}, \bar{q}_{t}^{*}, \tilde{q}_{t}^{*}$ are described by (22)-(26) and (ii) $\mu\left(\theta_{t}\right)=\hat{x}_{t}^{1}$ is the estimate (conditional expectation) of $X_{t}^{1}$ based on the common information $h_{t}^{0}$ and it can be computed recursively according to

$$
\hat{x}_{0}^{1}= \begin{cases}\mu\left(\pi_{X_{0}^{1}}\right) & \text { if } z_{0}=\emptyset,  \tag{27}\\ x_{0}^{1} & \text { if } z_{0}=x_{0}^{1} .\end{cases}
$$

$$
\begin{align*}
& \hat{x}_{t+1}^{1}= \\
& \left\{\begin{array}{l}
\sum_{m_{t}^{1} \in \mathcal{M}^{1}} \pi_{M^{1}}\left(m_{t}^{1}\right)\left[D\left(m_{t}^{0.1}\right)\right]_{1} \bullet\left[\begin{array}{c}
x_{t}^{0} \\
\hat{x}_{t}^{1} \\
\bar{u}_{t}^{* *} \\
\bar{u}_{t}^{*}\left(m_{t}^{1}\right)
\end{array}\right] \quad \text { if } z_{t+1}=\tilde{z}_{t}=\emptyset, \\
{\left[D\left(m_{t}^{0: 1}\right)\right]_{1} \bullet\left[\begin{array}{c}
x_{t}^{0} \\
\hat{x}_{t}^{1} \\
\bar{u}_{t}^{0 *} \\
\bar{q}_{t}^{*}\left(m_{t}^{1}\right)
\end{array}\right] \quad} \\
x_{t+1}^{1} \\
\text { if } z_{t+1}=\emptyset, \tilde{z}_{t}=m_{t}^{1}, \\
\text { if } z_{t+1}=x_{t+1}^{1},
\end{array}\right. \tag{28}
\end{align*}
$$

Proof. See Appendix $\boxed{X X}$ for a proof.
Remark 2. When the transmission from the controller $C^{1}$ to the controller $C^{0}$ is successful $\left(\gamma_{t}=1\right)$, the controller $C^{0}$ is aware of the state $x_{t}^{1}$ and local mode $m_{t}^{1}$. In this case, the term inside the minimization of (16) does not depend on function $\tilde{q}_{t}$ and also it does not depends on $\bar{q}_{t}(\ell)$ for all $\ell \neq m_{t}^{1}$. Hence, they can be chosen arbitrarily or set to be zero as described in (25) and (26).

Remark 3. If there is only possible value for the local mode, that is, $\left|\mathcal{M}^{1}\right|=1$, the controller $C^{0}$ knows this mode irrespective of the state of the link from the controller $C^{1}$ to the controller $C^{0}$. In this case, the optimal controller for this problem is described by (22) and (23) if $\mathcal{M}^{1}=\{1\}$.

## VI. Conclusion

We considered a discrete-time stochastically switched system consisting of two plants, global plant and local plant, and two controllers, global controller and local controller. We assumed the presence of an unreliable TCP-like communication channel through which the local controller can inform the global controller of the local plant's state and mode. We obtained explicit optimal strategies for the two controllers. In the optimal strategies, both controllers compute a common estimate of the local plant's state. The global controller's action is linear in the state of the global plant and the common estimated state, and the local controller's action is linear in the actual states of both plants and the common estimated state. Furthermore, the gain matrices for the global controller depend on the global mode and its observation about the local mode, while the gain matrices for the local controller depend on the actual modes of both plants and the global controller's observation about the local mode.

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## Appendix I

## SHORT-HANDS AND OpERATORS

We first define some shorthands for the simplicity of the presentation. For a vector $x$ and a matrix $G$, we define $Q F(G, x)=x^{\boldsymbol{\top}} G x=\operatorname{tr}\left(G x x^{\boldsymbol{\top}}\right)$. Furthermore, for a matrix $G=\left[\begin{array}{ll}G^{11} & G^{12} \\ G^{21} & G^{22}\end{array}\right]$, we define $S C\left(G, G^{22}\right):=G^{11}-$ $G^{12}\left(G^{22}\right)^{-1} G^{21}$ as the Schur complement of $G^{22}$ in $G$. For matrices $G^{1}, \ldots, G^{k}$, we define $H C\left(G^{1}, \ldots, G^{k}\right):=$ $\left[G^{1}, \ldots, G^{k}\right]$ to represent the horizontal concatenation of
$G^{1}, \ldots, G^{k}$. Furthermore, we use $\mathbb{1}_{E}(\cdot)$ to denote the indicator function of set $E$, that is, $\mathbb{1}_{E}(x)=1$ if $x \in E$, and 0 otherwise.

Operators:

- Let $G$ be a collection of matrices $G\left(m^{0}, \tilde{z}\right) \in \mathbb{R}^{d_{X} \times d_{X}}$ for all $m^{0} \in \mathcal{M}^{0}$ and $\tilde{z} \in \tilde{\mathcal{M}}^{1}=\mathcal{M}^{1} \cup\{\emptyset\}$. Then, define

$$
\begin{equation*}
\Pi(G)=\mathbb{E}\left[G\left(M_{t}^{0}, \tilde{Z}_{t}\right)\right]=\sum_{\gamma \in\{0,1\}} p(\gamma) \Pi(G, \gamma) . \tag{29}
\end{equation*}
$$

where for any $\gamma \in\{0,1\}$

$$
\begin{equation*}
\Pi(G, \gamma)=\mathbb{E}\left[G\left(M_{t}^{0}, \tilde{Z}_{t}\right) \mid \Gamma_{t}=\gamma\right] \tag{30}
\end{equation*}
$$

Note that

$$
\begin{aligned}
& \Pi(G, 0)=\sum_{m^{0} \in \mathcal{M}^{0}} G\left(m^{0}, \emptyset\right) \pi_{M^{0}}\left(m^{0}\right), \\
& \Pi(G, 1)=\sum_{m^{0} \in \mathcal{M}^{0}} \sum_{m^{1} \in \mathcal{M}^{1}} G\left(m^{0: 1}\right) \pi_{M^{0}}\left(m^{0}\right) \pi_{M^{1}}\left(m^{1}\right) .
\end{aligned}
$$

- For $n=1,2$, let $G^{n}$ be a collection of matrices $G^{n}\left(m^{0}, \tilde{z}\right) \in \mathbb{R}^{d_{X} \times d_{X}}$ for all $m^{0} \in \mathcal{M}^{0}$ and $\tilde{z} \in \tilde{\mathcal{M}}^{1}$. Then, define

$$
\begin{equation*}
\Psi\left(G^{1}, G^{2}\right)=p(0) \Pi\left(G^{1}, 0\right)+p(1) \Pi\left(G^{2}, 1\right) \tag{31}
\end{equation*}
$$

Matrices:

- For each $m^{1} \in \mathcal{M}^{1}, \quad L_{m^{1}} \in$ $\mathbb{R}^{\left(d_{X}+d_{U}^{0}+d_{U}^{1}\right) \times\left(d_{X}+d_{U}^{0}+\kappa^{1} d_{U}^{1}\right)}$ given by

$$
\begin{align*}
L_{m^{1}} & =H C\left(\left[\begin{array}{cc}
\mathbf{I}_{d_{X}} & \mathbf{0} \\
\mathbf{0} & \mathbf{I}_{d_{U}^{0}}^{0} \\
\mathbf{0} & \mathbf{0}
\end{array}\right], \mathbf{0}_{\left(d_{X}+d_{U}^{0}+d_{U}^{1}\right) \times\left(d_{X}+d_{U}^{0}+\left(m^{1}-1\right) d_{U}^{1}\right)},\right. \\
& {\left.\left[\begin{array}{c}
\mathbf{0} \\
\mathbf{0} \\
\mathbf{I}_{d_{U}^{1}}
\end{array}\right], \mathbf{0}_{\left(d_{X}+d_{U}^{0}+d_{U}^{1}\right) \times\left(d_{X}+d_{U}^{0}+\left(\kappa^{1}-m^{1}\right) d_{U}^{1}\right)}\right) . } \tag{32}
\end{align*}
$$

- $Q_{t}\left(M_{t}^{0: 1}\right)=\left[\begin{array}{ll}Q_{t}^{00}\left(M_{t}^{0: 1}\right) & Q_{t}^{01}\left(M_{t}^{0: 1}\right) \\ Q_{t}^{10}\left(M_{t}^{0: 1}\right) & Q_{t}^{11}\left(M_{t}^{0: 1}\right)\end{array}\right]$,
- $R_{t}\left(M_{t}^{0: 1}\right)=\left[\begin{array}{ll}R_{t}^{00}\left(M_{t}^{0: 1}\right) & R_{t}^{01}\left(M_{t}^{0: 1}\right) \\ R_{t}^{10}\left(M_{t}^{0: 1}\right) & R_{t}^{11}\left(M_{t}^{0: 1}\right)\end{array}\right]$,
- $D^{11}\left(m^{0: 1}\right)=\left[\begin{array}{ll}A^{11}\left(m^{0: 1}\right) & B^{11}\left(m^{0: 1}\right)\end{array}\right]$,
- $D^{\text {aug }}\left(m^{0: 1}\right)=\left[\begin{array}{ll}A\left(m^{0: 1}\right) & B\left(m^{0: 1}\right)\end{array}\right] L_{m^{1}}$,
- $D^{\emptyset}\left(m^{0}\right)=\sum_{m^{1} \in \mathcal{M}^{1}} D^{\text {aug }}\left(m^{0: 1}\right) \pi_{M^{1}}\left(m^{1}\right)$,
- $C_{t}\left(m^{0: 1}\right)=\left[\begin{array}{cc}Q_{t}\left(m^{0: 1}\right) & \mathbf{0} \\ \mathbf{0} & R_{t}\left(m^{0: 1}\right)\end{array}\right]$,
- $C_{t}^{\emptyset}\left(m^{0}\right)=\sum_{m^{1} \in \mathcal{M}^{1}} L_{m^{1}}^{\top} C_{t}\left(m^{0: 1}\right) L_{m^{1}} \pi_{M^{1}}\left(m^{1}\right)$,
- $C_{t}^{11}\left(m^{0: 1}\right)=\left[\begin{array}{cc}Q_{t}^{11}\left(m^{0: 1}\right) & \mathbf{0} \\ \mathbf{0} & R_{t}^{11}\left(m^{0: 1}\right)\end{array}\right]$.


## Appendix II

Recursions for $P_{t}, \tilde{P}_{t}$ and equations for $K_{t}, \tilde{K}_{t}$
For any $m^{0} \in \mathcal{M}^{0}$, $m^{1} \in \mathcal{M}^{1}$, and $\tilde{z} \in \tilde{\mathcal{M}}^{1}$, matrices $P_{t}\left(m^{0}, \tilde{z}\right)$ and $\tilde{P}_{t}\left(m^{0}, \tilde{z}\right)$ are recursively calculated as follows,

$$
\begin{align*}
& P_{T+1}\left(m^{0}, \tilde{z}\right)=\left[\begin{array}{ll}
P_{T+1}^{00}\left(m^{0}, \tilde{z}\right) & P_{T+1}^{01}\left(m^{0}, \tilde{z}\right) \\
P_{T+1}^{10}\left(m^{0}, \tilde{z}\right) & P_{T+1}^{11}\left(m^{0}, \tilde{z}\right)
\end{array}\right]=\mathbf{0}  \tag{33}\\
& \tilde{P}_{T+1}\left(m^{0}, \tilde{z}\right)=\mathbf{0} \tag{34}
\end{align*}
$$

