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Robust Exponential Stability of an Intermittent Transmission State Estimation Protocol

Sean Phillips, R. Scott Erwin, and Ricardo G. Sanfelice

Abstract— **The problem of distributed networked sensor agents jointly estimating the state of a plant given by a linear time-invariant system is studied. Each agent can only measure the output of the plant at intermittent time instances, at which times the agent also sends the received plant measurement and its estimate to its neighbors. At each agent, a decentralized observer is attached which utilizes the asynchronous incoming information being sent from its neighbors to drive its own estimate to the state of the plant. We provide sufficient conditions that guarantee global exponential stability of the zero estimation error set. Numerical illustrations are provided.**

I. INTRODUCTION

Distributed state estimation in networked systems has seen increased attention recently. A typical challenge in such system occurs when information is only intermittently available; namely, when information is transmitted to neighboring agents asynchronously and only at isolated time instances which are not known a priori. Furthermore, the amount of ordinary time elapsed between communication events for each agent can be different. For instance, an agent can receive information at a much faster rate than others.

Several observer architectures and design methods have been proposed in the literature. Results for when information is available periodically, discrete-time observers can be used [1], [2], [3]. Algorithms that treat the communication as impulsive events along the continuous dynamics of the plant have also been developed; see, e.g., [4], [5]. In [6], a distributed observer with undirected fixed communication topology and switching communication topology for periodic sampling time/communication events is presented. Distributed Kalman filtering is employed for achieving spatially-distributed estimation tasks in [7]. In [8], a continuous-discrete distributed observer design was presented for linear systems with discrete communication.

A distributed observer design that addresses the challenge of robustly reconstructing the state of the plant, at each agent, when information is arriving at intermittent (aperiodic) and asynchronous (between agents) time instants is considered. Namely, we construct a distributed observer, assigned to each agent, that uses available information it receives about the plant by way of direct measurements and communication from neighboring agents. We model the closed-loop system using the hybrid systems framework in [9]. With an appropriate change of coordinates, we show global exponential stability of the zero estimation error set. The main contribution of this work lay on the establishment of sufficient conditions for nominal and robust estimation over networks where information at each agent arrives at time instances triggered by each agent locally. Different from our previous work in [10], [11], this work assumes that information arrives to each agent asynchronously. Numerical examples that validate the results are presented throughout the paper.

The remainder of this paper is organized as follows. Notation along with a brief overview of hybrid systems and graph theory is given in Section II. In Section III, the problem description, modeling, and main results are presented.

II. PRELIMINARIES

A. Notation

Given a matrix A, $\text{eig}(A)$ is the set of all eigenvalues of A and $|A| := \max\{|\lambda|^{\frac{1}{2}} : \lambda \in \text{eig}(A^\top A)\}$. Given two vectors $u, v \in \mathbb{R}^n$, $|u| := \sqrt{u^\top u}$ and notation $[u^\top \ v^\top]^\top$ is equivalent to (u, v). Given a function $m : \mathbb{R}_{\geq 0} \to \mathbb{R}^n$, $|m|_\infty := \sup_{t \geq 0} |m(t)|$. The symbol $\mathbb{N}$ denotes the set of natural numbers including zero. Given a vector $x \in \mathbb{R}^n$ and a closed set $\mathcal{A} \subset \mathbb{R}^n$, the distance from x to $\mathcal{A}$ is defined as $|x|_\mathcal{A} = \inf_{z \in \mathcal{A}} |x - z|$. Given a symmetric matrix P, $\overline{\lambda}(P) := \max\{\lambda : \lambda \in \text{eig}(P)\}$ and $\underline{\lambda}(P) := \min\{\lambda : \lambda \in \text{eig}(P)\}$. Given matrices A, B with proper dimensions, we define the operator $\text{He}(A, B) := A^\top B + B^\top A$; $A \otimes B$ defines the Kronecker product; $\text{diag}(A, B)$ denotes a 2×2 block matrix with A and B being the diagonal entries; and $A * B$ defines the Khatri-Rao product[1] between A and B. Denote $\star$ as the symmetric block in a block-partitioned matrix. Given $N \in \mathbb{Z}_{\geq 1}$, $I_N \in \mathbb{R}^{N \times N}$ defines the identity matrix and $\mathbf{1}_N$ is the vector of N ones. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class-$\mathcal{K}$ function, also written $\alpha \in \mathcal{K}$, if α is zero at zero, continuous, strictly increasing; it is said to belong to class-$\mathcal{K}_\infty$, also written $\alpha \in \mathcal{K}_\infty$, if $\alpha \in \mathcal{K}$ and is unbounded; α is positive definite, also written $\alpha \in \mathcal{PD}$, if $\alpha(s) > 0$ for all $s > 0$ and $\alpha(0) = 0$. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a class-$\mathcal{KL}$ function, also written $\beta \in \mathcal{KL}$, if it is nondecreasing in its first argument, nonincreasing in its second argument, $\lim_{r \to 0^+} \beta(r, s) = 0$ for each $s \in \mathbb{R}_{\geq 0}$, and $\lim_{s \to \infty} \beta(r, s) = 0$ for each $r \in \mathbb{R}_{\geq 0}$. Given a function

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[1]For more information on Kronecker and Khatri-Rao products, see [12].

