# Biharmonic Distance and the Performance of Second-Order Consensus Networks with Stochastic Disturbances

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***Abstract*—We study second order consensus dynamics with random additive disturbances. We investigate three different performance measures: the steady-state variance of pairwise differences between vertex states, the steady-state variance of the deviation of each vertex state from the average, and the total steady-state variance of the system. We show that these performance measures are closely related to the biharmonic distance; the square of the biharmonic distance plays similar role in the system performance as resistance distances plays in the performance of first-order noisy consensus dynamics. We further define the new concepts of biharmonic Kirchhoff index and vertex centrality based on the biharmonic distance. Finally, we derive analytical results for the performance measures and concepts for complete graphs, star graphs, cycles, and paths, and we use this analysis to compare the asymptotic behavior of the steady-variance in first- and second-order systems.**

***Index Terms*—Distributed average consensus, network coherence, Laplacian spectral distance, biharmonic distances, Gaussian white noise**

## I. Introduction

Consensus dynamics have been studied intensively in the context of distributed networked systems because these dynamics represent a fundamental way of sharing information between agents in the network. Consensus algorithms can be widely applied to many real-world applications such as clock synchronization [1], [2], load balancing [3], sensor networks [4], formation control [5] and distributed optimization [6].

In consensus dynamics, when nodes are subject to external disturbances, these disturbances prevent the system from reaching consensus, instead making node states fluctuate around the current average [7]. Many works have explored analytical methods to quantify the steady-state variance of the deviations from the average. The vast majority of these have considered first-order consensus algorithms [7]–[12]. It has been shown that, in such systems, the total steady-state variance can be described by resistance distances in an associated electrical network [8], [9]. And, in turn, resistance distances are given by the covariance matrix of the vertex states in such a dynamical system [13].

Many real world systems can be more accurately modeled using second-order dynamics. For example, second-order consensus protocols are applied to formation control because they capture the kinematics of the vehicles [14]. Clock synchronization algorithms using second-order consensus scheme have also been studied [1]. While second-order dynamics have important applications, analysis of the effects of external perturbations on second-order systems remains limited when compared to recent work on first-order systems. Previous works have shown that the *total steady-state variance* in such systems are determined by the eigenvalues of the Laplacian matrix, and asymptotic behaviors for macroscopic and microscopic behaviors of the variance have been so studied in [7]. However, no unified metric for second-order systems that is similar to resistance distance for first-order systems has been previously proposed.

In this paper, we propose biharmonic distance as a tool to analyze second-order consensus dynamics with external perturbations. Biharmonic distance is defined based on the spectrum of the Laplacian matrix, and it has been used in computer graphics [15] as a metric that incorporates both local and global graph structure. We study three performance measures in second-order consensus systems: the variance of of the difference between the states of any pair of vertices, the variance between an individual vertex state and the system average, and the total variance of the system. For each of these performance measures, we show how it can be analyzed in terms of biharmonic distances. In addition, we introduce a new notion of vertex centrality based on a biharmonic vertex index. A vertex with higher biharmonic centrality has smaller steady-state variance. We then derive closed-form solutions for the biharmonic distances and related performance measures for complete graphs, star graphs, cycles, and paths. Finally, we use this analysis to compare the behavior of the steady-variance in first- and second-order systems.

*Related work:* Bamieh et al. introduced the concept of *network coherence*, a measure of the average steady-state variance of node states, for both first- and second-order consensus dynamics with stochastic external perturbations. This work showed a relationship between coherence and the spectrum of the Laplacian matrix and derived the asymptotic behavior of coherence in torus networks [7]. Several works have analyzed the coherence of first-order consensus in different classes

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of networks. Young et al. [8] elated network coherence to the Kirchhoff index of a graph and presented closed-form results for the coherence of cycle, path, and star graphs with first-order noisy consensus dynamics. Patterson and Bamieh analyzed coherence in several forms of fractal trees [9] and discussed the impact of fractal dimensions on network coherence, and Yi et al. investigated coherence in Farey graphs [11] and Koch graphs [16] as deterministic generated representatives of small-world networks and scale-free networks.

There have also been several recent works on analysis of coherence for second-order systems in different graph topologies. Namely, the second-order coherence of torus [7], fractals [9], and Koch graphs [16] have all been analyzed. However, none of these works have developed a general mathemtical connection between second-order coherence and a graph distance metric.

With respect to biharmonic distance, the recent work by Fitch and Leonard [10] used a slightly different definition of this distance to describe the centrality of multiple leaders in first-order consensus systems with leader nodes. We show that, while related, this different definition cannot be extended to describe coherence in leader-free second-order consensus networks.