$$
\begin{align*}
& E_{t}^{\emptyset}\left(m^{0}\right)=D^{\emptyset}\left(m^{0}\right)^{\top} \Pi\left(P_{t+1}\right) D^{\emptyset}\left(m^{0}\right),  \tag{35}\\
& F_{t}^{\emptyset}\left(m^{0}\right)= \\
& \sum_{m^{1} \in \mathcal{M}^{1}}\left[D^{\text {aug }}\left(m^{0: 1}\right)\right]_{1}^{\top} \Psi\left(\tilde{P}_{t+1}, P_{t+1}^{11}\right)\left[D^{\text {aug }}\left(m^{0: 1}\right)\right]_{\bullet} \bullet \pi_{M^{1}}\left(m^{1}\right) \\
& \quad-\left[D^{\emptyset}\left(m^{0}\right)\right]_{1 \bullet}^{\top} \Psi\left(\tilde{P}_{t+1}, P_{t+1}^{11}\right)\left[D^{\emptyset}\left(m^{0}\right)\right]_{1} \bullet  \tag{36}\\
& H_{t}\left(m^{0}, \tilde{z}\right)=\left[\begin{array}{ll}
H_{t}^{X X}\left(m^{0}, \tilde{z}\right) & H_{t}^{X U}\left(m^{0}, \tilde{z}\right) \\
H_{t}^{U X}\left(m^{0}, \tilde{z}\right) & H_{t}^{U U}\left(m^{0}, \tilde{z}\right)
\end{array}\right] \\
& = \begin{cases}C_{t}^{\emptyset}\left(m^{0}\right)+E_{t}^{\emptyset}\left(m^{0}\right)+F_{t}^{\emptyset}\left(m^{0}\right) & \text { if } \tilde{z}=\emptyset, \\
C_{t}\left(m^{0: 1}\right)+D\left(m^{0: 1}\right)^{\top} \Pi\left(P_{t+1}\right) D\left(m^{0: 1}\right) & \text { if } \tilde{z}=m^{1} .\end{cases}  \tag{37}\\
& P_{t}\left(m^{0}, \tilde{z}\right)=S C\left(H_{t}\left(m^{0}, \tilde{z}\right), H_{t}^{U U}\left(m^{0}, \tilde{z}\right)\right)  \tag{38}\\
& \tilde{H}_{t}\left(m^{0: 1}\right)=\left[\begin{array}{ll}
\tilde{H}_{t}^{X^{1} X^{1}}\left(m^{0: 1}\right) & \tilde{H}_{t}^{X^{1} U^{1}}\left(m^{0: 1}\right) \\
\tilde{H}_{t}^{U^{1} X^{1}}\left(m^{0: 1}\right) & \tilde{H}_{t}^{U^{1} U^{1}}\left(m^{0: 1}\right)
\end{array}\right] \\
& =C_{t}^{11}\left(m^{0: 1}\right)+D^{11}\left(m^{0: 1}\right)^{\top} \Psi\left(\tilde{P}_{t+1}, P_{t+1}^{11}\right) D^{11}\left(m^{0: 1}\right), \tag{39}
\end{align*}
$$

$\tilde{P}_{t}\left(m^{0}, \tilde{z}\right)=$
$\begin{cases}\sum_{m 1} \in \mathcal{M}^{1} \pi_{M}\left(m^{1}\right) S C\left(\tilde{H}_{t}\left(m^{0: 1}\right), \tilde{H}_{t}^{U^{1} U^{1}}\left(m^{0: 1}\right)\right) & \text { if } \tilde{z}=\emptyset, \\ S C\left(\tilde{H}_{t}\left(m^{0: 1}\right), \tilde{H}_{t}^{U^{1} U^{1}}\left(m^{0: 1}\right)\right) & \text { if } \tilde{z}=m^{1} .\end{cases}$

Furthermore, at each time $t$, the gain matrices $K_{t}\left(m^{0}, \tilde{z}\right)$ and $\tilde{K}_{t}\left(m^{0: 1}\right)$ for any $m^{0} \in \mathcal{M}^{0}, m^{1} \in \mathcal{M}^{1}$, and $\tilde{z} \in \tilde{\mathcal{M}}^{1}$ are calculated as follows,

$$
\begin{align*}
& K_{t}\left(m^{0}, \tilde{z}\right)=-\left(H_{t}^{U U}\left(m^{0}, \tilde{z}\right)\right)^{-1} H_{t}^{U X}\left(m^{0}, \tilde{z}\right)  \tag{41}\\
& \tilde{K}_{t}\left(m^{0: 1}\right)=-\left(\tilde{H}_{t}^{U^{1} U^{1}}\left(m_{t}^{0: 1}\right)\right)^{-1} \tilde{H}_{t}^{U^{1} X^{1}}\left(m_{t}^{0: 1}\right) \tag{42}
\end{align*}
$$

## Appendix III <br> Preliminary Results

Claim 1. Consider a feasible prescription strategy $\phi_{t}^{\diamond}=$ $\phi_{0: t-1}^{p r s} \in \Phi^{\text {prs }}$. Then, the random vectors $M_{t}^{0}, X_{t}^{0}$ and $X_{t}^{1}$ are conditionally independent given the common information $H_{t-1}^{0}$. That is, for any measurable sets $E^{0} \subset \mathbb{R}^{d_{x}^{0}}, E^{1} \subset$ $\mathbb{R}^{d_{X}^{1}}$, and $F \subset \mathcal{M}^{1}$,

$$
\begin{align*}
& \mathbb{P}^{\phi_{t}^{\circ}}\left(M_{t}^{0} \in F, X_{t}^{0} \in E^{0}, X_{t}^{1} \in E^{1} \mid H_{t-1}^{0}\right) \\
& =\mathbb{P}\left(M_{t}^{0} \in F\right) \mathbb{P}^{\mathbb{Q}_{t}^{\circ}}\left(X_{t}^{0} \in E^{0} \mid H_{t-1}^{0}\right) \mathbb{P}^{\phi_{t}^{\circ}}\left(X_{t}^{1} \in E^{1} \mid H_{t-1}^{0}\right) . \tag{43}
\end{align*}
$$

Proof. Note that given any realization $h_{t-1}^{0}$ of $H_{t-1}^{0}$, $X_{t}^{0}$ is a function of only $W_{t-1}^{0}$ while $X_{t}^{1}$ is a function of $X_{0}^{1}, W_{0: t-1}^{1}, M_{0: t-1}^{1}$. Then, the proof is easily resulted form the fact that random variables in the collection $\left\{M_{t}^{0}, X_{0}^{1}, W_{0: t-1}^{1}, M_{0: t-1}^{1}, W_{t-1}^{0}\right\}$ are independent.

Lemma 2. Consider a feasible prescription strategy $\phi_{t}^{\diamond}=$ $\phi_{0: t-1}^{p r s} \in \Phi^{\text {prs }}$. Then, for any $h_{t}^{0} \in \mathcal{H}_{t}^{0}$ and for any $m \in \mathcal{M}^{1}$, the $C^{0}$ 's belief $\mathbb{P}^{\phi_{t}^{\circ}}\left(M_{t}^{1}=m \mid h_{t}^{0}\right)$ on the local mode $M_{t}^{1}$ is $\omega^{\tilde{z}_{t}}(m)$ where

- If $\tilde{z}_{t}=m_{t}^{1}$, then $\omega^{m_{t}^{1}}(m)=\mathbb{1}_{\{m\}}\left(m_{t}^{1}\right)$.
- If $\tilde{z}_{t}=\emptyset$, then $\omega^{\emptyset}(m)=\pi_{M_{t}^{1}}(m)$.

Proof. Note that $\tilde{z}_{t}$ is included in $h_{t}^{0}$. Hence, when $\tilde{z}_{t}=m_{t}^{1}$, we have

$$
\begin{equation*}
\mathbb{P}^{\phi_{t}^{\circ}}\left(M_{t}^{1}=m \mid h_{t}^{0}\right)=\mathbb{P}\left(m_{t}^{1}=m\right)=\mathbb{1}_{\{m\}}\left(m_{t}^{1}\right) . \tag{44}
\end{equation*}
$$

When $\tilde{z}_{t}=\emptyset$, we have

$$
\begin{align*}
& \mathbb{P}^{\phi_{t}^{\circ}}\left(M_{t}^{1}=m \mid h_{t}^{0}\right)=\mathbb{P}^{\phi_{t}^{\circ}}\left(M_{t}^{1}=m \mid h_{t-1}^{0}, x_{t}^{0}, m_{t}^{0}, u_{t-1}^{0}\right) \\
& =\mathbb{P}^{\phi_{t}^{\circ}}\left(M_{t}^{1}=m\right)=\pi_{M_{t}^{1}}(m) \tag{45}
\end{align*}
$$

Note that $X_{t}^{0}=\left[D\left(m_{t-1}^{0: 1}\right)\right]_{0 \bullet} S_{t-1}+W_{t-1}^{0}$ where $S_{t-1}=$ $\operatorname{vec}\left(X_{t-1}^{0: 1}, U_{t-1}^{0: 1}\right)$. Hence, the second equality of (45) is true because 1) the collection $\left\{X_{t}^{0}, H_{t-1}^{0}, U_{t-1}^{0}\right\}$ depend on random variables from time 0 to $t-1$ and $M_{t}^{1}$ is independent of all previous random variables; 2) $M_{t}^{1}$ is independent of $M_{t}^{0}$.

## Appendix IV

Conditional independence of $X_{t}^{1}$ And $M_{t}^{1}$ GIVEn $H_{t}^{0}$
Lemma 3. Consider any feasible prescription strategy $\phi_{t}^{\circ}=$ $\phi_{0: t-1}^{p r s} \in \Phi^{\text {prs }}$. Then, $X_{t}^{1}$ and $M_{t}^{1}$ are conditionally independent given the common information $H_{t}^{0}$.
Proof. According to (7), for any measurable sets $E \subset \mathbb{R}^{d_{X}^{1}}$ and $F \subset \mathcal{M}^{1}$, we have,

$$
\begin{align*}
& \mathbb{P}^{\phi_{t}^{\diamond}}\left(X_{t}^{1} \in E, M_{t}^{1} \in F \mid H_{t}^{0}\right) \\
& =\mathbb{P}^{\phi_{t}^{\circ}}\left(X_{t}^{1} \in E, M_{t}^{1} \in F \mid H_{t-1}^{0}, X_{t}^{0}, M_{t}^{0}, Z_{t}, \tilde{Z}_{t}, U_{t-1}^{0}\right) \tag{46}
\end{align*}
$$

Now, consider two following cases. If $\Gamma_{t}^{0}=1$, we have
$\mathbb{P}^{\phi_{t}^{\circ}}\left(X_{t}^{1} \in E, M_{t}^{1} \in F \mid H_{t-1}^{0}, X_{t}^{0}, M_{t}^{0}, Z_{t}, \tilde{Z}_{t}, U_{t-1}^{0}\right) \mathbb{1}_{\left\{\Gamma_{t}=1\right\}}$
$=\mathbb{P}^{\phi_{t}^{\circ}}\left(X_{t}^{1} \in E, M_{t}^{1} \in F \mid H_{t-1}^{0}, X_{t}^{0}, M_{t}^{0}, X_{t}^{1}, M_{t}^{1}, U_{t-1}^{0}\right) \mathbb{1}_{\left\{\Gamma_{t}=1\right\}}$
$=\mathbb{P}^{\phi_{t}^{\diamond}}\left(X_{t}^{1} \in E \mid H_{t}^{0}\right) \mathbb{P}^{\phi_{\grave{\succcurlyeq}}^{\diamond}}\left(M_{t}^{1} \in F \mid H_{t}^{0}\right) \mathbb{1}_{\left\{\Gamma_{t}=1\right\}}$.
If $\Gamma_{t}^{0}=0$, we have

$$
\begin{align*}
& \mathbb{P}^{\phi_{t}^{\circ}}\left(X_{t}^{1} \in E, M_{t}^{1} \in F \mid H_{t-1}^{0}, X_{t}^{0}, M_{t}^{0}, Z_{t}, \tilde{Z}_{t}, U_{t-1}^{0}\right) \mathbb{1}_{\left\{\Gamma_{t}=0\right\}} \\
& =\mathbb{P}^{\phi_{t}^{\circ}}\left(X_{t}^{1} \in E, M_{t}^{1} \in F \mid H_{t-1}^{0}, X_{t}^{0}, M_{t}^{0}, U_{t-1}^{0}\right) \mathbb{1}_{\left\{\Gamma_{t}=0\right\}} \\
& =\mathbb{P}^{\phi_{t}^{\circ}}\left(X_{t}^{1} \in E \mid H_{t}^{0}\right) \mathbb{P}^{\phi_{t}^{\circ}}\left(M_{t}^{1} \in F \mid H_{t}^{0}\right) \mathbb{1}_{\left\{\Gamma_{t}=0\right\}} . \tag{48}
\end{align*}
$$

Note that $X_{t}^{0}=\left[D\left(M_{t-1}^{0: 1}\right)\right]_{0 \bullet} S_{t-1}+W_{t-1}^{0}, X_{t}^{1}=$ $\left[D\left(M_{t-1}^{0: 1}\right)\right]_{1} \cdot S_{t-1}+W_{t-1}^{1}$ where $S_{t-1}=\operatorname{vec}\left(X_{t-1}^{0: 1}, U_{t-1}^{0: 1}\right)$. Hence, the penultimate equality of (48) is true because 1) the collection $\left\{X_{t}^{0}, X_{t}^{1}, H_{t-1}^{0}, U_{t-1}^{0}\right\}$ depend on random variables from time 0 to $t-1$ and $M_{t}^{1}$ is independent of all previous random variables; 2) $M_{t}^{1}$ is independent of $M_{t}^{0}$. This completes the proof of Lemma 3 .