$f : \mathbb{R}^n \to \mathbb{R}^m$, the domain of f is denoted by $\operatorname{dom} f$. Given a closed set S, $\mathbb{T}_S(x)$ denotes its tangent cone S at x; see, e.g., [9, Definition 5.12].

B. Preliminaries on graph theory

A directed graph (digraph) is defined as $\Gamma = (\mathcal{V}, \mathcal{E}, \mathcal{G})$. The set of nodes of the digraph are indexed by the elements of $\mathcal{V} = \{1, 2, \ldots, N\}$ and the edges are the pairs in the set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$. Each edge directly links two nodes, i.e., an edge from i to k, denoted by (i, k), implies that agent i can receive information from agent k. The adjacency matrix of the digraph Γ is denoted by $\mathcal{G} \in \mathbb{R}^{N \times N}$, where its (i, k)-th entry g_{ik} is equal to one if $(i, k) \in \mathcal{E}$ and zero otherwise. Without loss of generality, we assume that an agent may be connected to itself; i.e., the edge (i, i) is contained in the edge set $\mathcal{E}$ and the corresponding element in the adjacency matrix $\mathcal{G}$ is $g_{ii} = 1$. Let the cardinality of the edge set $\mathcal{E}$ be defined as $\bar{\mathcal{E}}$. The in-degree and out-degree of agent i are defined by $d_i^{in} = \sum_{k=1}^{N} g_{ik}$ and $d_i^{out} = \sum_{k=1}^{N} g_{ki}$. The in-degree matrix $\mathcal{D}$ is the diagonal matrix with entries $D_{ii} = d_i^{in}$ for all $i \in \mathcal{V}$. Each element $\ell_{ik} \in \mathbb{R}$ in the Laplacian matrix of the graph Γ, denoted by $\mathcal{L} \in \mathbb{R}^{N \times N}$, is defined as $\ell_{ik} = -g_{ik}$ for each $i, k \in \mathcal{V}$ such that $i \neq k$ and $\ell_{ii} = d_i^{in}$ for each diagonal element $i \in \mathcal{V}$. The set of indices corresponding to the neighbors that can send information to the i-th agent is denoted by $\mathcal{N}(i) := \{k \in \mathcal{V} : (i, k) \in \mathcal{E}\}$.

C. Preliminary on Hybrid Systems

This section introduces the main notions and definitions on hybrid systems used throughout this work. More information on such systems can be found in [9]. A hybrid system $\mathcal{H}$ has data (C, f, D, G) and can be represented in the compact form

$$\mathcal{H} : \begin{cases} \dot{\xi} & = f(\xi) & \xi \in C, \\ \xi^+ & \in G(\xi) & \xi \in D, \end{cases} \tag{1}$$

where $\xi \in \mathbb{R}^n$ is the state. The data of the hybrid system is given by (C, f, D, G). The flow map, defined as $f : \mathbb{R}^n \to \mathbb{R}^n$, is a single-valued map capturing the continuous dynamics, which are allowed to occur in the flow set $C \subset \mathbb{R}^n$. The set-valued mapping $G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defines the jump map and determines the value of ξ after jumps, which is denoted by ξ^+. Jumps are allowed to occurs in the jump set, defined as $D \subset \mathbb{R}^n$. Solutions ϕ to $\mathcal{H}$ are parameterized by (t, j), where $t \in \mathbb{R}_{\geq 0}$ counts ordinary time and $j \in \mathbb{N}$ counts the number of jumps. The domain $\operatorname{dom} \phi \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a hybrid time domain if for every $(T, J) \in \operatorname{dom} \phi$, the set $\operatorname{dom} \phi \cap ([0, T] \times \{0, 1, \ldots, J\})$ can be written as the union of sets $\cup_{j=0}^{J} (I_j \times \{j\})$, where $I_j := [t_j, t_{j+1}]$ for a time sequence $0 = t_0 \leq t_1 \leq t_2 \leq \cdots \leq t_{J+1}$. The t_j's with $j > 0$ define the time instants when the state of the hybrid system jumps and j counts the number of jumps.

A hybrid system $\mathcal{H} = (C, f, D, G)$ with data in (1) is said to satisfy the *hybrid basic conditions* if it satisfies the conditions in [9, Assumption 6.5].