The remainder of this paper is organized as follows. In Section II, we introduce notation and the system dynamics studied in this paper. In Section III, we first describe the notion of biharmonic distance and its definition. We then introduce graph indices and vertex centrality based on biharmonic distance. In Section IV, we show that biharmonic distance plays a important role in perturbed second-order consensus dynamics, and we give relationships between coherence performance measures and the biharmonic distance and its derived indices. In Section V, we compare the relationships between first-order noisy consensus dynamics and resistance distance and second-order noisy consensus dynamics and biharmonic distance.Section VI gives closed-form solutions for the coherence performance measures for complete graphs, star graphs, cycles, and paths. In Section VII, we further investigate these performance measures using numerical examples. Finally, we conclude the paper in Section VIII.

## II. Preliminaries

### A. Concepts and Notation

Let $\mathcal{G}$ be an undirected connected graph, and let $\mathcal{V} = \{0, 1 \ldots, N-1\}$ and $\mathcal{E}$ be the vertex set and edge set that constitute $\mathcal{G}$ as $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$. Let $N = |\mathcal{V}|$ and $M = |\mathcal{E}|$. Define $A$ as the $N \times N$ (0-indexed) adjacency matrix of $\mathcal{G}$, in which $a_{ij} = 1$ if $\{i, j\} \in E$ and $a_{ij} = 0$ otherwise. Let $D$ be the diagonal matrix where $d_{ii}$ is equal to the degree of vertex $i$, *i.e.*, $d_{ii} = \sum_{i=0}^{N-1} a_{ij}$. Define $L = D - A$ as the Laplacian matrix of graph $G$. We use $\lambda_i$ and $u_i$ to denote the $i$-th eigenvalue and eigenvector of $L$, $i \in \{0, 1, \ldots, N-1\}$, where $0 = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_{N-1}$. The all-one vector of order $N$ is denoted by $\mathbf{1}_N$. Therefore, $u_0 = \frac{1}{\sqrt{N}} \mathbf{1}_N$. Then, $L$ can be diagonalized as $L = U \Lambda U^\top$, where $\Lambda \in \mathbb{R}^{N \times N}$ is diagonal and $\Lambda_{ii} = \lambda_i$, $U \in \mathbb{R}^{N \times N}$, with its $i$th column being $u_i$. In addition, we denote by $L^\dagger$ the pseudo-inverse of $L$, and define $L^{2\dagger} = (L^\dagger)^2$.

### B. System Dynamics

Each vertex in the network has a scalar-valued state. Let $x_1(t)$ be the $N$-vector that contains the states of all vertices; $x_{1j}(t)$ represents the state of vertex $j$, $j \in \{0, 1 \ldots, N-1\}$. Then, we define $x_2(t)$ as the first derivative of $x_1(t)$ with respect to $t$, that is, $x_2(t) = \dot{x}_1(t)$. A vertex $j$ adjust its state by setting $\dot{x}_{2j}(t)$ according to the differences of its state ($x_{1j}(t)$ and $x_{2j}(t)$) and the states of its neighbors. The following equation gives the noisy second-order consensus algorithm:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & I \\ -L & -L \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} w(t), \quad (1)$$

where $0$, $I$, and $L$ are all $N \times N$ matrices, and $w(t)$ is a $2N$-vector of uncorrelated Gaussian white noise processes.

### C. Performance Measures

Because the state of each vertex is disturbed by Gaussian noise, the networked system can never reach exact consensus. Therefore, we are interested in the expected deviations of the states of the vertices. In particular, we are interested in three performance measures related to these deviations, which we define below.

First, we want to know how far the states of two vertices are driven away by disturbances. Therefore we study the steady-state of the variance of this pairwise deviation.

**Definition II.1.** *For any two vertices $j, k \in \mathcal{V}$, the* pairwise variance *$H_{\mathrm{SO}}(j, k)$ is the steady-state variance of the difference between $x_{1j}$ and $x_{1k}$, i.e.,*

$$H_{\mathrm{SO}}(j, k) = \lim_{t \to \infty} \mathbb{E}[(x_{1j}(t) - x_{1k}(t))^2]. \quad (2)$$

We note that in a $d$-dimensional torus $\mathbb{Z}_N^d$, $H_{\mathrm{SO}}(j, j-1)$ is the second-order microscopic coherence defined in [7], and $H_{\mathrm{SO}}(j, j + \frac{N}{2})$ is the second-order long-range coherence defined in [7]. Thus, our pairwise variance performance measure is a generalization of these two performance measures.

We are also interested in the variance of the difference between the state of a vertex and the (current) average value in the network. Let $\bar{x}_1(t)$ be the average state $\bar{x}_1(t) = \frac{1}{N} \mathbf{1}_N^\top x_1(t)$.

**Definition II.2.** *For a vertex $j \in \mathcal{V}$, the* vertex variance *$H_{\mathrm{SO}}(j)$ is the steady-state variance of the difference between $x_{1j}(t)$ and $\bar{x}_1(t)$, i.e.,*

$$H_{\mathrm{SO}}(j) = \lim_{t \to \infty} \mathbb{E}[(x_{1j}(t) - \bar{x}_1(t))^2]. \quad (3)$$

Finally, we are interested in the total variance of the system.