## Appendix V

Recursive Update for Common belief $\theta_{t}$
Lemma 4. For any feasible prescription strategy $\phi^{p r s} \in$ $\Phi^{\text {prs }}$ and for any $h_{t}^{0} \in \mathcal{H}_{t}^{0}$, the common belief $\theta_{t}$ can be updated according to

$$
\begin{equation*}
\theta_{t+1}=\psi_{t}\left(\theta_{t}, u_{t}^{p r s}, x_{t}^{0}, m_{t}^{0}, \tilde{z}_{t}, z_{t+1}\right) \tag{49}
\end{equation*}
$$

where $u_{t}^{\text {prs }}=\left(u_{t}^{0}, \rho_{t}\right)=\phi_{t}^{\text {prs }}\left(h_{t}^{0}\right)$ and for any measurable set $E \subset \mathbb{R}^{d_{X}^{1}}$,

$$
\theta_{0}(E)= \begin{cases}\pi_{X_{0}^{1}}(E) & \text { if } z_{0}=\emptyset  \tag{50}\\ \mathbb{1}_{E}\left(x_{0}^{1}\right) & \text { if } z_{0}=x_{0}^{1}\end{cases}
$$

- If $z_{t+1}=x_{t+1}^{1}$, then

$$
\begin{equation*}
\left[\psi_{t}\left(\theta_{t}, u_{t}^{p r s}, x_{t}^{0}, m_{t}^{0}, \tilde{z}_{t}, x_{t+1}^{1}\right)\right](E)=\mathbb{1}_{E}\left(x_{t+1}^{1}\right) \tag{51}
\end{equation*}
$$

- If $z_{t+1}=\emptyset$, then

$$
\begin{align*}
& {\left[\psi_{t}\left(\theta_{t}, u_{t}^{p r s}, x_{t}^{0}, m_{t}^{0}, \tilde{z}_{t}, \emptyset\right)\right](E)=} \\
& \sum_{m_{t}^{1} \in \mathcal{M}^{1}} \iint \mathbb{1}_{E}\left(\left[D\left(m_{t}^{0: 1}\right)\right]_{1} \bullet \operatorname{vec}\left(x_{t}^{0: 1}, u_{t}^{0}, \rho_{t}\left(x_{t}^{1}, m_{t}^{1}\right)\right)+w_{t}^{1}\right) \\
& \quad \times \theta_{t}\left(d x_{t}^{1}\right) \omega^{\tilde{z}_{t}}\left(m_{t}^{1}\right) \pi_{W_{t}^{1}}\left(d w_{t}^{1}\right) \tag{52}
\end{align*}
$$

Proof. To prove Lemma 4 we proceed as follows. For any feasible prescription strategy $\phi^{p r s} \in \Phi^{p r s}$ and for any $h_{t}^{0} \in$ $\mathcal{H}_{t}^{0}$, we recursively define $\nu_{t}\left(h_{t}^{0}\right) \in \Delta\left(\mathbb{R}^{d_{X}^{1}}\right)$ as follows:

For any measurable set $E \subset \mathbb{R}^{d_{X}^{1}}$,

$$
\left[\nu_{0}\left(h_{0}^{0}\right)\right](E)= \begin{cases}\pi_{X_{0}^{1}}(E) & \text { if } z_{0}=\emptyset  \tag{53}\\ \mathbb{1}_{E}\left(x_{0}^{1}\right) & \text { if } z_{0}=x_{0}^{1}\end{cases}
$$

For any measurable set $E \subset \mathbb{R}^{d_{X}^{1}}$,

$$
\begin{equation*}
\left[\nu_{t+1}\left(h_{t+1}^{0}\right)\right](E)=\left[\psi_{t}\left(\nu_{t}\left(h_{t}^{0}\right), u_{t}^{p r s}, x_{t}^{0}, m_{t}^{0}, \tilde{z}_{t}, z_{t+1}\right)\right](E) \tag{54}
\end{equation*}
$$

where $u_{t}^{p r s}=\left(u_{t}^{0}, \rho_{t}\right)=\phi_{t}^{p r s}\left(h_{t}^{0}\right)$ and $\psi_{t}^{n}\left(\nu_{t}\left(h_{t}^{0}\right), u_{t}^{p r s}, x_{t}^{0}, m_{t}^{0}, \tilde{z}_{t}, z_{t+1}\right)$ is defined as follows:

- If $z_{t+1}=x_{t+1}^{1}$, then

$$
\begin{equation*}
\left[\psi_{t}\left(\nu_{t}\left(h_{t}^{0}\right), u_{t}^{p r s}, x_{t}^{0}, m_{t}^{0}, \tilde{z}_{t}, x_{t+1}^{1}\right)\right](E)=\mathbb{1}_{E}\left(x_{t+1}^{1}\right) \tag{55}
\end{equation*}
$$

- If $z_{t+1}=\emptyset$, then

$$
\begin{gather*}
{\left[\psi_{t}\left(\nu_{t}\left(h_{t}^{0}\right), u_{t}^{p r s}, x_{t}^{0}, m_{t}^{0}, \tilde{z}_{t}, \emptyset\right)\right](E)=} \\
\sum_{m_{t}^{1} \in \mathcal{M}^{1}} \iint \mathbb{1}_{E}\left(\left[D\left(m_{t}^{0: 1}\right)\right]_{1} \bullet \operatorname{vec}\left(x_{t}^{0: 1}, u_{t}^{0}, \rho_{t}\left(x_{t}^{1}, m_{t}^{1}\right)\right)+w_{t}^{1}\right) \\
\quad \times\left[\nu_{t}\left(h_{t}^{0}\right)\right]\left(d x_{t}^{1}\right) \omega^{\tilde{z}_{t}}\left(m_{t}^{1}\right) \pi_{W_{t}^{1}}\left(d w_{t}^{1}\right) \tag{56}
\end{gather*}
$$

Then, we show that $\nu_{t}$ is a conditional probability of $X_{t}^{1}$ given $H_{t}^{0}$, that is $\left[\nu_{t}\left(H_{t}^{0}\right)\right](E)=\mathbb{P}^{\phi_{t}^{\circ}}\left(X_{t}^{1} \in E \mid H_{t}^{0}\right)$ where $\phi_{t}^{\diamond}=\phi_{0: t-1}^{p r s}$.

First, note that from (53)-(56), $\left[\nu_{t}(\cdot)\right](E): \mathcal{H}_{t}^{0} \mapsto \mathbb{R}$ is a measurable function. To show that $\left[\nu_{t}\left(H_{t}^{0}\right)\right](E)=$ $\mathbb{P}^{\phi_{t}^{\diamond}}\left(X_{t}^{1} \in E \mid H_{t}^{0}\right)$, first note that for any $t$

$$
\begin{align*}
& \mathbb{P}^{\phi_{t}^{\circ}}\left(X_{t}^{1} \in E \mid H_{t}^{0}\right)=\mathbb{P}^{\phi_{t}^{\circ}}\left(X_{t}^{1} \in E \mid H_{t-1}^{0}, X_{t}^{0}, M_{t}^{0}, Z_{t}, \tilde{Z}_{t}\right) \\
& =\mathbb{P}^{\phi_{t}^{\circ}}\left(X_{t}^{1} \in E \mid H_{t-1}^{0}, Z_{t}^{n}, \tilde{Z}_{t},\right) \tag{57}
\end{align*}
$$

where the last equality is true due to Claim 1
We now prove by induction that

$$
\begin{equation*}
\left[\nu_{t}\left(H_{t}^{0}\right)\right](E)=\mathbb{P}^{\phi_{t}^{\circ}}\left(X_{t}^{1} \in E \mid H_{t-1}^{0}, Z_{t}, \tilde{Z}_{t}\right) \tag{58}
\end{equation*}
$$

At time $t=0$, since $\Gamma_{0} \in\{0,1\}$, consider two cases:

- If $\Gamma_{0}=1$,

$$
\begin{align*}
& \mathbb{P}\left(X_{0}^{1} \in E \mid Z_{0}, \tilde{Z}_{0}\right) \mathbb{1}_{\left\{\Gamma_{0}=1\right\}}=\mathbb{P}\left(X_{0}^{1} \in E \mid X_{0}^{1}, M_{0}^{1}\right) \mathbb{1}_{\left\{\Gamma_{0}=1\right\}} \\
& =\mathbb{P}\left(X_{0}^{1} \in E \mid X_{0}^{1}\right) \mathbb{1}_{\left\{\Gamma_{0}=1\right\}}=\mathbb{1}_{E}\left(X_{0}^{1}\right) \mathbb{1}_{\left\{\Gamma_{0}=1\right\}} \tag{59}
\end{align*}
$$

- If $\Gamma_{0}=0$,

$$
\begin{align*}
& \mathbb{P}\left(X_{0}^{1} \in E \mid Z_{0}, \tilde{Z}_{0}\right) \mathbb{1}_{\left\{\Gamma_{0}=0\right\}}=\mathbb{P}\left(X_{0}^{1} \in E\right) \mathbb{1}_{\left\{\Gamma_{0}=0\right\}} \\
& =\pi_{X_{0}^{1}}(E) \mathbb{1}_{\left\{\Gamma_{0}=0\right\}} . \tag{60}
\end{align*}
$$

Hence, (58) holds at time 0. Assume that (58) holds at time $t$. This means that

$$
\begin{equation*}
\mathbb{P}^{\phi_{t}^{\circ}}\left(d x_{t}^{1} \mid H_{t}^{0}\right)=\left[\nu_{t}\left(H_{t}^{0}\right)\right]\left(d x_{t}^{1}\right) \tag{61}
\end{equation*}
$$

At time $t+1$, since $\Gamma_{t+1} \in\{0,1\}$, consider two cases:

- If $\Gamma_{t+1}=1$, similar to (59) we obtain

$$
\begin{align*}
& \mathbb{P}^{\phi_{t+1}^{\varrho}}\left(X_{t+1}^{1} \in E \mid H_{t}^{0}, Z_{t+1}, \tilde{Z}_{t+1}\right) \mathbb{1}_{\left\{\Gamma_{t+1}=1\right\}} \\
& =\mathbb{1}_{E}\left(X_{t+1}^{1}\right) \mathbb{1}_{\left\{\Gamma_{t+1}=1\right\}}=\left[\nu_{t+1}\left(H_{t+1}^{0}\right)\right](E) \mathbb{1}_{\left\{\Gamma_{t+1}=1\right\}} \tag{62}
\end{align*}
$$

- If $\Gamma_{t+1}=0$,

$$
\begin{align*}
& \mathbb{P}^{\phi_{t+1}^{\circ}}\left(X_{t+1}^{1} \in E \mid H_{t}^{0}, Z_{t+1}, \tilde{Z}_{t+1}\right) \mathbb{1}_{\left\{\Gamma_{t+1}^{n}=0\right\}} \\
& =\mathbb{P}_{t+1}^{\phi_{t+1}^{\circ}}\left(f_{t}^{1}\left(X_{t}^{0: 1}, M_{t}^{0: 1}, \phi_{t}^{p r s}\left(H_{t}^{0}\right), W_{t}^{1}\right) \in E \mid H_{t}^{0}\right) \mathbb{1}_{\left\{\Gamma_{t+1}=0\right\}} \\
& =\sum_{m_{t}^{1}} \iint \mathbb{1}_{E}\left(f_{t}^{1}\left(x_{t}^{1}, X_{t}^{0}, m_{t}^{1}, M_{t}^{0}, \phi_{t}^{p r s}\left(H_{t}^{0}\right), w_{t}^{1}\right)\right) \times \\
& \mathbb{P}^{\phi_{t}^{\circ}}\left(d x_{t}^{1} \mid H_{t}^{0}\right) \mathbb{P}^{\phi_{t}^{\diamond}}\left(d w_{t}^{1} \mid H_{t}^{0}\right) \mathbb{P}^{\phi_{t}^{\diamond}}\left(M_{t}^{1}=m_{t}^{1} \mid H_{t}^{0}\right) \mathbb{1}_{\left\{\Gamma_{t+1}=0\right\}} \\
& =\sum_{m_{t}^{1}} \iint \mathbb{1}_{E}\left(f_{t}^{1}\left(X_{t}^{0}, x_{t}^{1}, M_{t}^{0}, m_{t}^{1}, \phi_{t}^{p r s}\left(H_{t}^{0}\right), w_{t}^{1}\right)\right) \times \\
& \quad\left[\nu_{t}\left(H_{t}^{0}\right)\right]\left(d x_{t}^{1}\right) \pi_{W_{t}^{1}}\left(d w_{t}^{1}\right) \omega^{\tilde{Z}_{t}}\left(m_{t}^{1}\right) \mathbb{1}_{\left\{\Gamma_{t+1}=0\right\}} \\
& =\left[\nu_{t+1}\left(H_{t+1}^{0}\right)\right](E) \mathbb{1}_{\left\{\Gamma_{t+1}=0\right\}}, \tag{63}
\end{align*}
$$

where we have defined

$$
\begin{align*}
& f_{t}^{1}\left(x_{t}^{0: 1}, m_{t}^{0: 1}, u_{t}^{\text {prs }}, w_{t}^{1}\right) \\
& =\left[D\left(m_{t}^{0: 1}\right)\right]_{1} \bullet \operatorname{vec}\left(x_{t}^{0: 1}, u_{t}^{0}, \rho_{t}\left(x_{t}^{1}, m_{t}^{1}\right)\right)+w_{t}^{1} \tag{64}
\end{align*}
$$