Definition 2.1: (global exponential stability) Let a hybrid system $\mathcal{H}$ be defined on $\mathbb{R}^n$. Let $\mathcal{A} \subset \mathbb{R}^n$ be closed. The set $\mathcal{A}$ is said to be *globally exponentially stable* (GES) for $\mathcal{H}$ if
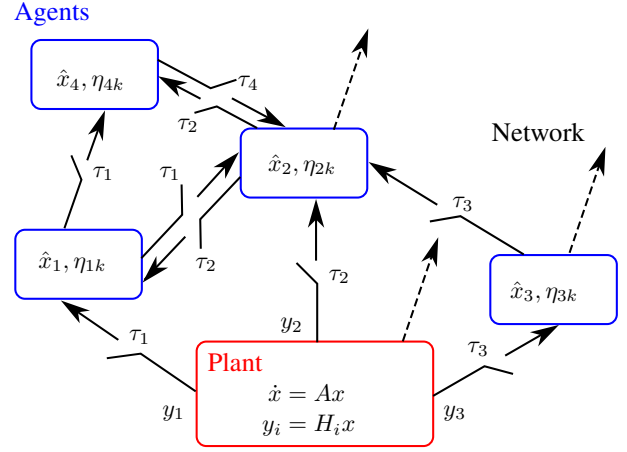

Fig. 1. Decentralized network architecture under consideration. When the timer state resets (i.e., $\tau_i = 0$), the i-th agent receives the output of the plant y_i and transmits it, along with its current estimate $\hat{x}_i$, to its neighbors, which, in turn, updates the corresponding information state η_{ik} for each $k \in \mathcal{N}(i)$.

there exist $\kappa, \alpha > 0$ such that every maximal solution ϕ to $\mathcal{H}$ is complete and satisfies

$$|\phi(t, j)|_{\mathcal{A}} \leq \kappa \exp(-\alpha(t + j)) |\phi(0, 0)|_{\mathcal{A}}$$

for each $(t, j) \in \operatorname{dom} \phi$. $\qquad \square$

III. DISTRIBUTED HYBRID ESTIMATION PROTOCOL AND NOMINAL PROPERTIES

A. Problem Formulation and Proposed Solution

In this paper, we consider the problem of designing a distributed observer to estimate the state of a plant over a network of N agents. Wherein each agent may have a local heterogeneous sensor and memory. The plant has continuous-time dynamics

$$\dot{x} = Ax \tag{2}$$

where x is the state and $A \in \mathbb{R}^{n_x \times n_x}$ is the state matrix. The N agents are connected via a directed graph and each agent runs a local observer estimating the state x of (2). Each agent in the network can measure the output of the plant and can transmit this measurement and estimate to its neighbors at time instances given by the sequence $\{t_s^i\}_{s=1}^{\infty}$. Namely, at each such time instant $t \in \{t_s^i\}_{s=1}^{\infty}$, the i-th agent receives a measurement of the output of the plant and transmits it to its neighbors. The measurement is given by

$$y_i(t) = H_i x(t) \tag{3}$$

where $H_i \in \mathbb{R}^{n_{p_i} \times n_x}$ is a local output matrix of the plant for agent i.

The event times in the sequence $\{t_s^i\}_{s=0}^{\infty}$ are independently defined for each agent. Given positive scalars $T_2^i \geq T_1^i$ for each $i \in \mathcal{V}$, the only restriction imposed on such times is that they must satisfy

$$\begin{aligned} t_{s+1}^i - t_s^i &\in [T_1^i, T_2^i] \qquad \forall s \in \{1, 2, \ldots\}, \\ t_1^i &\leq T_2^i . \end{aligned} \tag{4}$$

The scalars T_1^i and T_2^i are the nominal parameters that define the lower and upper bounds, respectively, of the time allowed to elapse between consecutive events. The parameters T_1^i and T_2^i are assumed to be known, but are not necessarily the same for each agent.

Due to the impulsive nature of the communication and measurement events $\{t_s^i\}_{s=0}^\infty$ satisfying (4), for each $i \in \mathcal{V}$, we define a decreasing timer to trigger such events. Inspired by [13], the timer at the i-th agent, denoted by $\tau_i \in [0, T_2^i]$, decreases with ordinary time and upon reaching zero is reset to a point in the interval $[T_1^i, T_2^i]$. Namely, the dynamics of τ_i are given by

$$\begin{cases} \dot{\tau}_i = -1 & \tau_i \in [0, T_2^i], \\ \tau_i^+ \in [T_1^i, T_2^i] & \tau_i = 0. \end{cases} \quad (5)$$

Note that the domain of solutions to this system, denoted ϕ_{τ_i}, are such that the jump times t_j satisfy $t_{j+1} - t_j \in [T_1^i, T_2^i]$ for each $j \geq 1$ and $t_1 \leq T_2^i$; i.e., the sequence of times satisfy (4).

We propose a distributed hybrid observer that is capable of asymptotically reconstructing the state of the plant x locally at each agent, with stability and by only exchanging information from the plant and its neighbors at communication events $\{t_s^i\}_{s=1}^\infty$ satisfying (4). Each observer runs an algorithm at the i-th agent that generates an estimate of the state x, which is denoted $\hat{x}_i \in \mathbb{R}^{n_x}$, and utilizes d_i^{in} information states, denoted η_{ik} for each $k \in \mathcal{N}(i)$, stored locally at the i-th agent. Let $\tau = (\tau_1, \tau_2, \ldots, \tau_N) \in [0, T_2^1] \times [0, T_2^2] \times \cdots \times [0, T_2^N] =: \mathcal{T}$. When no timer state has expired, i.e., when $\tau \in \mathcal{T} \setminus \{0\}$, no new information has arrived and the observer states $\hat{x}_i$ are each continuously updated by the following general differential equations:

$$\begin{aligned} \dot{\hat{x}}_i &= A\hat{x}_i + \sum_{k \in \mathcal{N}(i)} \eta_{ik} \\ \dot{\eta}_{ik} &= f_{ik}(\hat{x}_i, \eta_{ik}) \qquad \forall k \in \mathcal{N}(i) \end{aligned} \quad (6)$$

for each $i \in \mathcal{V}$, where $f_{ik} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \to \mathbb{R}^{n_x}$ defines the continuous evolution of the information state. When data arrives from measurements of the plant or from neighboring agents, the estimation state and corresponding information state η_{ik} are updated. For example, when $\tau_k = 0$ and $i \in \mathcal{N}(k)$ (where agent i is a neighbor of agent k) which triggers agent k to transmit information to agent i and the rest of its neighbors, the estimation and corresponding information states are updated by

$$\begin{aligned} \hat{x}_i^+ &= \hat{x}_i \\ \eta_{ik}^+ &= G_{ik}(\hat{x}_i, \hat{x}_k, y_k) \qquad \forall i \in \mathcal{N}(k). \end{aligned} \quad (7)$$

where $G_{ik} : \mathbb{R}^{n_x} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_{p_k}} \to \mathbb{R}^{n_x}$ defines the function which combines the received information. The continuous and discrete dynamics in (6) and (7), respectively, along with (5). The interconnection of these systems leads to a hybrid dynamical system $\mathcal{H}$ as in (1) capturing the dynamics of modeling the entire networked system.

B. Distributed Estimation Protocol and Hybrid Modeling

In this section, we define the particular form of the information states under consideration. In particular, the continuous evolution of the states η_{ik} in (6) is governed by the following data:

$$\dot{\eta}_{ik} = h_{ik}\eta_{ik} =: f_{ik}(\hat{x}_i, \eta_{ik}) \quad (8)$$

where $h_{ik} \in \mathbb{R}$. When the k-th agent takes a measurement of the output of the plant given by (3), which is when $\tau_k = 0$, the k-th agent transmits y_k and the current value of the estimation state $\hat{x}_k$ to its neighbors updating η_{ik} impulsively by

$$\eta_{ik}^+ = K_{ik}(H_k\hat{x}_k - y_k) + \gamma(\hat{x}_i - \hat{x}_k) =: G_{ik}(\hat{x}_i, \hat{x}_k, y_k) \quad (9)$$

for each $i \in \mathcal{N}(k)$, where $\gamma \in \mathbb{R}$, $K_{ik} \in \mathbb{R}^{n_x \times n_{p_k}}$ define the parameters of the observer. In this way, we can easily use the properties of Kronecker products, bidirectional graphs and Laplacian matrices to model the system.

1) Change of Coordinates: Inspired by [13], for each $i \in \mathcal{V}$ and each $k \in \mathcal{N}(i)$, consider the change of coordinates

$$\begin{aligned} e_i &= \hat{x}_i - x, \\ \theta_{ik} &= K_{ik}H_k e_k + \gamma(e_i - e_k) - \eta_{ik}. \end{aligned} \quad (10)$$

Then, the continuous dynamics of e_i are given by

$$\dot{e}_i = Ae_i + \sum_{k \in \mathcal{N}(i)} (K_{ik}H_k e_k + \gamma(e_i - e_k) - \theta_{ik}) \quad (11)$$

and, with $e = (e_1, e_2, \ldots, e_N)$ and $\theta = (\theta_1, \theta_2, \ldots, \theta_N)$, implies that

$$\dot{e} = (I \otimes A + K_{\mathcal{G}} + \gamma\mathcal{L} \otimes I)e - \mathcal{D}\theta \quad (12)$$

where $\mathcal{D} = \text{diag}(I \otimes \mathbf{1}_{d_{in}^1}^\top, \ldots, I \otimes \mathbf{1}_{d_{in}^N}^\top)$ with the in-degree of the i-th agent d_{in}^i, $K_{\mathcal{G}} = (KH) * \mathcal{G}$, and $K \in \mathbb{R}^{n_x N \times n_p N}$ is an $N \times N$ block matrix where the (i, k)-th entry is given by $K_{ik} \in \mathbb{R}^{n_x \times n_{p_k}}$ for each $i, k \in \mathcal{V}$ such that $(i, k) \in \mathcal{E}$ and a matrix full of zeros elsewhere with $n_p = \sum_{i \in \mathcal{V}} n_{p_i}$, and $H = \text{diag}(H_1, H_2, \ldots, H_N)$.