**Definition II.3.** *For a network $\mathcal{G}$, the* total variance *$H_{\mathrm{SO}}(\mathcal{G})$ is the total steady-state variance of the deviation of each vertex state from the current average, i.e.,*

$$H_{\mathrm{SO}}(\mathcal{G}) = \lim_{t \to \infty} \sum_{j=0}^{N-1} \mathbb{E}[(x_{1j}(t) - \bar{x}_1(t))^2]. \quad (4)$$

In a $d$-dimensional torus $\mathbb{Z}_N^d$, $H_{\mathrm{SO}}(\mathcal{G})$ is the variance of the deviation from average defined in [7].

## III. Biharmonic Distance

Several slightly different definitions of biharmonic distance have been proposed in related literature [10], [15], [17]. In this paper we follow the definition in [15] and [17], which is as follows.

**Definition III.1.** *The biharmonic distance $d_B(j,k)$ between two vertices $j$ and $k$ in a undirected graph $\mathcal{G}$ is:*

$$d_B^2(j,k) = L_{jj}^{2\dagger} + L_{kk}^{2\dagger} - 2L_{jk}^{2\dagger} = \sum_{i=1}^{N-1} \frac{1}{\lambda_i^2}(u_{ij} - u_{ik})^2 \,. \quad (5)$$

Note that this definition is equal to the square root of the one used by Fitch and Leonard in [10].

Biharmonic distance is a metric, as shown in the following theorem. While this result has been previously proved [15], we include a proof for the convenience of the reader.

**Theorem III.1.** *The biharmonic distance $d_B(j,k)$ is a $\mathcal{V} \times \mathcal{V} \to \mathbb{R}$ metric, which is equivalent to satisfying the following properties:*

- *Non-negativity: $d_B(j,k) \geqslant 0$,*
- *Nullity: $d_B(j,k) = 0$ if and only if $j = k$,*
- *Symmetry: $d_B(j,k) = d_B(k,j)$, and*
- *Triangle inequality $d_B(j,r) + d_B(r,k) \geqslant d_B(j,k)$.*

*Proof:* The non-negativity and symmetry are easily obtained from Definition III.1 along with the fact that $L$ is positive semi-definite. Assume $d_B(j,k) = 0$ for $j \neq k$, then $u_{ij} = u_{ik}$ for all $i \in \{0, 1, \ldots, N-1\}$. Since $L = U\Lambda U^\top$, $L_{jj} = \sum_{i=1}^{N-1} \lambda_i u_{ij} u_{ij}$ and $L_{jk} = \sum_{i=1}^{N-1} \lambda_i u_{ij} u_{ik}$. This leads to $L_{jj} = L_{jk}$ for $j \neq k$, which contradicts with the definition of the Laplacian matrix.

The triangle inequality can be proved as follows. Define a vector,

$$v_j = \sum_{i=1}^{N-1} \frac{u_{ij}}{\lambda_i} u_i \in \mathbb{R}^N \text{ for } j = 0, 1, \ldots, N-1 \,.$$

We note again that $u_i$ is the $i$th eigenvector, and $u_{ij}$ is the $j$th entry of $u_i$. Then it follows that the Euclidean distance $\|v_j - v_k\|_2$ between $v_j$ and $v_k$ is

$$\|v_j - v_k\|_2 = \left\| \sum_{i=1}^{N-1} \frac{(u_{ij} - u_{ik})}{\lambda_i} u_i \right\|_2 = \sqrt{\sum_{i=1}^{N-1} \frac{(u_{ij} - u_{ik})^2}{\lambda_i^2}} \,,$$

which means $d_B(j,k)$ is equal to $\|v_j - v_k\|_2$. Since the Euclidean distance in $\mathbb{R}^N$ is a metric and, therefore, satisfies triangle inequality, $d_B(j,k)$ also satisfies the triangle inequality. ■

We observe that $v_j, j \in \{1, \ldots, N\}$ assigns a position to vertex $j$ in $\mathbb{R}^N$ Euclidean space that preserves biharmonic distance.

**Definition III.2.** *We define an $N$-dimensional mapping of of the vertices in $\mathcal{G}$, $\mathcal{F} : \mathcal{V} \to \mathbb{R}^N$. For any vertex $j$, $\mathcal{F}(j) = v_j = L^\dagger e_j$. $v_j$ is a* biharmonic embeddings *of graph $\mathcal{G}$ in $\mathbb{R}^N$.*

Based on the definition of biharmonic distance, we also define the following graph indices.