Note that in (63), the first equality is true due to the disintegration theorem [35] an the fact that $M_{t}^{1}, X_{t}^{1}$, and $W_{t}^{1}$ are conditionally independent given $H_{t}^{0}$ from Lemma 3, and the second equality is true because of (61) and Lemma 2

Hence, (58) holds at time $t+1$ and from (54), we have $\theta_{t+1}=\psi_{t}\left(\theta_{t}, u_{t}^{p r s}, x_{t}^{0}, m_{t}^{0}, \tilde{z}_{t}, z_{t+1}\right)$. This completes the proof of Lemma 4

## Appendix VI

## Proof of Theorem 1

For any $\phi^{\text {prs }} \in \Phi^{\text {prs }}$ and any realization $h_{t}^{0} \in \mathcal{H}_{t}^{0}$, let the realization of the common belief $\Theta_{t}$ be $\theta_{t}=$ $\mathbb{P}^{\phi_{0: t-1}^{p r s}}\left(d x_{t}^{1} \mid h_{t}^{0}\right)$ defined by Lemma 4 Suppose the prescription strategy $\phi^{\text {prs* }} \in \Phi^{\text {prs }}$ achieves the minimum of (16) for $x_{t}^{0}, m_{t}^{0}, \theta_{t}, \tilde{z}_{t}, t=0, \ldots, T$, and let $u_{t}^{p r s *}=\left(u_{t}^{0 *}, \rho_{t}^{*}\right)=$ $\phi^{p r s *}\left(h_{t}^{0}\right)$ for any realization $h_{t}^{0} \in \mathcal{H}_{t}^{0}$.

We prove by induction that $V_{t}\left(x_{t}^{0}, m_{t}^{0}, \theta_{t}, \tilde{z}_{t}\right)$ is a measurable function with respect to $h_{t}^{0}$, and for any $h_{t}^{0} \in \mathcal{H}_{t}^{0}$ and for any $\phi_{t}^{\diamond}:=\phi_{0: t-1}^{p r s} \in \Phi^{p r s}$ we have

$$
\begin{align*}
& \mathbb{E}^{\phi_{t}^{\prime}}\left[\sum_{s=t}^{T} c_{s}^{p r s}\left(X_{s}^{0: 1}, M_{s}^{0: 1}, U_{s}^{p r s}\right) \mid h_{t}^{0}\right] \\
& =V_{t}\left(x_{t}^{0}, m_{t}^{0}, \mathbb{P}^{\phi_{t}^{\diamond}}\left(d x_{t}^{1} \mid h_{t}^{0}\right), \tilde{z}_{t}\right)  \tag{65}\\
& \leq \mathbb{E}^{\phi^{p r s}}\left[\sum_{s=t}^{T} c_{s}^{p r s}\left(X_{s}^{0: 1}, M_{s}^{0: 1}, U_{s}^{p r s}\right) \mid h_{t}^{0}\right] \tag{66}
\end{align*}
$$

where $\phi_{t}^{\prime}=\left\{\phi_{t}^{\diamond}, \phi_{t: T}^{p r s *}\right\}$. Note that the above equation at $t=0$ gives the optimality of $\phi^{p r s *}$ for Problem 2

We first consider (65). At $T+1$, (65) is true (all terms are defined to be 0 at $T+1$ ). Assume $V_{t+1}\left(x_{t+1}^{0}, m_{t+1}^{0}, \theta_{t+1}, \tilde{z}_{t+1}\right)$ is a measurable function with respect to $h_{t+1}^{0}$ and 65) is true at $t+1$, that is, for any $h_{t+1}^{0} \in \mathcal{H}_{t+1}^{0}$ and for any $\phi_{t+1}^{\diamond} \in \Phi^{\text {prs }}$

$$
\begin{align*}
& \mathbb{E}^{\phi_{t+1}^{\prime}}\left[\sum_{s=t+1}^{T} c_{s}^{p r s}\left(X_{s}^{0: 1}, M_{s}^{0: 1}, U_{s}^{p r s}\right) \mid h_{t+1}^{0}\right] \\
& =V_{t+1}\left(x_{t+1}^{0}, m_{t+1}^{0}, \mathbb{P}^{\phi_{t+1}^{0}}\left(d x_{t+1}^{1} \mid h_{t+1}^{0}\right), \tilde{z}_{t+1}\right) \tag{67}
\end{align*}
$$

where $\phi_{t+1}^{\prime}=\left\{\phi_{t+1}^{\diamond}, \phi_{t+1: T}^{p r s *}\right\}$. Since (67) is true for any $\phi_{t+1}^{\diamond} \in \Phi^{p r s}$, by choosing $\phi_{t}^{p r s}$ to be $\phi_{t}^{p r s *}$, we get $\phi_{t+1}^{\prime}=$ $\left\{\phi_{t+1}^{\diamond}, \phi_{t+1: T}^{p r s *}\right\}=\left\{\phi_{t}^{\diamond}, \phi_{t: T}^{p r s *}\right\}=\phi_{t}^{\prime}$. Then, from 67), we get,

$$
\begin{align*}
& \mathbb{E}^{\phi_{t+1}^{\prime}}\left[\sum_{s=t+1}^{T} c_{s}^{p r s}\left(X_{s}^{0: 1}, M_{s}^{0: 1}, U_{s}^{p r s}\right) \mid h_{t+1}^{0}\right] \\
& =V_{t+1}\left(x_{t+1}^{0}, m_{t+1}^{0}, \mathbb{P}^{\phi_{t}^{\diamond}, \phi_{t}^{p r s *}}\left(d x_{t+1}^{1} \mid h_{t+1}^{0}\right), \tilde{z}_{t+1}\right) . \tag{68}
\end{align*}
$$

At time $t$, from the tower property of conditional expectation we have

$$
\begin{align*}
& \mathbb{E}^{\phi_{t}^{\prime}}\left[\sum_{s=t}^{T} c_{s}^{p r s}\left(X_{s}^{0: 1}, M_{s}^{0: 1}, U_{s}^{p r s}\right) \mid h_{t}^{0}\right] \\
& =\mathbb{E}^{\phi_{t}^{\prime}}\left[c_{t}^{p r s}\left(X_{t}^{0: 1}, M_{t}^{0: 1}, U_{t}^{p r s}\right) \mid h_{t}^{0}\right] \\
& +\mathbb{E}^{\phi_{t}^{\prime}}\left[\mathbb{E}^{\phi_{t}^{\prime}}\left[\sum_{s=t+1}^{T} c_{s}^{p r s}\left(X_{s}^{0: 1}, M_{s}^{0: 1}, U_{s}^{p r s}\right) \mid H_{t+1}^{0}\right] \mid h_{t}^{0}\right] . \tag{69}
\end{align*}
$$

Note that the first term in (69) is equal to

$$
\begin{align*}
& \mathbb{E}^{\phi_{t}^{\prime}}\left[c_{t}^{p r s}\left(X_{t}^{0: 1}, M_{t}^{0: 1}, U_{t}^{p r s}\right) \mid h_{t}^{0}\right] \\
& =\sum_{m_{t}^{1} \in \mathcal{M}^{1}} \int c_{t}^{p r s}\left(x_{t}^{0: 1}, m_{t}^{0: 1}, u_{t}^{p r s *}\right) \theta_{t}\left(d x_{t}^{1}\right) \omega^{\tilde{z}_{t}}\left(m_{t}^{1}\right)  \tag{70}\\
& =: \mathbb{E}\left[c_{t}^{p r s}\left(x_{t}^{0}, X_{t}^{1}, m_{t}^{0}, M_{t}^{1}, u_{t}^{p r s}\right) \mid x_{t}^{0}, m_{t}^{0}, \theta_{t}, \tilde{z}_{t}, u_{t}^{p r s *}\right] \tag{71}
\end{align*}
$$

where the first equality is true because of Lemma 3] Furthermore, we can write (70) as (71) because of Lemma 3 where in (71), $X_{t}^{1}$ is a random vector distributed according to $\theta_{t}$.

According to 68), the second term in 69) can be written as

$$
\begin{align*}
& \mathbb{E}^{\phi_{t}^{\prime}}\left[V_{t+1}\left(X_{t+1}^{0}, M_{t+1}^{0}, \mathbb{P}^{\phi_{t}^{\diamond}, \phi_{t}^{p r s *}}\left(d x_{t+1}^{1} \mid H_{t+1}^{0}\right), \tilde{Z}_{t+1}\right) \mid h_{t}^{0}\right] \\
& =\mathbb{E}^{\phi_{t}^{\prime}}\left[V_{t+1}\left(X_{t+1}^{0}, M_{t+1}^{0}, \psi_{t}^{\circ *}\left(Z_{t+1}\right), \tilde{Z}_{t+1}\right) \mid h_{t}^{0}\right] \\
& =\sum_{\gamma_{t+1} \in\{0,1\}} \mathbb{E}^{\phi_{t}^{\prime}}\left[V_{t+1}\left(X_{t+1}^{0}, M_{t+1}^{0}, \psi_{t}^{\circ *}\left(Z_{t+1}\right), \tilde{Z}_{t+1}\right)\right. \\
& \left.\quad \mid h_{t}^{0}, \Gamma_{t+1}=\gamma_{t+1}\right] p\left(\gamma_{t+1}\right), \tag{72}
\end{align*}
$$

where we have defined

$$
\begin{equation*}
\psi_{t}^{\circ *}\left(Z_{t+1}\right):=\psi_{t}\left(\theta_{t}, u_{t}^{p r s *}, x_{t}^{0}, m_{t}^{0}, \tilde{z}_{t}, Z_{t+1}\right) \tag{73}
\end{equation*}
$$

Note that

$$
\begin{array}{r}
\mathbb{E}^{\phi_{t}^{\prime}}\left[V_{t+1}\left(X_{t+1}^{0}, M_{t+1}^{0}, \psi_{t}^{\circ *}(\emptyset), \emptyset\right) \mid h_{t}^{0}, \Gamma_{t+1}=0\right]=\sum_{m_{t+1}^{0} \in \mathcal{M}^{0}} \\
\int V_{t+1}\left(x_{t+1}^{0}, m_{t+1}^{0}, \alpha_{t}^{1 *}, \emptyset\right) \alpha_{t}^{0 *}\left(d x_{t+1}^{0}\right) \pi_{M^{0}}\left(m_{t+1}^{0}\right) . \tag{74}
\end{array}
$$

In (74), $\alpha_{t}^{1 *}=\psi_{t}^{\circ}(\emptyset)$ and $\alpha_{t}^{0 *}=\tilde{\psi}_{t}\left(x_{t}^{0}, m_{t}^{0}, u_{t}^{0 *}\right)$ where for any $E \subset \mathbb{R}^{d_{X}^{0}}$ we have

$$
\begin{align*}
& {\left[\tilde{\psi}_{t}\left(x_{t}^{0}, m_{t}^{0}, u_{t}^{0}\right)\right](E):=\mathbb{P}^{\phi_{t}^{\circ}}\left(X_{t+1}^{0} \in E \mid h_{t}^{0}\right)} \\
& \quad=\mathbb{P}\left(A^{00}\left(m_{t}^{0}\right) x_{t}^{0}+B^{00}\left(m_{t}^{0}\right) u_{t}^{0}+W_{t}^{0} \in E\right) \tag{75}
\end{align*}
$$

and the last equality of (74) is true because $\mathbb{P}^{\phi_{t}^{\prime}}\left(d x_{t+1}^{1} \mid h_{t}^{0}, \Gamma_{t+1}=0\right)=\alpha_{t}^{1 *}\left(d x_{t+1}^{1}\right)$ from Lemma
4 Furthermore,

$$
\begin{align*}
& \mathbb{E}^{\phi_{t}^{\prime}}\left[V_{t+1}\left(X_{t+1}^{0}, M_{t+1}^{0}, \psi_{t}^{\circ *}\left(X_{t+1}^{1}\right), M_{t+1}^{1}\right) \mid h_{t}^{0}, \Gamma_{t+1}=1\right] \\
& =\mathbb{E}^{\phi_{t}^{\prime}}\left[V_{t+1}\left(X_{t+1}^{0}, M_{t+1}^{0}, \psi_{t}^{\circ *}\left(X_{t+1}^{1}\right), M_{t+1}^{1}\right) \mid h_{t}^{0}, \Gamma_{t+1}=0\right] \\
& =\sum_{\substack{m_{t+1}^{0} \in \mathcal{M}^{0} \\
m_{t+1}^{1} \in \mathcal{M}^{1}}} \iint V_{t+1}\left(x_{t+1}^{0}, m_{t+1}^{0}, \delta\left(x_{t+1}^{1}\right), m_{t+1}^{1}\right) \\
& \quad \times \prod_{n \in\{0,1\}} \alpha_{t}^{n *}\left(d x_{t+1}^{n}\right) \pi_{M^{n}}\left(m_{t+1}^{n}\right), \tag{76}
\end{align*}
$$

The first equality in (76) is true because $X_{t+1}^{1: 2}$ and $M_{t+1}^{1: 2}$ are independent of $\Gamma_{t+1}$.