Let $\eta = (\eta_1, \eta_2, \ldots, \eta_N)$ and, for each $i \in \mathcal{V}$, θ_i (likewise, η_i) contains the states θ_{ik} (η_{ik}, respectively) states for each $(i, k) \in \mathcal{E}$ in ascending order of the index k. Then. we have

$$\theta = K_\theta e - \eta \quad (13)$$

where $K_\theta = \bar{K}H + \gamma\Pi$, and $\bar{K} = (\bar{K}_{11}, \bar{K}_{12}, \ldots, \bar{K}_{ik}, \ldots, \bar{K}_{NN})$ where, for each $(i, k) \in \mathcal{E}$, $\bar{K}_{ik} = v_k \otimes K_{ik}$, Π is a stack of matrices $(v_i - v_k) \otimes I$ for each $i, k \in \mathcal{V}$ such that $(i, k) \in \mathcal{E}$ corresponds to the ordering of the θ states, and v_i is the ith canonical vector.

During jumps, namely, if there exists $k \in \mathcal{V}$ such that $\tau_k = 0$, then a jump occurs. At such points, the dynamics of e_i are given by

$$e_i^+ = e_i \quad (14)$$

and, for each $i \in \mathcal{N}(k)$, the definition of θ_{ik} in (10) with (9) lead to

$$\theta_{ik}^+ = 0, \quad (15)$$

otherwise, $\theta_{ik}^+ = \theta_{ik}$.

In this following example, we consider the case of $N = 3$ and construct many of the matrices in (12) and (13).

Example 3.1: Consider the case $N = 3$, and

$$\mathcal{G} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

through the change in coordinates, we have that $\theta = (\theta_1, \theta_2, \theta_3)$, and $\theta_1 = (\theta_{11}, \theta_{12})$, $\theta_2 = (\theta_{22}, \theta_{23}), \theta_3 = (\theta_{31}, \theta_{32}, \theta_{33})$. The matrix K in (12) is given by

$$K = \begin{bmatrix} K_{11} & K_{12} & 0 \\ 0 & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{bmatrix}. \tag{16}$$

Noting from $\mathcal{G}$, the in-degree of each agent is given by $d_{in}^1 = 2$, $d_{in}^2 = 2$, $d_{in}^1 = 3$, which leads to

$$\mathcal{D} = \begin{bmatrix} I & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & I & I \end{bmatrix}.$$

From (13), the matrices $\bar{K}$ is given by $(\bar{K}_{11}, \bar{K}_{12}, \bar{K}_{22}, \bar{K}_{23}, \bar{K}_{31}, \bar{K}_{32}, \bar{K}_{33})$ and the elements of Π are defined $(v_i - v_k) \otimes I$, namely, the matrices are given by

$$\bar{K} = \begin{bmatrix} K_{11} & 0 & 0 \\ 0 & K_{12} & 0 \\ 0 & K_{22} & 0 \\ 0 & 0 & K_{23} \\ K_{31} & 0 & 0 \\ 0 & K_{32} & 0 \\ 0 & 0 & K_{33} \end{bmatrix}, \qquad \Pi = \begin{bmatrix} 0 & 0 & 0 \\ I & -I & 0 \\ 0 & 0 & 0 \\ 0 & I & -I \\ -I & 0 & I \\ 0 & -I & I \\ 0 & 0 & 0 \end{bmatrix}.$$

2) Closed-loop Hybrid System: We define the hybrid system $\mathcal{H} = (C, f, D, G)$ with state $\xi = (\chi, \tau) \in \mathcal{X} := \mathbb{R}^{n_x N} \times \mathbb{R}^{n_x d} \times \mathcal{T}$ resulting from the change of coordinates in (10) where $d = \sum_{i=1}^N d_{in}^i$, and $\chi = (e, \theta)$. Recall that $\tau = (\tau_1, \tau_2, \ldots, \tau_N) \in \mathcal{T} = [0, T_2^1] \times [0, T_2^2] \times \cdots \times [0, T_2^N]$. Then, the hybrid system $\mathcal{H}$ has data given by

$$f(\xi) = (A_f \chi, -\mathbf{1}_N) \qquad \xi \in C := \mathcal{X} \tag{17}$$

and

$$G(\xi) := \{G_i(\xi) : \xi \in D_i, i \in \mathcal{V}\} \qquad \xi \in D = \bigcup_{i \in \mathcal{V}} D_i \tag{18}$$

where $D_i := \{\xi \in \mathcal{X} : \tau_i = 0\}$ and

$$G_i(\xi) := \{(e, \theta', \tau') : \xi = (e, \theta, \tau), \tau_i' \in [T_1^i, T_2^i],$$
$$\theta_{ki}' = 0 \ \forall k \in \mathcal{N}(i)\}$$

Using the change of coordinates in (10) along with the continuous-time dynamics in (12) and (8), the matrix A_f in (17) is given by

$$A_f = \begin{bmatrix} A_\theta & -\mathcal{D} \\ K_\theta A_\theta - h K_\theta & h - K_\theta \mathcal{D} \end{bmatrix}, \tag{19}$$

$A_\theta = (I \otimes A) + K_{\mathcal{G}} + \gamma(\mathcal{L} \otimes I)$, $K_\theta = \bar{K}H + \gamma\Pi$ and $h = \text{diag}(h_{11}I_n, h_{12}I_n, \ldots, h_{NN}I_n)$, where the subscripts of h_{ik} are for each $i, k \in \mathcal{V}$ such that $(i, k) \in \mathcal{E}$.