**Definition III.3.** *The* biharmonic Kirchhoff index *$D_B^2(\mathcal{G})$ of a graph $\mathcal{G}$ is*

$$D_B^2(\mathcal{G}) = \sum_{\substack{j,k \in V \\ j<k}} d_B^2(j,k) \,. \quad (6)$$

**Definition III.4.** *The* biharmonic vertex index *$D_B^2(j)$ of a node $j$ in a graph $\mathcal{G}$ is*

$$D_B^2(j) = \sum_{k \in V} d_B^2(j,k) \,. \quad (7)$$

We can derive from the definition of $d_B(j,k)$ that

$$D_B^2(\mathcal{G}) = N \cdot \sum_{i=1}^{N-1} \frac{1}{(\lambda_i)^2} \,. \quad (8)$$

Finally, for a vertex $j$ in graph $\mathcal{G}$, we can define its centrality based on biharmonic distances.

**Definition III.5.** *The* biharmonic centrality *of vertex $j$ in graph $\mathcal{G}$ is*

$$C_B(j) = \left( \frac{1}{N} D_B^2(j) \right)^{-1} \,. \quad (9)$$

## IV. Biharmonic Distance in Second-order Consensus Dynamics with Disturbances

The equation (1) gives the dynamics of the second-order consensus algorithm with stochastic perturbations. The deviation of the state of vertex $j$ from the average of all states is given by $y_j(t) = x_{1j}(t) - \bar{x}_1(t)$. Let $y(t)$ be a $N \times 1$ vector representing all vertices' deviations from average,

$$y(t) = [\Pi \mid 0]\, x(t) = \Pi x_1(t) \,,$$

where $\Pi = I_N - \frac{1}{N}\mathbf{1}_N \mathbf{1}_N^\top$. The performance measures we study in this paper can all be expressed in terms of of $y(t)$. Specifically,

$$\begin{aligned} H_{\text{SO}}(j,k) &= \lim_{t\to\infty} \mathbb{E}[((x_{1j}(t) - \bar{x}_1(t)) - (x_{1k}(t) - \bar{x}_1(t)))^2] \\ &= \lim_{t\to\infty} \mathbb{E}[(y_j(t) - y_k(t))^2] \quad (10) \\ H_{\text{SO}}(j) &= \lim_{t\to\infty} \mathbb{E}[(x_{1j}(t) - \bar{x}_1(t))^2] = \lim_{t\to\infty} \mathbb{E}[y_j(t)^2] \quad (11) \\ H_{\text{SO}}(\mathcal{G}) &= \lim_{t\to\infty} \sum_{j=0}^{N-1} \mathbb{E}[y_j(t)^2] \,. \quad (12) \end{aligned}$$

However, the system described by (1) is only marginally stable [8]. To obtain a stable system, we only consider the dynamics in the subspace that is orthogonal to the subspace spanned by $\mathbf{1}_N$. We define $Q$ as a $(N-1) \times N$ matrix whose rows are the eigenvectors of $L$, excluding $\mathbf{1}_N$. We recall that $L$ can be diagonalized as $U\Lambda U^\top$, where $U$ is a unitary matrix and $\Lambda$ is a diagonal matrix. Then, $Q^\top$ is the submatrix of $U$ formed by eliminating the first column. It is easy to confirm that $Q\mathbf{1}_N = 0$, $QQ^\top = I_{N-1}$, $Q^\top Q = \Pi$, and $LQ^\top Q = L$. Then, we define

$$z_1(t) = [\ Q \mid 0\ ]\, x(t) = Qx_1(t) \,,$$

and note that $y(t) = Q^\top z_1(t)$. It indicates that we can write expressions for our performance measures using $z_1(t)$. Let $z_2(t) = \dot{z}_1(t)$. Then (1) leads to

$$\begin{bmatrix} Q & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & Q \\ -QLQ^\top Q & -QLQ^\top Q \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ Q \end{bmatrix} w(t) \,,$$

Therefore, we obtain a stable system:

$$\begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & I_{N-1} \\ -\bar{\Lambda} & -\bar{\Lambda} \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ Q \end{bmatrix} w(t) \,,$$

where $\bar{\Lambda} = QLQ^\top = QU\Lambda(QU)^\top = diag(\lambda_1, \ldots, \lambda_{N-1})$.

We can always find the unitary (orthogonal) permutation matrix $V \in \{0,1\}^{(2N-2)\times(2N-2)}$ such that

$$\begin{bmatrix} \dot{z}_1(t) \\ \dot{z}_2(t) \end{bmatrix} = V^\top K V \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ Q \end{bmatrix} w(t) \,, \tag{13}$$

where $K$ is the block diagonal matrix,

$$K = \left[\begin{array}{c|c|c} P_1 & & \\ \hline & \ddots & \\ \hline & & P_{N-1} \end{array}\right] , \tag{14}$$

with each $P_i$ defined as:

$$P_i = \begin{bmatrix} 0 & 1 \\ -\lambda_i & -\lambda_i \end{bmatrix} .$$

Hereafter, we use the system dynamics in (13) to develop expressions for the performance measures defined in Section II-C.