Now, by combining (74) and (76), (72) can be written as

$$
\begin{align*}
& \mathbb{E}^{\phi_{t}^{\prime}}\left[V_{t+1}\left(X_{t+1}^{0}, M_{t+1}^{0}, \psi_{t}^{\circ *}\left(Z_{t+1}\right), \tilde{Z}_{t+1}\right) \mid h_{t}^{0}\right] \\
& =\mathbb{E}\left[V_{t+1}\left(X_{t+1}^{0}, M_{t+1}^{0}, \psi_{t}^{\circ *}\left(Z_{t+1}\right), \tilde{Z}_{t+1}\right) \mid x_{t}^{0}, m_{t}^{0}, \theta_{t}, \tilde{z}_{t}, u_{t}^{p r s *}\right] . \tag{77}
\end{align*}
$$

Now, from (71) and (77), the right hand side of (69) is $V_{t}\left(x_{t}^{0}, m_{t}^{0}, \theta_{t}, \tilde{z}_{t}\right)$ from the definition of the value function (16). Hence, (65) is true at time $t$. The measurability of $V_{t}\left(x_{t}^{0}, m_{t}^{0}, \theta_{t}, \tilde{z}_{t}\right)$ with respect to $h_{t}^{0}$ is also resulted from the fact that $V_{t}\left(x_{t}^{0}, m_{t}^{0}, \mathbb{P}^{\phi_{t}^{\varnothing}}\left(d x_{t}^{1} \mid h_{t}^{0}\right), \tilde{z}_{t}\right)$ is equal to the conditional expectation $\mathbb{E}^{\phi_{t}^{\prime}}\left[\sum_{s=t}^{T} c_{s}^{p r s}\left(X_{s}^{0: 1}, M_{s}^{0: 1}, U_{s}^{p r s}\right) \mid h_{t}^{0}\right]$ which is measurable with respect to $h_{t}^{0}$.

Now let's consider (66). At $T+1$, (66) is true (all terms are defined to be 0 at $T+1$ ). Assume (66) is true at $t+1$. Let $u_{t}^{\text {prs }}=\left(u_{t}^{0}, \rho_{t}\right)=\phi^{\text {prs }}\left(h_{t}^{0}\right)$. Following an argument similar to that of (69)-(76),

$$
\begin{align*}
& \mathbb{E}^{\phi^{p r s}}\left[\sum_{s=t}^{T} c_{s}\left(X_{s}^{0: N}, U_{s}^{0: N}\right) \mid h_{t}^{0}\right] \geq \\
& \mathbb{E}\left[c_{t}^{p r s}\left(x_{t}^{0}, X_{t}^{1}, m_{t}^{0}, M_{t}^{1}, u_{t}^{p r s}\right) \mid x_{t}^{0}, m_{t}^{0}, \theta_{t}, \tilde{z}_{t}, u_{t}^{p r s}\right] \\
& +\mathbb{E}\left[V_{t+1}\left(X_{t+1}^{0}, M_{t+1}^{0}, \psi_{t}^{0}\left(Z_{t+1}\right), \tilde{Z}_{t+1}\right) \mid x_{t}^{0}, m_{t}^{0}, \theta_{t}, \tilde{z}_{t}, u_{t}^{p r s}\right] \\
& \geq V_{t}\left(x_{t}^{0}, m_{t}^{0}, \theta_{t}, \tilde{z}_{t}\right) . \tag{78}
\end{align*}
$$

where the last inequality follows from the definition of the value function (16). This completes the proof of the induction step, and the proof of the theorem.

## Appendix VII

Proof of Lemman
To show (20), let $\overline{\mathcal{P}}$ denote the set $\overline{\mathcal{P}}=\left\{\bar{q}_{t} \circ h_{2}+\tilde{q}_{t}\right.$ : $\left.\bar{q}_{t} \in \overline{\mathcal{Q}}, \tilde{q}_{t} \in \tilde{\mathcal{Q}}\left(\theta_{t}\right)\right\}$. Then, we want to show that for any $\theta_{t} \in \Delta\left(\mathbb{R}^{d_{X}^{1}}\right), \overline{\mathcal{P}}=\mathcal{P}$ where $\mathcal{P}=\left\{\rho: \mathbb{R}^{d_{X}^{1}} \times \mathcal{M}^{1} \rightarrow\right.$ $\mathbb{R}^{d_{U}^{1}}$ measurable $\}$.

First, assume $\rho_{t} \in \mathcal{P}$. For any $\theta_{t} \in \Delta\left(\mathbb{R}^{d_{X}^{1}}\right)$, define $\bar{q}_{t}(\cdot)=\int \rho_{t}\left(x_{t}^{1}, \cdot\right) \theta_{t}\left(d x_{t}^{1}\right)$ and $\tilde{q}_{t}=\rho_{t}-\bar{q}_{t} \circ h_{2}$. Then, we have $\rho_{t}=\bar{q}_{t} \circ h_{2}+\tilde{q}_{t}$. Note that $\bar{q}_{t} \in$ $\overline{\mathcal{Q}}$ and since $\rho_{t}$ is measurable, $\tilde{q}_{t}$ is measurable. Furthermore, $\int \tilde{q}_{t}\left(x_{t}^{1}, m_{t}^{1}\right) \theta_{t}\left(d x_{t}^{1}\right)=\int \rho_{t}\left(x_{t}^{1}, m_{t}^{1}\right) \theta_{t}\left(d x_{t}^{1}\right)-$ $\int_{\tilde{Q}} \rho_{t}\left(x_{t}^{1}, m_{t}^{1}\right) \theta_{t}\left(d x_{t}^{1}\right)=0$ for any $m_{t}^{1} \in \mathcal{M}^{1}$. Hence, $\tilde{q}_{t} \in$ $\tilde{\mathcal{Q}}\left(\theta_{t}\right)$. This concludes that $\rho_{t} \in \overline{\mathcal{P}}$.
For the reverse direction, assume that $\rho_{t} \in \overline{\mathcal{P}}$. Then, it can be written as $\rho_{t}=\bar{q}_{t} \circ h_{2}+\tilde{q}_{t}$ where $\bar{q}_{t} \in \overline{\mathcal{Q}}$ and $\tilde{q}_{t} \in \tilde{\mathcal{Q}}\left(\theta_{t}\right)$. Since, $\bar{q}_{t}$ and $\tilde{q}_{t}$ are measurable, $\rho_{t}$ is a measurable function from $\mathbb{R}^{d_{X}^{1}} \times \mathcal{M}^{1}$ to $\mathbb{R}^{d_{U}^{1}}$ and hence, $\rho_{t} \in \mathcal{P}$. This completes the proof.

## Appendix VIII Proof of Theorem 3

The proof is done by induction.

- At time $T+1$ :

It is clear that (21) is true because $P_{T+1}\left(m_{T+1}^{0}, \tilde{z}_{T+1}\right)=$ $\tilde{P}_{T+1}\left(m_{T+1}^{0}, \tilde{z}_{T+1}\right)=\mathbf{0}$ for any $m_{T+1}^{0} \in \mathcal{M}^{0}$, and $\tilde{z}_{T+1} \in$ $\tilde{\mathcal{M}}^{1}$, and $e_{T+1}=0$.

- At time $t+1$ :

Suppose (21) is true, that is, for any $x_{t}^{0} \in \mathbb{R}^{d_{X}^{0}}, m_{t}^{0} \in \mathcal{M}^{0}$, $\theta_{t} \in \Delta\left(\mathbb{R}^{d_{X}^{1}}\right)$, and $\tilde{z}_{t} \in \tilde{\mathcal{M}}^{1}$,

$$
\begin{align*}
& V_{t+1}\left(x_{t+1}^{0}, m_{t+1}^{0}, \theta_{t+1}, \tilde{z}_{t+1}\right)= \\
& \quad Q F\left(P_{t+1}\left(m_{t+1}^{0}, \tilde{z}_{t+1}\right), \operatorname{vec}\left(x_{t+1}^{0}, \mu\left(\theta_{t+1}\right)\right)\right) \\
& \quad+\operatorname{tr}\left(\tilde{P}_{t+1}\left(m_{t+1}^{0}, \tilde{z}_{t+1}\right) \operatorname{cov}\left(\theta_{t+1}\right)\right)+e_{t+1} \tag{79}
\end{align*}
$$

and the matrices $P_{t+1}\left(m_{t+1}^{0}, \tilde{z}_{t+1}\right)$ and $\tilde{P}_{t+1}\left(m_{t+1}^{0}, \tilde{z}_{t+1}\right)$ are all positive semi-definite (PSD) for any $m_{t+1}^{0} \in \mathcal{M}^{0}$, and $\tilde{z}_{t+1} \in \tilde{\mathcal{M}}^{1}$.

- At time $t$ :

Let's now compute the value function at $t$ given by (16) in Theorem 1 1 In order to do so, we need to calculate

where the first and second term are as defined in (71) and (77), respectively and $\psi_{t}^{\circ}\left(Z_{t+1}\right)=$ $\psi_{t}\left(\theta_{t}, u_{t}^{p r s}, x_{t}^{0}, m_{t}^{0}, \tilde{z}_{t}, Z_{t+1}\right)$.

To this end, we consider two following cases.
A. $\quad \tilde{z}_{t}=\emptyset$

This corresponds to the case that $\gamma_{t}=0$ from (6). In the following we calculate $\mathbb{T}_{1}(\emptyset)$ and $\mathbb{T}_{2}(\emptyset)$.

Calculating $\mathbb{T}_{1}(\emptyset)$ :
Let $S_{t}^{\theta_{t}, m_{t}^{1}}:=\operatorname{vec}\left(x_{t}^{0}, X^{\theta_{t}}, u_{t}^{0}, \bar{q}_{t}\left(m_{t}^{1}\right)+\tilde{q}_{t}\left(X^{\theta_{t}}, m_{t}^{1}\right)\right)$ where $X^{\theta_{t}}$ is a random vector with distribution $\theta_{t}$. Then, according to (9) and (13),

$$
\begin{align*}
\mathbb{T}_{1}(\emptyset) & =\mathbb{E}\left[c_{t}^{p r s}\left(x_{t}^{0}, X_{t}^{1}, m_{t}^{0}, M_{t}^{1}, u_{t}^{p r s}\right) \mid x_{t}^{0}, m_{t}^{0}, \theta_{t}, \emptyset, u_{t}^{p r s}\right] \\
& =\sum_{m_{t}^{1} \in \mathcal{M}^{1}} \int c_{t}^{p r s}\left(x_{t}^{0: 1}, m_{t}^{0: 1}, u_{t}^{p r s}\right) \theta_{t}\left(d x_{t}^{1}\right) \pi_{M^{1}}\left(m_{t}^{1}\right) \\
& =\sum_{m_{t}^{1} \in \mathcal{M}^{1}} \mathbb{E}\left[c_{t}^{p r s}\left(x_{t}^{0}, X^{\theta_{t}}, m_{t}^{0: 1}, u_{t}^{p r s}\right)\right] \pi_{M^{1}}\left(m_{t}^{1}\right) \\
& =\sum_{m_{t}^{1} \in \mathcal{M}^{1}} \mathbb{E}\left[Q F\left(C_{t}\left(m_{t}^{0: 1}\right), S_{t}^{\theta_{t}, m_{t}^{1}}\right)\right] \pi_{M^{1}}\left(m_{t}^{1}\right) \\
& =\sum_{m_{t}^{1} \in \mathcal{M}^{1}}\left[Q F\left(C_{t}\left(m_{t}^{0: 1}\right), \mathbb{E}\left[S_{t}^{\theta_{t}, m_{t}^{1}}\right]\right)\right. \\
& \left.+\operatorname{tr}\left(C_{t}\left(m_{t}^{0: 1}\right) \operatorname{cov}\left(S_{t}^{\theta_{t}, m_{t}^{1}}\right)\right)\right] \pi_{M^{1}}\left(m_{t}^{1}\right) \tag{81}
\end{align*}
$$

where the second equality is true because of (71).
Note that in the minimization of Theorem 2, we are looking for only $\tilde{q}_{t} \in \tilde{\mathcal{Q}}\left(\theta_{t}\right)$. Hence, according to (19), $\mathbb{E}\left[\tilde{q}_{t}\left(X^{\theta_{t}}, m_{t}^{1}\right)\right]=0$ and

$$
\begin{align*}
\mathbb{E}\left[S_{t}^{\theta_{t}, m_{t}^{1}}\right] & =\operatorname{vec}\left(x_{t}^{0}, \mu\left(\theta_{t}\right), u_{t}^{0}, \bar{q}_{t}\left(m_{t}^{1}\right)\right)  \tag{82}\\
\operatorname{cov}\left(S_{t}^{\theta_{t}, m_{t}^{1}}\right) & =\mathbf{c o v}\left(\mathbf{v e c}\left(0, X^{\theta_{t}}, 0, \tilde{q}_{t}\left(X^{\theta_{t}}, m_{t}^{1}\right)\right)\right) . \tag{83}
\end{align*}
$$