Remark 3.2: Note that C and D are closed and that f is continuous and G is outer semicontinuous and locally bounded on D. Therefore, the hybrid system $\mathcal{H}$ satisfies the *hybrid basic conditions* given in [9, Definition 6.5]. Note that satisfying the hybrid basic conditions imply that the hybrid system $\mathcal{H}$ is well-posed and with asymptotic stability of a compact set is robust to small enough perturbations.

The objective of each agent in the hybrid system is to estimate the state of the plant, i.e., to drive the difference between the estimates and the plant to zero asymptotically. The definition of η and θ also imply that these states will converge to zero as the error e converges to zero. Therefore, in the (e, θ, τ) coordinates, the set to asymptotically stabilize is given by

$$\mathcal{A} = \{0_{n_x N}\} \times \{0_{n_x d}\} \times \mathcal{T}. \tag{20}$$

In the following sections, we provide conditions guaranteeing that this set is exponentially stable for the hybrid system $\mathcal{H}$ with data in (17) and (18).

From the definition of this hybrid system, solutions to $\mathcal{H}$ jump when there exist $i \in \mathcal{V}$ such that $\tau_i = 0$ as defined below (18). Moreover, for each such point, τ_i is updated to a point within the interval $[T_1^i, T_2^i]$,

Lemma 3.3: ([11, Lemma 3.5]) Given positive scalars T_1^i and T_2^i such that $T_1^i \leq T_2^i$ for each $i \in \mathcal{V}$, every solution ϕ to $\mathcal{H}$ with data in (17) and (18) satisfies the following

1) *every maximal solution is complete, i.e., the domain of every maximal solution is unbounded.*
2) *for each $(t, j) \in \text{dom} \phi$, $\left(\frac{j}{N} - 1\right) T_1^{\min} \leq t \leq \frac{j}{N} T_2^{\max}$ where $T_1^{\min} := \min_{i \in \mathcal{V}} T_1^i$ and $T_2^{\max} := \max_{i \in \mathcal{V}} T_2^i$*
3) *for all $j \in \mathbb{Z}_{\geq 1}$ such that $(t_{(j+1)N}, (j + 1)N), (t_{jN}, jN) \in \text{dom} \phi$, $t_{(j+1)N} - t_{jN} \in [T_1^{\min}, T_2^{\max}]$.*

We use this result to establish global exponential stability of $\mathcal{A}$ for the hybrid system $\mathcal{H}$ in Theorem 3.4.

C. Main Results

With the change of coordinates in (10), we use the Lyapunov function candidate

$$V(\xi) = e^\top P e + \theta^\top R(\tau) \theta \tag{21}$$

for all $\xi \in \mathcal{X}$, where $P = P^\top > 0$ and $\tau \mapsto R(\tau)$ is continuously differentiable and positive definite for all $\tau \in \mathcal{T}$; see [14] for a similar construction. This choice of V satisfies $V(\xi) = 0$ for each $\xi \in \mathcal{A}$, and $V(\xi) > 0$ for every $\xi \in \mathcal{X} \setminus \mathcal{A}$ and is continuously differentiable. Therefore, V is an appropriate Lyapunov function candidate, as defined in [15, Definition 3.16]. Moreover, due to the choice of the change of coordinates in (10), this function satisfies, regardless of which timer τ_k expires, the property that $V(\xi^+) - V(\xi)$ is nonpositive for each $\xi \in D$. The injection of η_{ik} in the flows of the local estimate in (6), and the continuous dynamics of η_{ik} with flow map (8) further permit a decrease of V during flows. By virtue of the aforementioned Lyapunov function candidate, we arrive to the following result.

Theorem 3.4: Let $0 < T_1^i \leq T_2^i$ for each $i \in \mathcal{V}$ and a directed graph Γ be given. The hybrid system $\mathcal{H}$ with data in (17) and (18) *has the set $\mathcal{A}$ in (20) globally exponentially stable if there exists scalars $\sigma > 0$, $\gamma \in \mathbb{R}$, $h_{ik} \in \mathbb{R}$ and matrices $K_{ik} \in \mathbb{R}^{n_x \times n_{p_i}}$ for each $i, k \in \mathcal{V}$ such that $(i,k) \in \mathcal{E}$, and positive definite symmetric matrices $P \in \mathbb{R}^{n_x N \times n_x N}$, $R_{ik} \in \mathbb{R}^{n_x \times n_x}$ such that $M(\nu) < 0$ for each $\nu \in \mathcal{T}$, where*[2]

$$M(\nu) := \begin{bmatrix} \text{He}(PA_\theta) & -PD + (K_\theta A_\theta - hK_\theta)^\top R(\nu) \\ \star & \text{He}(R(\nu)(h - K_\theta D)) - \sigma R(\nu) \end{bmatrix} \tag{22}$$

$R(\nu) = \text{diag}(R_1(\nu), R_2(\nu), \ldots, R_N(\nu))$, $R_i(\nu) = \text{diag}(\exp(\sigma\nu_1)R_{11}, \exp(\sigma\nu_2)R_{12} \ldots, \exp(\sigma\nu_N)R_{1N})$.