### A. Pairwise Variance

**Theorem IV.1.** *The pairwise variance of the difference between states of vertices $j$ and $k$ with dynamics (1) can be expressed by the spectrum of the Laplacian matrix of graph $\mathcal{G}$ as*

$$H_{\mathrm{SO}}(j,k) = \sum_{i=1}^{N-1} \frac{(u_{ij} - u_{ik})^2}{2\lambda_i^2} . \tag{15}$$

*Proof:* We start by expressing $H_{\mathrm{SO}}(j,k)$ in terms of $z_1(t)$,

$$\begin{aligned} H_{\mathrm{SO}}(j,k) &= \lim_{t\to\infty} \mathbb{E}\left[y(t)^\top (e_j - e_k)(e_j^\top - e_k^\top) y(t)\right] \\ &= \lim_{t\to\infty} \mathbb{E}\left[(Q^\top z_1(t))^\top (e_j - e_k)(e_j^\top - e_k^\top) Q^\top z_1(t)\right] \\ &= \lim_{t\to\infty} \mathbb{E}\left[z_1(t)^\top Q (e_j - e_k)(e_j^\top - e_k^\top) Q^\top z_1(t)\right] \\ &= \lim_{t\to\infty} \mathbb{E}\left[\mathbf{tr}((e_j - e_k)^\top Q^\top z_1(t) z_1(t)^\top Q (e_j - e_k))\right] , \end{aligned}$$

where $e_j$ is the $j$th canonical basis vector of $\mathbb{R}^N$. We define the output of the system as

$$\begin{aligned} \phi(t) &= (e_j - e_k)^\top Q^\top [I_{N-1} | 0_{N-1}] z(t) \\ &= (e_j - e_k)^\top Q^\top z_1(t) \,. \end{aligned} \tag{16}$$

Then, we define $\Sigma(t) = \mathbb{E}[\phi(t)\phi(t)^\top]$; therefore, $H_{\mathrm{SO}}(j,k) = \lim_{t\to\infty}[\mathbf{tr}\,(\Sigma(t))] = [\mathbf{tr}\,(\lim_{t\to\infty}\Sigma(t))] =: [\mathbf{tr}\,(\Sigma)]$.

For the state-space system given by (13) and (16), the square of the $\mathcal{H}_2$ norm of the system is

$$\mathcal{H}_2^2 = \int_0^\infty B^\top \mathrm{e}^{-M^\top t} Z \mathrm{e}^{-Mt} B \mathrm{d}t \,, \tag{17}$$

in which

$$B = \begin{bmatrix} 0 \\ Q \end{bmatrix} \tag{18}$$

$$M = \begin{bmatrix} 0 & I \\ -\bar{\Lambda} & -\bar{\Lambda} \end{bmatrix} \text{ and} \tag{19}$$

$$Z = \begin{bmatrix} Q(e_j - e_k)(Q(e_j - e_k))^\top & 0 \\ 0 & 0 \end{bmatrix} . \tag{20}$$

It follows that $H_{\mathrm{SO}}(j,k) = \mathcal{H}_2^2 = \mathbf{tr}\left(B^\top \Sigma B\right)$. $\Sigma$ is the solution of the following Lyapunov equation,

$$M^\top \Sigma + \Sigma M + Z = 0 \,. \tag{21}$$

The equation is equivalent to

$$VM^\top \Sigma V^\top + V\Sigma M V^\top = -VZV^\top \text{ or}$$
$$(VM^\top V^\top)(V\Sigma V^\top) + (V\Sigma V^\top)(VMV^\top) = -VZV^\top$$

where $V$ was defined in (13) as a (unitary) permutation matrix. We denote by $K = VMV^\top$ and $\Theta = V\Sigma V^\top$. Then equation (21) can be written as

$$\begin{aligned} K^\top\Theta + \Theta K &= -VZV^\top \\ &= -\left[\begin{array}{c|c|c} Z_{11} & \cdots & Z_{1(N-1)} \\ \hline \vdots & \ddots & \vdots \\ \hline Z_{(N-1)1} & \cdots & Z_{(N-1)(N-1)} \end{array}\right] , \end{aligned} \tag{22}$$

for $i, m \in \{1, \ldots, N-1\}$,

$$\begin{aligned} Z_{im} &= \begin{bmatrix} (Q_{ij}Q_{mj} - Q_{ij}Q_{mk} - Q_{ik}Q_{mj} + Q_{ik}Q_{mk}) & 0 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} (u_{ij}u_{mj} - u_{ij}u_{mk} - u_{ik}u_{mj} + u_{ik}u_{mk}) & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

We note that $K$ is block-diagonal. Substituting (14) into diagonal blocks of (22) yields $P_i^\top \Theta_{ii} + \Theta_{ii} P_i = Z_{ii}$. Since $Z_{ii}$ and $P_i$ are symmetric, $\Theta_{ii}$ is also symmetric. We write $\Theta_{ii}$ as

$$\Theta_{ii} = \begin{bmatrix} X_{ii} & \Psi_{ii} \\ \Psi_{ii} & Y_{ii} \end{bmatrix} .$$