Let $\bar{S}_{t}^{\theta_{t}}=\operatorname{vec}\left(x_{t}^{0}, \mu\left(\theta_{t}\right), u_{t}^{0}, \bar{q}_{t}(1), \ldots, \bar{q}_{t}\left(\kappa^{1}\right)\right)$. Then,

$$
\begin{equation*}
\mathbb{E}\left[S_{t}^{\theta_{t}, m_{t}^{1}}\right]=L_{m_{t}^{1}} \bar{S}_{t}^{\theta_{t}} \tag{84}
\end{equation*}
$$

where $L_{m_{t}^{1}}$ is as defined in (32).
Furthermore, according to (83), we can write

$$
\begin{equation*}
\operatorname{tr}\left(C_{t}\left(m_{t}^{0: 1}\right) \operatorname{cov}\left(S_{t}^{\theta_{t}, m_{t}^{1}}\right)\right)=\operatorname{tr}\left(C_{t}^{11}\left(m_{t}^{0: 1}\right) \operatorname{cov}\left(\tilde{S}_{t}^{\theta_{t}, m_{t}^{1}}\right)\right), \tag{85}
\end{equation*}
$$

where we defined $\tilde{S}_{t}^{\theta_{t}, m_{t}^{1}}:=\operatorname{vec}\left(X^{\theta_{t}}, \tilde{q}_{t}\left(X^{\theta_{t}}, m_{t}^{1}\right)\right)$. Now, using (84) and (85), (81) can be written as,

$$
\begin{align*}
\mathbb{T}_{1}(\emptyset) & =Q F\left(C_{t}^{\emptyset}\left(m_{t}^{0}\right), \bar{S}_{t}^{\theta_{t}}\right) \\
& +\sum_{m_{t}^{1} \in \mathcal{M}^{1}} \operatorname{tr}\left(C_{t}^{11}\left(m_{t}^{0: 1}\right) \operatorname{cov}\left(\tilde{S}_{t}^{\theta_{t}, m_{t}^{1}}\right)\right) \pi_{M^{1}}\left(m_{t}^{1}\right) . \tag{86}
\end{align*}
$$

## Calculating $\mathbb{T}_{2}(\emptyset)$ :

$\overline{\text { We first calculate }} \mathbb{T}_{2}\left(\tilde{z}_{t}\right)$ for any $\tilde{z}_{t} \in \overline{\mathcal{M}}^{1}$, and then simplify it for the case $\tilde{z}_{t}=\emptyset$. According to (77),

$$
\begin{align*}
\mathbb{T}_{2}\left(\tilde{z}_{t}\right)= & \sum_{\gamma_{t+1} \in\{0,1\}} p\left(\gamma_{t+1}\right) \sum_{\substack{m_{t+1}^{0} \in \mathcal{M}^{0} \\
m_{t+1}^{t} \in \mathcal{M}^{1}}} \\
& \iint \underbrace{V_{t+1}\left(x_{t+1}^{0}, m_{t+1}^{0}, N B\left(\gamma_{t+1}, \alpha_{t}^{1}, x_{t+1}^{1}\right), \tilde{z}_{t+1}\right)}_{\mathbb{T}_{3}} \\
& \times \prod_{n \in\{0,1\}} \alpha_{t}^{n}\left(d x_{t+1}^{n}\right) \pi_{M^{n}}\left(m_{t+1}^{n}\right) . \tag{87}
\end{align*}
$$

where $\alpha_{t}^{1}=\psi_{t}^{\circ}(\emptyset)=\psi_{t}\left(\theta_{t}, u_{t}^{p r s}, x_{t}^{0}, m_{t}^{0}, \tilde{z}_{t}, \emptyset\right)$, $\alpha_{t}^{0}=\tilde{\psi}_{t}\left(x_{t}^{0}, m_{t}^{0}, u_{t}^{0}\right)$ as defined in (75), and we defined $N B\left(\gamma_{t+1}, \alpha_{t}^{1}, x_{t+1}^{1}\right):=\left(1-\gamma_{t+1}\right) \alpha_{t}^{1}+\gamma_{t+1} \delta\left(x_{t+1}^{1}\right)$. Further, we can write $x_{t+1}^{0}=\mu\left(\alpha_{t}^{0}\right)+\left(x_{t+1}^{0}-\mu\left(\alpha_{t}^{0}\right)\right)$. Considering these and after some algebra, we can get,

$$
\begin{align*}
& \iint \mathbb{T}_{3} \prod_{n \in\{0,1\}} \alpha_{t}^{n}\left(d x_{t+1}^{n}\right) \\
& =\gamma_{t+1} Q F\left(P_{t+1}\left(m_{t+1}^{0}, m_{t+1}^{1}\right), \operatorname{vec}\left(\mu\left(\alpha_{t}^{0}\right), \mu\left(\alpha_{t}^{1}\right)\right)\right) \\
& +\left(1-\gamma_{t+1}\right) Q F\left(P_{t+1}\left(m_{t+1}^{0}, \emptyset\right), \operatorname{vec}\left(\mu\left(\alpha_{t}^{0}\right), \mu\left(\alpha_{t}^{1}\right)\right)\right) \\
& +\gamma_{t+1} \operatorname{tr}\left(P_{t+1}^{00}\left(m_{t+1}^{0}, m_{t+1}^{1}\right) \operatorname{cov}\left(\alpha_{t}^{0}\right)\right) \\
& +\left(1-\gamma_{t+1}\right) \operatorname{tr}\left(P_{t+1}^{00}\left(m_{t+1}^{0}, \emptyset\right) \operatorname{cov}\left(\alpha_{t}^{0}\right)\right) \\
& +\gamma_{t+1} \operatorname{tr}\left(P_{t+1}^{11}\left(m_{t+1}^{0}, m_{t+1}\right) \operatorname{cov}\left(\alpha_{t}^{1}\right)\right) \\
& +\left(1-\gamma_{t+1}\right) \operatorname{tr}\left(\tilde{P}_{t+1}\left(m_{t+1}^{0}, \emptyset\right) \operatorname{cov}\left(\alpha_{t}^{1}\right)\right)+e_{t+1} . \tag{88}
\end{align*}
$$

Note that $\alpha_{t}^{1}$ depends on $\tilde{z}_{t}$ and $\theta_{t}$. To make this dependence apparent, we write $\alpha_{t}^{1}$ as $\psi_{t}^{\triangleleft}\left(\tilde{z}_{t}, \theta_{t}\right)$. Then, from (88) and using operators $\Pi$ and $\Psi$ defined in (29) and (31), (87) can be written as

$$
\begin{align*}
& \mathbb{T}_{2}\left(\tilde{z}_{t}\right)=\underbrace{Q F\left(\Pi\left(P_{t+1}\right), \operatorname{vec}\left(\mu\left(\alpha_{t}^{0}\right), \mu\left(\psi_{t}^{\triangleleft}\left(\tilde{z}_{t}, \theta_{t}\right)\right)\right)\right)}_{\mathbb{T}_{4}\left(\tilde{z}_{t}\right)} \\
& +\underbrace{\operatorname{tr}\left(\Pi\left(P_{t+1}^{00}\right) \operatorname{cov}\left(\alpha_{t}^{0}\right)\right)}_{\mathbb{T}_{5}} \\
& +\underbrace{\operatorname{tr}\left(\Psi\left(\tilde{P}_{t+1}, P_{t+1}^{11}\right) \operatorname{cov}\left(\psi_{t}^{\triangleleft}\left(\tilde{z}_{t}, \theta_{t}\right)\right)\right)}_{\mathbb{T}_{6}\left(\tilde{z}_{t}\right)}+e_{t+1} . \tag{89}
\end{align*}
$$

Now, to calculate $\mathbb{T}_{2}(\emptyset)$ from (89), we need to calculate mean and covariance of $\alpha_{t}^{0}$ and $\psi_{t}^{\triangleleft}\left(\emptyset, \theta_{t}\right)$.

Calculating mean and covariance of $\alpha_{t}^{0}$ and $\psi_{t}^{\triangleleft}\left(\emptyset, \theta_{t}\right)$ :
Let $S_{t}^{\theta_{t}, M_{t}^{1}}:=\operatorname{vec}\left(x_{t}^{0}, X^{\theta_{t}}, u_{t}^{0}, \bar{q}_{t}\left(M_{t}^{1}\right)+\tilde{q}_{t}\left(X^{\theta_{t}}, M_{t}^{1}\right)\right)$ where $X^{\theta_{t}}$ is a random vector independent of $M_{t}^{1}$ and with distribution $\theta_{t}$. Then, we define $Y_{t}^{\theta_{t}, M_{t}^{1}}:=$ $\left[D\left(m_{t}^{0}, M_{t}^{1}\right)\right]_{1} \bullet S_{t}^{\theta_{t}, M_{t}^{1}}+W_{t}^{1}$ and $\tilde{Y}_{t}:=A^{00}\left(m_{t}^{0}\right) x_{t}^{0}+$ $B^{00}\left(m_{t}^{0}\right) u_{t}^{0}+W_{t}^{0}$. From (56) in Lemma 4 we know that $Y_{t}^{\theta_{t}, M_{t}^{1}}$ has distribution $\psi_{t}^{\triangleleft}\left(\emptyset, \theta_{t}\right)$, and furthermore, from (75) we know that $\tilde{Y}_{t}$ has distribution $\alpha_{t}^{0}$. Then, using the fact that from (84), we can write $\mathbb{E}\left[S_{t}^{\theta_{t}, m_{t}^{1}}\right]=L_{m_{t}^{1}} \bar{S}_{t}^{\theta_{t}}$, we have

$$
\begin{align*}
\mu\left(\alpha_{t}^{0}\right) & =\mathbb{E}\left[\tilde{Y}_{t}\right]=A^{00}\left(m_{t}^{0}\right) x_{t}^{0}+B^{00}\left(m_{t}^{0}\right) u_{t}^{0} \\
& =\sum_{m_{t}^{1} \in \mathcal{M}^{1}}\left[D\left(m_{t}^{0: 1}\right)\right]_{0 \bullet} L_{m_{t}^{1}} \bar{S}_{t}^{\theta_{t}} \pi_{M^{1}}\left(m_{t}^{1}\right),  \tag{90}\\
\operatorname{cov}\left(\alpha_{t}^{0}\right) & =\mathbf{\operatorname { c o v } ( \tilde { Y } _ { t } ) = \operatorname { c o v } ( W _ { t } ^ { 0 } ) ,}  \tag{91}\\
\mu\left(\psi_{t}^{\triangleleft}\left(\emptyset, \theta_{t}\right)\right)= & \mathbb{E}\left[Y_{t}^{\theta_{t}, M_{t}^{1}}\right] \\
= & \sum_{m_{t}^{1} \in \mathcal{M}^{1}}\left[D\left(m_{t}^{0: 1}\right)\right]_{1} L_{m_{t}^{1}} \bar{S}_{t}^{\theta_{t}} \pi_{M^{1}}\left(m_{t}^{1}\right),  \tag{92}\\
\operatorname{cov}\left(\psi_{t}^{\triangleleft}\left(\emptyset, \theta_{t}\right)\right)= & \operatorname{cov}\left(Y_{t}^{\theta_{t}, M_{t}^{1}}\right)=\mathbb{E}\left[\operatorname{cov}\left(Y_{t}^{\theta_{t}, M_{t}^{1}} \mid M_{t}^{1}\right)\right] \\
& +\operatorname{cov}\left(\mathbb{E}\left[Y_{t}^{\theta_{t}, M_{t}^{1}} \mid M_{t}^{1}\right]\right), \tag{93}
\end{align*}
$$

where the last equality of (90) is true because $\left[D\left(m_{t}^{0: 1}\right)\right]_{0 \bullet}=$ $\left[\begin{array}{llll}A^{00}\left(m_{t}^{0}\right) & \mathbf{0} & B^{00}\left(m_{t}^{0}\right) & \mathbf{0}\end{array}\right]$ and further, (93) is true be-
cause of "Law of total variance" and

$$
\begin{align*}
& \mathbb{E}\left[\operatorname{cov}\left(Y_{t}^{\theta_{t}, M_{t}^{1}} \mid M_{t}^{1}\right)\right]=\sum_{m_{t}^{1} \in \mathcal{M}^{1}} \operatorname{cov}\left(Y_{t}^{\theta_{t}, m_{t}^{1}} \mid m_{t}^{1}\right) \pi_{M^{1}}\left(m_{t}^{1}\right)= \\
& \sum_{m_{t}^{1} \in \mathcal{M}^{1}} D^{11}\left(m_{t}^{0: 1}\right) \operatorname{cov}\left(\tilde{S}_{t}^{\theta_{t}, m_{t}^{1}}\right) D^{11}\left(m_{t}^{0: 1}\right)^{\top} \pi_{M^{1}}\left(m_{t}^{1}\right)+\operatorname{cov}\left(W_{t}^{1}\right), \tag{94}
\end{align*}
$$