Proof Sketch The property that $\mathcal{A}$ is globally exponentially stable under condition (22) can be established by using the Lyapunov function V in (21). Due to the definition of $\mathcal{A}$ in (20) there exist positive scalars α_1, α_2 satisfying $\alpha_1 |\xi|_\mathcal{A}^2 \leq V(\xi) \leq \alpha_2 |\xi|_\mathcal{A}^2$. Moreover, in light of the strict inequality of M in (22), there exists $\beta > 0$ such that, for each $\xi \in C$, $\langle \nabla V(\xi), f(\xi) \rangle \leq -\frac{\beta}{\alpha_2} V(\xi)$. For each $\xi \in D$, $g \in G(\xi)$, it can be seen that $V(g) - V(\xi) \leq 0$. Direct integration of V over a solution ϕ along with the properties in Lemma 3.3 lead to $M(\nu) < 0$ for all $\nu \in \mathcal{T}$. Moreover, by Lemma 3.3, every maximal solution is complete implying that the set $\mathcal{A}$ is globally exponentially stable for the hybrid system $\mathcal{H}$. ∎

Due to the fact that $M(\nu) < 0$ needs to be checked over an infinite number of points in the compact set $\mathcal{T}$, the following result provides relaxed conditions which ensure $M(\nu) < 0$ is satisfied for every $\nu \in \mathcal{T}$.

Proposition 3.5: Let $T_2^i > 0$ for each $i \in \mathcal{V}$ be given. The inequality $M(\nu) < 0$ in (22) with M defined in (22) holds for each $\nu \in \mathcal{T}$ if there exist $\sigma > 0$ and positive definite symmetric matrices P, R_{ik} such that

$$M(0) < 0, \qquad M(T_2) < 0 \tag{23}$$

where $T_2 = (T_2^1, T_2^2, \ldots, T_2^N)$.

Remark 3.6: For the case when $N = 1$, the resulting observer is a single Luenberger-like observer for the case of intermittent measurements and the matrix inequality in (23) reduces to the condition in Theorem 1 in [13].

Remark 3.7: Note that the off-diagonal block matrices in (23) involve the multiplication of $KA_\theta - hK$, which contain cross terms involving K_{ik}, γ, and h_{ik}. The presence of these terms makes the problem nonlinear and difficult to solve numerically. However, LMI conditions can be established following the ideas in [13].

Example 3.8: Consider a plant with system matrix

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{24}$$

where the state $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ has oscillatory dynamics for (x_1, x_2) and trivial dynamics for x_3. Note that such a plant has eigenvalues at $\{\pm 1i, 0\}$ and, for every initial condition outside of the origin, its states never converge to a set point. In this example, we consider three scenarios which show that the estimation states $\hat{x}_i$ converge to the state of the plant exponentially: 1) the case of all-to-all network but each agent cannot reconstruct the state individually; 2) a strongly connected network with the same output matrices as in 1); and, lastly, 3) an all-to-all connection, but the second agent cannot measure the plant.

Consider the case of three agents that are all-to-all connected, i.e.,

$$\mathcal{G} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$
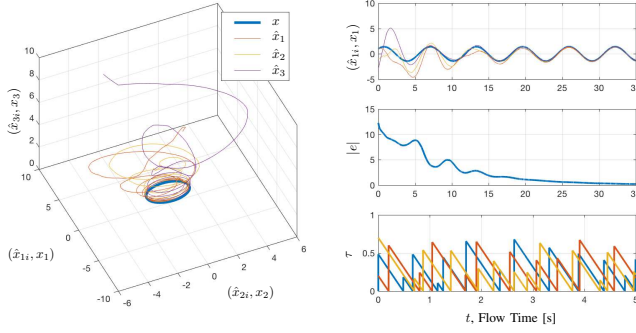
and measure y_i according to (3) with

$$H_1 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \quad H_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \quad H_3 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$