Then,

$$\begin{bmatrix} 0 & \lambda_i \\ 1 & \lambda_i \end{bmatrix} \begin{bmatrix} X_{ii} & \Psi_{ii} \\ \Psi_{ii} & Y_{ii} \end{bmatrix} + \begin{bmatrix} X_{ii} & \Psi_{ii} \\ \Psi_{ii} & Y_{ii} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ \lambda_i & \lambda_i \end{bmatrix} = \begin{bmatrix} (u_{ij} - u_{ik})^2 & 0 \\ 0 & 0 \end{bmatrix} ,$$

which leads to

$$Y_{ii} = \frac{(u_{ij} - u_{ik})^2}{2\lambda_i^2} .$$

Then, we derive that

$$\begin{aligned} H_{\mathrm{SO}}(j,k) = \mathcal{H}_2^2 &= \mathbf{tr}\left(B^\top \Sigma B\right)^{\frac{1}{2}} = \mathbf{tr}\left(B^\top V^\top \Theta V B\right) \\ &= \sum_{i=1}^{N-1} (Y_{ii}) = \sum_{i=1}^{N-1} \frac{(u_{ij} - u_{ik})^2}{2\lambda_i^2} . \end{aligned} \tag{23}$$

■

Applying (5), we immediately obtain the following theorem.

**Theorem IV.2.** *For any vertex pair $j$ and $k$ in a network $\mathcal{G}$ with dynamics (1),*

$$H_{\mathrm{SO}}(j,k) = \frac{1}{2} d_B^2(j,k) \tag{24}$$

This theorem shows that the pairwise variance between vertices $j$ and $k$ is proportional to the square of their biharmonic distance.

### *B. Vertex Variance*

We first give an expression for the vertex variance in terms of the eigenvalues and eigenvectors of $L$.

**Theorem IV.3.** *For any vertex $j$ in network $G$ with dynamics (1)*

$$H_{\mathrm{SO}}(j) = \sum_{i=1}^{N-1} \frac{u_{ij}^2}{2\lambda_i^2}\,. \tag{25}$$

*Proof:* First, we derive an expression for the vertex variance in terms of $z_1(t)$,

$$\begin{aligned} H_{\mathrm{SO}}(j) &= \lim_{t\to\infty} \mathbb{E}\left[y(t)^\top e_j e_j^\top y(t)\right] \\ &= \lim_{t\to\infty} \mathbb{E}\left[(Q^\top z_1(t))^\top e_j e_j^\top Q^\top z_1(t)\right] \\ &= \lim_{t\to\infty} \mathbb{E}\left[z_1(t)^\top Q e_j e_j^\top Q^\top z_1(t)\right] \\ &= \lim_{t\to\infty} \mathbb{E}\left[\mathbf{tr}(e_j^\top Q^\top z_1(t) z_1(t)^\top Q e_j)\right]\,. \end{aligned}$$

With this, we define the output for the dynamics (13) as,

$$\phi(t) = e_j^\top Q^\top [I_{N-1}|0_{N-1}] z(t) = e_j^\top Q^\top z_1(t)\,. \tag{26}$$

Again, we define $\Sigma(t) = \mathbb{E}[\phi(t)\phi(t)^\top]$, therefore $H_{\mathrm{SO}}(j) = \lim_{t\to\infty}[\mathbf{tr}\,(\Sigma(t))] = [\mathbf{tr}\,(\lim_{t\to\infty}\Sigma(t))] =: [\mathbf{tr}\,(\Sigma)]$.

For the state-space system given by (13) and (26), the square of $\mathcal{H}_2$ norm of the system is also defined by (17), in which $B$ and $M$ are given by (18) and (19), $Z$ is expressed by

$$Z = \begin{bmatrix} Qe_j(Qe_j)^\top & 0 \\ 0 & 0 \end{bmatrix}.$$

It follows that $H_{\mathrm{SO}}(j) = \mathcal{H}_2^2 = \mathbf{tr}\left(B^\top \Sigma B\right)$. $\Sigma$ is the solution of the following Lyapunov equation,

$$M^\top \Sigma + \Sigma M + Z = 0\,, \tag{27}$$

The equation is equivalent to

$$\begin{aligned} &K^\top\Theta + \Theta K = -VZV^\top \\ &= -\left[\begin{array}{c|c|c} Z_{11} & \cdots & Z_{1(N-1)} \\ \hline \vdots & \ddots & \vdots \\ \hline Z_{(N-1)1} & \cdots & Z_{(N-1)(N-1)} \end{array}\right], \end{aligned} \tag{28}$$

where

$$Z_{im} = \begin{bmatrix} Q_{ij}Q_{mj} & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} u_{ij}u_{mj} & 0 \\ 0 & 0 \end{bmatrix},$$

for $i, m \in \{1, \ldots, N-1\}$. We recall that $K = VMV^\top$ and $\Theta = V\Sigma V^\top$.