$\operatorname{cov}\left(\mathbb{E}\left[Y_{t}^{\theta_{t}, M_{t}^{1}} \mid M_{t}^{1}\right]\right)=\mathbf{c o v}\left(\left[D\left(m_{t}^{0}, M_{t}^{1}\right)\right]_{1} \bullet \mathbb{E}\left[S_{t}^{\theta_{t}, M_{t}^{1}} \mid M_{t}^{1}\right]\right)=$ $\sum_{m_{t}^{1} \in \mathcal{M}^{1}}\left[D\left(m_{t}^{0: 1}\right)\right]_{1} \bullet L_{m_{t}^{1}} \bar{S}_{t}^{\theta_{t}}\left(L_{m_{t}^{1}} \bar{S}_{t}^{\theta_{t}}\right)^{\top}\left[D\left(m_{t}^{0: 1}\right)\right]_{1}^{\top} \pi_{M^{1}}\left(m_{t}^{1}\right)$
$-\left[\tilde{D}^{\emptyset}\left(m_{t}^{0}\right)\right]_{\bullet}\left[\tilde{D}^{\emptyset}\left(m_{t}^{0}\right)\right]_{1}^{\top}$,
where we defined $\tilde{D}^{\emptyset}\left(m_{t}^{0}\right)$
$\sum_{m_{t}^{1} \in \mathcal{M}^{1}}\left[D\left(m_{t}^{0: 1}\right)\right]_{1} \bullet L_{m_{t}^{1}} \bar{S}_{t}^{\theta_{t}} \pi_{M^{1}}\left(m_{t}^{1}\right)$.
Now that we have calculated the mean and covariance of $\alpha_{t}^{0}$ and $\psi_{t}^{\triangleleft}\left(\emptyset, \theta_{t}\right)$ in (90)-(95), using the matrices $E_{t}^{\emptyset}, F_{t}^{\emptyset}$ defined in (35) and (36), $\mathbb{T}_{4}(\emptyset), \mathbb{T}_{5}$, and $\mathbb{T}_{6}(\emptyset)$ can be respectively written as follows:

$$
\begin{align*}
\mathbb{T}_{4}(\emptyset) & =Q F\left(E_{t}^{\emptyset}\left(m_{t}^{0}\right), \bar{S}_{t}^{\theta_{t}}\right)  \tag{96}\\
\mathbb{T}_{5} & =\operatorname{tr}\left(\Pi\left(P_{t+1}^{00}\right) \operatorname{cov}\left(W_{t}^{0}\right)\right)  \tag{97}\\
\mathbb{T}_{6}(\emptyset) & =\sum_{m_{t}^{1} \in \mathcal{M}^{1}} \operatorname{tr}\left(G_{t}\left(m_{t}^{0: 1}\right) \operatorname{cov}\left(\tilde{S}_{t}^{\theta_{t}, m_{t}^{1}}\right)\right) \pi_{M^{1}}\left(m_{t}^{1}\right) \\
& +Q F\left(F_{t}^{\emptyset}\left(m_{t}^{0}\right), \bar{S}_{t}^{\theta_{t}}\right)+\mathbf{t r}\left(\Psi\left(\tilde{P}_{t+1}, P_{t+1}^{11}\right) \operatorname{cov}\left(W_{t}^{1}\right)\right) \tag{98}
\end{align*}
$$

where we have defined $G_{t}\left(m_{t}^{0: 1}\right)=$ $D^{11}\left(m^{0: 1}\right)^{\top} \Psi\left(\tilde{P}_{t+1}, P_{t+1}^{11}\right) D^{11}\left(m^{0: 1}\right)$. Now, considering (96)-(98), (89) can be written as

$$
\begin{align*}
& \mathbb{T}_{2}(\emptyset)=Q F\left(E_{t}^{\emptyset}\left(m_{t}^{0}\right)+F_{t}^{\emptyset}\left(m_{t}^{0}\right), \bar{S}_{t}^{\theta_{t}}\right)+ \\
& \sum_{m_{t}^{1} \in \mathcal{M}^{1}} \operatorname{tr}\left(G_{t}\left(m_{t}^{0: 1}\right) \operatorname{cov}\left(\tilde{S}_{t}^{\theta_{t}, m_{t}^{1}}\right)\right) \pi_{M^{1}}\left(m_{t}^{1}\right)+e_{t} \tag{99}
\end{align*}
$$

where

$$
\begin{align*}
e_{t} & =e_{t+1}+\operatorname{tr}\left(\Pi\left(P_{t+1}^{00}\right) \operatorname{cov}\left(W_{t}^{0}\right)\right) \\
& +\operatorname{tr}\left(\Psi\left(\tilde{P}_{t+1}, P_{t+1}^{11}\right) \operatorname{cov}\left(W_{t}^{1}\right)\right) . \tag{100}
\end{align*}
$$

Calculating $\mathbb{T}(\emptyset)$ :
Now that we have calculated $\mathbb{T}_{1}(\emptyset)$ and $\mathbb{T}_{2}(\emptyset)$, we use them to simplify $\mathbb{T}(\emptyset)$, and then by substituting it into the value function of (16) we get

$$
\begin{align*}
& V_{t}\left(x_{t}^{0}, m_{t}^{0}, \theta_{t}, \tilde{z}_{t}\right)=e_{t}+\min _{u_{t}^{0} \in \mathbb{R}_{U}^{d_{U}^{0}} \overline{q_{t} \in \overline{\mathcal{Q}}, \tilde{q}_{t} \in \tilde{\mathcal{Q}}\left(\theta_{t}\right)}}\{ \\
& \quad Q F\left(H_{t}\left(m_{t}^{0}, \emptyset\right), \bar{S}_{t}^{\theta_{t}}\right) \\
& \left.\quad+\sum_{m_{t}^{1} \in \mathcal{M}^{1}} \operatorname{tr}\left(\tilde{H}_{t}\left(m_{t}^{0: 1}\right) \operatorname{cov}\left(\tilde{S}_{t}^{\theta_{t}, m_{t}^{1}}\right)\right) \pi_{M^{1}}\left(m_{t}^{1}\right)\right\} . \tag{101}
\end{align*}
$$

Note that $\bar{S}_{t}^{\theta_{t}}=\operatorname{vec}\left(x_{t}^{0}, \mu\left(\theta_{t}\right), u_{t}^{0}, \bar{q}_{t}(1), \ldots, \bar{q}_{t}\left(\kappa^{1}\right)\right)$ depends only on $u_{t}^{0}$ and $\bar{q}_{t}$. Furthermore, we have $\tilde{S}_{t}^{\theta_{t}, m_{t}^{1}}=$ $\operatorname{vec}\left(X^{\theta_{t}}, \tilde{q}_{n}\left(X^{\theta_{t}}, m_{t}^{1}\right)\right)$, and hence $\operatorname{cov}\left(\tilde{S}_{t}^{\theta_{t}, m_{t}^{1}}\right)$ depends only on the choice of $\tilde{q}_{t}\left(\cdot, m_{t}^{1}\right)$. Consequently, in order to
solve the optimization problem in the (101), we need to solve the two optimization problems

$$
\begin{align*}
& \min _{u_{t}^{0} \in \mathbb{R}_{U, \bar{q}_{t} \in \overline{\mathcal{Q}}}^{d}} Q F\left(H_{t}\left(m_{t}^{0}, \emptyset\right), \bar{S}_{t}^{\theta_{t}}\right),  \tag{102}\\
& \min _{\tilde{q}_{t} \in \tilde{\mathcal{Q}}\left(\theta_{t}\right)} \sum_{m_{t}^{1} \in \mathcal{M}^{1}} \operatorname{tr}\left(\pi_{M^{1}}\left(m_{t}^{1}\right) \tilde{H}_{t}\left(m_{t}^{0: 1}\right) \operatorname{cov}\left(\tilde{S}_{t}^{\theta_{t}, m_{t}^{1}}\right)\right) \tag{103}
\end{align*}
$$

Since $H_{t}\left(m_{t}^{0}, \tilde{z}_{t}\right)$ is PD, it follows from [1, Lemma 4] that the optimal solution of (102) is given by (22) and

$$
\begin{align*}
& \min _{u_{t}^{0} \in \mathbb{R}_{U}^{d_{U}^{0}, \bar{q}_{t} \in \overline{\mathcal{Q}}}} Q F\left(H_{t}\left(m_{t}^{0}, \emptyset\right), \bar{S}_{t}^{\theta_{t}}\right) \\
& =Q F\left(P_{t}\left(m_{t}^{0}, \emptyset\right), \operatorname{vec}\left(x_{t}^{0}, \mu\left(\theta_{t}\right)\right)\right) . \tag{104}
\end{align*}
$$

Furthermore, $P_{t}\left(m^{0}, \emptyset\right)$ is PSD because it is Schur complement of $H_{t}\left(m^{0}, \tilde{z}\right)$ with respect to $H_{t}^{U U}\left(m^{0}, \emptyset\right)$. Similarly, since $\tilde{H}_{t}\left(m_{t}^{0: 1}\right)$ is also PD, from [1, Lemma 4], the optimal solution of (103) is given by (26) and

$$
\begin{align*}
& \min _{\tilde{q}_{t} \in \tilde{\mathcal{Q}}\left(\theta_{t}\right)} \sum_{m_{t}^{1} \in \mathcal{M}^{1}} \operatorname{tr}\left(\pi_{M^{1}}\left(m_{t}^{1}\right) \tilde{H}_{t}\left(m_{t}^{0: 1}\right) \operatorname{cov}\left(\tilde{S}_{t}^{\theta_{t}, m_{t}^{1}}\right)\right) \\
& \sum_{m_{t}^{1} \in \mathcal{M}^{1}} \min _{\substack{\left.\tilde{q}_{t}\left(x_{t}^{1}, m_{t}^{1}\right) \theta_{t}\left(d x_{t}^{1}\right)=0 \\
\tilde{q}_{t}, \cdot m_{t}^{1}\right):}} \operatorname{tr}\left(\pi_{M^{1}}\left(m_{t}^{1}\right) \tilde{H}_{t}\left(m_{t}^{0: 1}\right) \operatorname{cov}\left(\tilde{S}_{t}^{\theta_{t}, m_{t}^{1}}\right)\right) \\
& =\operatorname{tr}\left(\tilde{P}_{t}\left(m_{t}^{0}, \emptyset\right) \operatorname{cov}\left(\theta_{t}\right)\right) . \tag{105}
\end{align*}
$$

Note that from (26), we have $\int \tilde{q}_{t}\left(x_{t}^{1}, m_{t}^{1}\right) \theta_{t}\left(d x_{t}^{1}\right)=0$ for all $m_{t}^{1} \in \mathcal{M}^{1}$ and hence, $\tilde{q}_{t} \in \tilde{\mathcal{Q}}\left(\theta_{t}\right)$. Furthermore, $\tilde{P}_{t}\left(m^{0}, \emptyset\right)$ is PSD because it is convex combination of Schur complements of $\tilde{H}_{t}\left(m^{0}, \emptyset\right)$ with respect to $\tilde{H}_{t}^{U^{1} U^{1}}\left(m^{0}, \emptyset\right)$.