Since the pair (H_i, A) is not observable for each i-th agent, no single agent can estimate the full state of the plant running an observer like that in [13]. However, when communication between agents is allowed, our observer is able to reconstruct the state of the plant x. In fact, given $T_1^i = 0.1$ and $T_2^i = 0.7$ for each $i \in \mathcal{V}$, by solving for the conditions in Theorem 3.4, we obtain the following parameters: $K_{11} = \begin{bmatrix} 1.2 & -0.9 & 0.3 \end{bmatrix}^\top$, $K_{12} = \begin{bmatrix} 0 & -0.1 & -0.3 \end{bmatrix}^\top$, $K_{13} = \begin{bmatrix} -0.1 & -0.4 & -0.1 \end{bmatrix}^\top$ $K_{21} = \begin{bmatrix} -0.5 & -0.1 & -0.1 \end{bmatrix}^\top$ $K_{22} = \begin{bmatrix} 1 & -0.1 & -0.4 \end{bmatrix}^\top$, $K_{23} = \begin{bmatrix} -0.3 & -0.1 & -0.3 \end{bmatrix}^\top$, $K_{31} = \begin{bmatrix} -0.2 & -0.5 & -1 \end{bmatrix}^\top$, $K_{32} = \begin{bmatrix} -0.2 & -0.1 & -0.1 \end{bmatrix}^\top$, $K_{32} = \begin{bmatrix} 0.4 & -0.2 & -0.1 \end{bmatrix}^\top$, $h_{11} = -1$, $h_{12} = h_{13} = h_{22} = h_{31} = h_{32} = h_{33} - 0.1$, $h_{21} = -0.5$, $h_{23} = -0.4$, $\sigma = 0.3$ and $\gamma = -0.2$. The numerical solution shown in Figure 2(a) indicates that the estimates $\hat{x}_i$ for each $i \in \mathcal{V}$ converge to the state of the plant x exponentially.
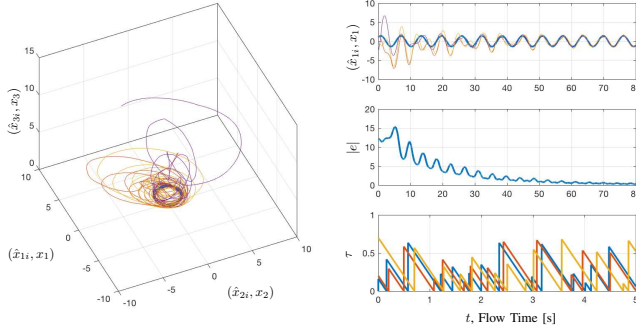
We can reduce the number of links between the agents and still satisfy the conditions in Theorem 3.4 using the same parameters K_{ik}, h_{ik}, and γ previously proposed. In Figure 2(b), we use the gains from the initial simulation in (a) while forcing $g_{21} = g_{32} = 0$. In particular, when edges $(2,1)$ and $(3,2)$ are removed from the edge set, each agent has less information to use in their observer, which, as can be seen in Figure 2(b), makes the estimate converge slower to the state of the plant.[3]

More interestingly, due to the communication topology between the agents under the previous network, the case when a single agent may not receive any measurements, but when it is connected to neighbors the consensus terms in (9) allows the agent to reconstruct the state of the plant. For instance, consider the previous system model but with $H_2 \equiv 0$. Then, values of the gains K_{ik}, h_{ik} and γ can be found such that the conditions in Theorem 3.4 satisfied; see

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[2] In partitioned symmetric block matrices, the symbol $\star$ stands for symmetric blocks in the matrix.

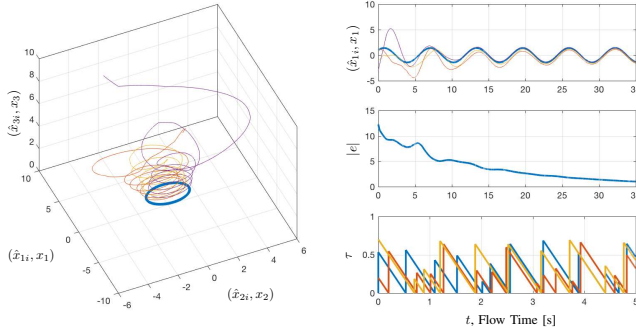[3] Code at `https://github.com/HybridSystemsLab/EstimAsyncTrans`

(a) Completely connected graph network, each agent cannot measure the full state of the plant.

(b) Same network as in (a), but with two links removed. The graph remains strongly connected; however, each agent has less information. Exponential convergence ensues.

(c) Completely connected graph as in (a), but Agent 2 does not receive measurements of the output the plant. Convergence is still possible, however, in this scenario the convergence is slower than (a) and (b).

Fig. 2. Numerical Simulation of three agents estimating a plant with oscillatory behavior in (x_1, x_2) and trivial dynamics in x_3. In each simulation case, even though each pair (H_i, A) is not detectable individually, through the interconnection, the estimates $\hat{x}_i$ still converge to the state of the plant x as indicated by $|e|$ converging to zero.

Figure 2(c). Note that it is solely through communication that reconstruction of the plant is achieved.

Remark 3.9: Due to form the memory states defined in (9), as mentioned in Example 3.8, even though a single agent in a network may not be able to measure the state directly, it can still maintain an estimate of the plant using neighboring information to reconstruct the state. More specifically, the $\gamma(x_i - x_k)$ component of (9), sometimes considered as a consensus term, takes the difference between the estimates of itself and its neighbors to drive its estimate to the true state solely based on the estimates of its neighbors exponentially.

IV. CONCLUSION

In this paper, a distributed state observer was developed to accurately reconstruct the state under intermittent communication and measurement is proposed. At each agent, the estimation algorithm stores information received using multiple memory states, which are updated asynchronously between agents. Sufficient conditions were presented in the form of matrix inequalities which ensure global exponential stability of the estimation error set. Future directions of research include investigating nonlinear dynamics and measurements, dynamic time varying network graphs, and delays in the communication structure.

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