Substituting (14) into diagonal blocks of (28) yields $P_i^\top\Theta_{ii} + \Theta_{ii}P_i = Z_{ii}$. Similar to the pairwise case, we assume

$$\Theta_{ii} = \begin{bmatrix} X_{ii} & \Psi_{ii} \\ \Psi_{ii} & Y_{ii} \end{bmatrix}. \tag{29}$$

By solving $P_i^\top\Theta_{ii} + \Theta_{ii}P_i = Z_{ii}$ we derive

$$Y_{ii} = \frac{u_{ij}^2}{2\lambda_i^2}\,.$$

Then we obtain

$$\begin{aligned} H_{\mathrm{SO}}(j) = \mathcal{H}_2^2 &= \mathbf{tr}\left(B^\top \Sigma B\right) = \mathbf{tr}\left(B^\top V^\top \Theta V B\right) \\ &= \sum_{i=1}^{N-1} (Y_{ii}) = \sum_{i=1}^{N-1} \frac{u_{ij}^2}{2\lambda_i^2}\,. \end{aligned} \tag{30}$$

■

We next use Theorem IV.3 to derive an expression for the vertex variance in terms of biharmonic distances.

**Theorem IV.4.** *For any vertex $j$ in network $\mathcal{G}$ with dynamics (1), the variance of difference between the state of a vertex and the system averge is decided by the spectrum of the Laplacian marix of the graph, that is*

$$H_{\mathrm{SO}}(j) = \frac{1}{2N}\left(D_B^2(j) - \frac{1}{N} D_B^2(\mathcal{G})\right)\,. \tag{31}$$

*Proof:* The biharmonic distance from vertex $j$ to all other vertices is

$$\begin{aligned} D_B^2(j) &= \sum_{k=0}^{N-1} d_B^2(j,k) = \sum_{k=0}^{N-1}\sum_{i=1}^{N-1} \frac{1}{\lambda_i^2}(u_{ij} - u_{ik})^2 \\ &= \sum_{i=1}^{N-1}\sum_{k=0}^{N-1} \frac{u_{ij}^2 - 2u_{ij}u_{ik} + u_{ik}^2}{\lambda_i^2} \\ &= N \sum_{i=1}^{N-1} \frac{u_{ij}^2}{\lambda_i^2} + \sum_{i=1}^{N-1} \frac{1}{\lambda_i^2}\,. \end{aligned} \tag{32}$$

Substituting (8) and (25) into (32), we obtain

$$H_{\mathrm{SO}}(j) = \frac{D_B^2(j)}{2N} - \frac{D_B^2(\mathcal{G})}{2N^2}\,. \tag{33}$$

■

### *C. Total Variance*

Finally, we present expressions for the total variance in terms of the spectrum of the Laplacian matrix.

**Theorem IV.5.** *The total steady-state variance $H_{\mathrm{SO}}(\mathcal{G})$ of system (1) is*

$$H_{\mathrm{SO}}(\mathcal{G}) = \sum_{i=0}^{N-1} \frac{1}{2\lambda_i^2}\,. \tag{34}$$

*Proof:* Since,

$$H_{\mathrm{SO}}(\mathcal{G}) = \sum_{j=0}^{N-1} H_{\mathrm{SO}}(j)\,,$$

we immediately obtain

$$H_{\mathrm{SO}}(\mathcal{G}) = \sum_{j=0}^{N-1}\sum_{i=0}^{N-1}\frac{u_{ij}^2}{2\lambda_i^2} = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1}\frac{u_{ij}^2}{2\lambda_i^2} = \sum_{i=0}^{N-1}\frac{1}{2\lambda_i^2}.$$

■

In similar fashion, we use (8) to obtain the following theorem about the relationship between the total variance and biharmonic distances.

**Theorem IV.6.** *For a network $\mathcal{G}$ with dynamics (1), the total variance is given by the biharmonic Kirchhoff index of the graph, specifically,*

$$H_{\mathrm{SO}}(G) = \frac{1}{2N}D_B^2(\mathcal{G})\,. \tag{35}$$

## V. Resistance Distance in First-order Consensus Dynamics with Disturbances

In this section, we briefly review first-order consensus dynamics with stochastic disturbances and the relationship between resistance distance and the total steady-state variance The first-order consensus system is formulated as

$$\dot{x}(t) = -Lx(t) + w(t)\,, \tag{36}$$

where $x(t) \in \mathbb{R}^N$ represents the states of the vertices, and $w(t) \in \mathbb{R}^N$ is a vector of uncorrelated Gaussian white noise processes. The total steady-state variance of the system is

$$H_{\mathrm{FO}}(\mathcal{G}) = \lim_{t\to\infty}\sum_{j=1}^{N}\mathbb{E}[(x_j(t)-\bar{x}(t))^2]\,, \tag{37}$$

where $\bar{x}(t) = \frac{1}{N}\mathbf{1}_N^\top x(t)$.