Finally, substituting (104) and (105) into (101), we observed that $V_{t}$ has the form described in (21). This completes the proof of the induction step for the case $\tilde{z}_{t}=\emptyset$ and the proof of the theorem for this case. Next we consider the case $\tilde{z}_{t}=l_{t}^{1}$.

## B. $\tilde{z}_{t}=l$ for some $l \in \mathcal{M}^{1}$

This corresponds to the case that $\gamma_{t}=1$ from (6). Hence, in this case $z_{t}=x_{t}^{1}$ and from (55), $\theta_{t}=\delta\left(x_{t}^{1}\right)$. In the following we calculate $\mathbb{T}_{1}(l)$ and $\mathbb{T}_{2}(l)$.

## Calculating $\mathbb{T}_{1}(l)$ :

Remember that we defined $S_{t}^{\theta_{t}, m_{t}^{1}}=$ $\operatorname{vec}\left(x_{t}^{0}, X^{\theta_{t}}, u_{t}^{0}, \bar{q}_{t}\left(m_{t}^{1}\right)+\tilde{q}_{t}\left(X^{\theta_{t}}, m_{t}^{1}\right)\right)$ where $X^{\theta_{t}}$ is a random vector with distribution $\theta_{t}$. In this case, we have

$$
\begin{align*}
& \mathbb{E}\left[S_{t}^{\delta\left(x_{t}^{1}\right), l}\right]=\operatorname{vec}\left(x_{t}^{0}, x_{t}^{1}, u_{t}^{0}, \bar{q}_{t}(l)\right),  \tag{106}\\
& \operatorname{cov}\left(S_{t}^{\delta\left(x_{t}^{1}\right), l}\right)=\mathbf{0} . \tag{107}
\end{align*}
$$

According to (9) and (13),

$$
\begin{align*}
\mathbb{T}_{1}\left(l_{t}^{1}\right) & =\mathbb{E}\left[c_{t}^{p r s}\left(x_{t}^{0}, X_{t}^{1}, m_{t}^{0}, M_{t}^{1}, u_{t}^{p r s}\right) \mid x_{t}^{0}, m_{t}^{0}, \delta\left(x_{t}^{1}\right), l, u_{t}^{p r s}\right] \\
& =\mathbb{E}\left[Q F\left(C_{t}\left(m_{t}^{0}, l_{t}^{1}\right), S_{t}^{\delta\left(x_{t}^{1}\right), l}\right)\right] \\
& =Q F\left(C_{t}\left(m_{t}^{0}, l\right), \mathbb{E}\left[S_{t}^{\delta\left(x_{t}^{1}\right), l}\right]\right) \\
& +\operatorname{tr}\left(C_{t}\left(m_{t}^{0}, l\right) \operatorname{cov}\left(S_{t}^{\delta\left(x_{t}^{1}\right), l}\right)\right) \\
& =Q F\left(C_{t}\left(m_{t}^{0}, l\right), \operatorname{vec}\left(x_{t}^{0}, x_{t}^{1}, u_{t}^{0}, \bar{q}_{t}(l)\right)\right) \tag{108}
\end{align*}
$$

where the second equality is true because of (71).

Calculating $\mathbb{T}_{2}(l)$ :
Let $Y_{t}^{\delta\left(x_{t}^{1}\right), l}:=\left[D\left(m_{t}^{0}, l\right)\right]_{1} \cdot S_{t}^{\delta\left(x_{t}^{1}\right), l}+W_{t}^{1}$. Then, from (56) in Lemma 4, we know that $Y_{t}^{\delta\left(x_{t}^{1}\right), l}$ has distribution $\psi_{t}^{\triangleleft}\left(l, \delta\left(x_{t}^{1}\right)\right)$ and we have,

$$
\begin{align*}
& \left.\mu\left(\psi_{t}^{\triangleleft}\left(l, \delta\left(x_{t}^{1}\right)\right)\right)=\mathbb{E}\left[Y_{t}^{\delta\left(x_{t}^{1}\right), l}\right]=\left[D\left(m_{t}^{0}, l\right)\right]\right]_{1} \mathbb{E}\left[S_{t}^{\delta\left(x_{t}^{1}\right), l}\right]  \tag{109}\\
& \operatorname{cov}\left(\psi_{t}^{\triangleleft}\left(l, \delta\left(x_{t}^{1}\right)\right)\right)=\operatorname{cov}\left(W_{t}^{1}\right) . \tag{110}
\end{align*}
$$

Furthermore, in this case from (90) and (91),

$$
\begin{equation*}
\mu\left(\alpha_{t}^{0}\right)=\left[D\left(m_{t}^{0}, l\right)\right]_{0} \mathbb{E}\left[S_{t}^{\delta\left(x_{t}^{1}\right), l}\right], \quad \operatorname{cov}\left(\alpha_{t}^{0}\right)=\operatorname{cov}\left(W_{t}^{0}\right) . \tag{111}
\end{equation*}
$$

Now, considering (109), (110), and (111), $\mathbb{T}_{2}(l)$ from (89) can be calculate as follows,

$$
\begin{equation*}
\mathbb{T}_{2}(l)=Q F\left(D\left(m_{t}^{0}, l\right)^{\top} \Pi\left(P_{t+1}\right) D\left(m_{t}^{0}, l\right), \mathbb{E}\left[S_{t}^{\delta\left(x_{t}^{1}\right), l}\right]\right)+e_{t} \tag{112}
\end{equation*}
$$

where $e_{t}$ is as described in (100).
Calculating $\mathbb{T}(l)$ :
Now that we have calculated $\mathbb{T}_{1}(l)$ and $\mathbb{T}_{2}(l)$, we use them to simplify $\mathbb{T}(l)$, and then by substituting it into the value function of (16) we get

$$
\begin{align*}
& V_{t}\left(x_{t}^{0}, m_{t}^{0}, \theta_{t}, \tilde{z}_{t}\right)=e_{t}+\min _{u_{t}^{0} \in \mathbb{R}_{U, \bar{q}_{t} \in \overline{\mathcal{Q}}, \tilde{q}_{t} \in \tilde{\mathcal{Q}}\left(\theta_{t}\right)}\{ }^{\left.Q F\left(H_{t}\left(m_{t}^{0}, l\right) \operatorname{vec}\left(x_{t}^{0}, x_{t}^{1}, u_{t}^{0}, \bar{q}_{t}(l)\right)\right)\right\} .} \$\{
\end{align*}
$$

Note that the term inside the minimization does not depend on function $\tilde{q}_{t}$ and also it does not depends on $\bar{q}_{t}\left(m_{t}^{1}\right)$ for all $m_{t}^{1} \in \mathcal{M}^{1} \backslash\{l\}$. Hence, they can be chosen arbitrarily or set to be zero as described in (25) and (26). Since $H_{t}\left(m_{t}^{0}, l\right)$ is PD, it follows from [1, Lemma 4] that the optimal solution of (102) is given by (24) and

$$
\begin{align*}
& \min Q F\left(H_{t}\left(m_{t}^{0}, l\right) \operatorname{vec}\left(x_{t}^{0}, x_{t}^{1}, u_{t}^{0}, \bar{q}_{t}(l)\right)\right) \\
& =Q F\left(P_{t}\left(m_{t}^{0}, l\right), \operatorname{vec}\left(x_{t}^{0}, \mu\left(\delta\left(x_{t}^{1}\right)\right)\right)\right) \tag{114}
\end{align*}
$$

Finally, substituting (114) into (113), we observed that $V_{t}$ has the form described in (21). This completes the proof of the induction step for the case $\tilde{z}_{t}=l$ and the proof of the theorem for this case.

## Appendix IX

## Proof of Theorem 4

Let $\hat{x}_{t}^{1}$ be the estimate (conditional expectation) of $X_{t}^{1}$ based on the common information $h_{t}^{0}$. Then, for any realization of the common belief $\theta_{t}, \hat{x}_{t}^{1}=\mu\left(\theta_{t}\right)$. To show (27) and (28), note that at time $t=0$, for any realization $h_{t}^{0}$ of $H_{t}^{0}$,

$$
\begin{align*}
& \hat{x}_{0}^{1}=\mu\left(\theta_{0}\right)=\int y \theta_{0}(d y) \\
& = \begin{cases}\int y \pi_{X_{0}^{1}}(d y)=\mu\left(\pi_{X_{0}^{1}}\right) & \text { if } z_{0}=\emptyset \\
\int y \mathbb{1}_{\{y\}}\left(x_{0}^{1}\right)(d y)=x_{0}^{1} & \text { if } z_{0}=x_{0}^{1}\end{cases} \tag{115}
\end{align*}
$$

Therefore, (27) is true. Furthermore, at time $t+1$ and for any realization $h_{t+1}^{0}$ of $H_{t+1}^{0}$, let $\theta_{t+1}$ be the corresponding common belief and $u_{t}^{\text {prs* }}=\left(u_{t}^{0 *}, \bar{q}_{t}^{*}, \tilde{q}_{t}^{*}\right)$, then
$\hat{x}_{t+1}^{1}=\mu\left(\theta_{t+1}\right)=\int y\left[\psi_{t}\left(\theta_{t}, u_{t}^{p r s *}, x_{t}^{0}, m_{t}^{0}, \tilde{z}_{t}, z_{t+1}\right)\right](d y)$.

If $z_{t+1}=x_{t+1}^{1}$, then $\hat{x}_{t+1}^{1}=\int y \mathbb{1}_{\{y\}}\left(x_{t+1}^{1}\right)(d y)=x_{t+1}^{1}$.
If $z_{t+1}=\emptyset$, then,

$$
\begin{align*}
\hat{x}_{t+1}^{1} & =\int y \sum_{m_{t}^{1} \in \mathcal{M}^{1}} \iint \mathbb{1}_{\{y\}}\left(f_{t}^{1}\left(x_{t}^{0: 1}, m_{t}^{0: 1}, u_{t}^{p r s *}, w_{t}^{1}\right)\right) \\
& \times \theta_{t}\left(d x_{t}^{1}\right) \omega^{\tilde{z}_{t}}\left(m_{t}^{1}\right) \pi_{W_{t}^{1}}\left(d w_{t}^{1}\right)(d y) \\
& =\sum_{m_{t}^{1} \in \mathcal{M}^{1}} \iint f_{t}^{1}\left(x_{t}^{0: 1}, m_{t}^{0: 1}, u_{t}^{p r s *}, w_{t}^{1}\right) \\
& \times \theta_{t}\left(d x_{t}^{1}\right) \omega^{\tilde{z}_{t}}\left(m_{t}^{1}\right) \pi_{W_{t}^{1}}\left(d w_{t}^{1}\right) \\
& =\sum_{m_{t}^{1} \in \mathcal{M}^{1}}\left[D\left(m_{t}^{0: 1}\right)\right]_{1} \bullet \operatorname{vec}\left(x_{t}^{0}, \hat{x}_{t}^{1}, \bar{u}_{t}^{0 *}, \bar{q}_{t}^{*}\left(m_{t}^{1}\right)\right) \omega^{\tilde{z}_{t}}\left(m_{t}^{1}\right), \tag{116}
\end{align*}
$$

where the third equality is true because

$$
\begin{aligned}
& \int y \mathbb{1}_{\{y\}}\left(f_{t}^{1}\left(x_{t}^{0: 1}, m_{t}^{0: 1}, u_{t}^{p r s *}, w_{t}^{1}\right)\right) d y \\
& =f_{t}^{1}\left(x_{t}^{0: 1}, m_{t}^{0: 1}, u_{t}^{p r s *}, w_{t}^{1}\right),
\end{aligned}
$$

Furthermore, the last equality of (116) is true because $\tilde{q}_{t} \in$ $\tilde{\mathcal{Q}}(\theta)$ and $W_{t}^{1}$ is a zero mean random vector. Therefore, (28) is true and the proof is complete.


[^0]:    ${ }^{1}$ One can follow the argument of [1, Lemma 1] to see that the results of this paper hold for any $H_{t}^{1}=H_{t}^{0} \cup \hat{H}_{t}^{1}$ where $\left\{X_{t}^{1}, M_{t}^{1}\right\} \subseteq \hat{H}_{t}^{1} \subseteq$ $\left\{X_{0: t}^{1}, M_{0: t}^{1}, U_{0: t-1}^{1}\right\}$. For simplicity of presentation, we restrict to $\hat{H}_{t}^{1}=$ $\left\{X_{t}^{1}, M_{t}^{1}\right\}$.

[^1]:    ${ }^{2}$ See Appendix $\square$ for the exact description of transformation $\psi_{t}$.