The total steady-state variance $H_{\mathrm{FO}}$ can be expressed in terms of resistance distances in an electrical network. We first formalize the notion of resistance distance and the Kirchhoff index.

**Definition V.1.** *The resistance distance $d_R(j,k)$ between two vertices $j$ and $k$ in an undirected graph $\mathcal{G}$ is defined as*

$$d_R(j,k) = L_{jj}^\dagger + L_{kk}^\dagger - 2L_{jk}^\dagger = \sum_{i=1}^{N-1}\frac{1}{\lambda_i}(u_{ij}-u_{ik})^2\,. \tag{38}$$

**Definition V.2.** *The* Kirchhoff index $D_R(\mathcal{G})$ *of a graph $\mathcal{G}$ is defined as*

$$D_R(\mathcal{G}) = \sum_{\substack{j,k\in V\\ j<k}} d_R(j,k)\,. \tag{39}$$

It has been shown [7], [18] that the Kirchhoff index is related to the total steady-state variance of system (36) as

$$H_{\mathrm{FO}}(\mathcal{G}) = \frac{1}{2N}D_R(\mathcal{G})\,. \tag{40}$$

We also note that the notion of the *information centrality* of a vertex can be expressed in terms of resistance distances. If we defined the sum of resistance distances between all vertices to a vertex $j$ as

$$D_R(j) = \sum_{k\in V} d_R(j,k)\,, \tag{41}$$

then the *information centrality* of vertex $j$ in graph $\mathcal{G}$ is [19]

$$C_R(j) = \left(\frac{1}{N}D_R(j)\right)^{-1}\,. \tag{42}$$

Finally, we define the resistance embedding of a graph.

**Definition V.3.** *Let $\mathcal{F}_R : \mathcal{V} \to \mathbb{R}^N$ be an $n$-dimensional mapping of $\mathcal{G}$, such that for any vertex $j$, $\mathcal{F}_R(j) = \mu_j = L^{\dagger/2}e_j$. $\mu_j$ is a* resistance embedding *of graph $\mathcal{G}$ in $\mathbb{R}^N$.*

## VI. Analytical Examples

In this section we give examples for biharmonic distance, connectivity and centrality in networks with special topology. Closed form expressions are derived for all cases. We also compare the asymptotic behavior of the steady-state variance of first- and second-order systems.

We note that in some of these examples eigenvectors, are given as complex vectors (although they can be given as real vectors by an unitary linear transform). Therefore, we calculate the biharmonic distances using the following expression:

$$d_B^2(j,k) = L_{jj}^{2\dagger} + L_{kk}^{2\dagger} - 2L_{jk}^{2\dagger} = \sum_{n=1}^{N-1}\frac{1}{\lambda_n^2}|u_{nj}-u_{nk}|^2\,, \tag{43}$$

which is a slight variation of the definition in (5). We note that $i$ is used to indicate the imaginary unit in this section.

### A. Complete Graph

A complete graph is a network in which every vertex is connected to every other vertex. We consider a complete graph of $N$ vertices. Its Laplacian matrix of it is

$$L_N^{cp} = \begin{pmatrix} N-1 & -1 & \cdots & -1 & -1 \\ -1 & N-1 & \cdots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -1 & \cdots & N-1 & -1 \\ -1 & -1 & \cdots & -1 & N-1 \end{pmatrix}.$$

Matrix $L_N^{cp}$ is diagonalized by a discrete Fourier transform. It can be verified that its eigenvalues and eigenvectors are given by

$$\lambda_0 = 0 \tag{44}$$

$$\lambda_n = N, \qquad n = 1,2,\cdots,N-1 \tag{45}$$

$$u_{nm} = \frac{1}{\sqrt{N}}e^{i2\pi nm/N}, \qquad n,m = 0,1,\cdots,N-1\,. \tag{46}$$

**Proposition VI.1.** *In a complete graph $\mathcal{G} = (\mathcal{V},\mathcal{E})$ with $N$ vertices, let $j,k \in \mathcal{V}$, $j \neq k$. The biharmonic distance between $j$ and $k$ is*

$$d_B(j,k) = \frac{\sqrt{2}}{N}\,. \tag{47}$$

*Proof:* By substituting the eigenvalues and eigenvectors in (44) - (46) into (43), we obtain

$$\begin{aligned} d_B^2(j,k) &= \sum_{n=1}^{N-1}\frac{|u_{nj}-u_{nk}|^2}{N^2} \\ &= \frac{1}{N^3}\sum_{n=1}^{N-1}4\sin^2\frac{(j-k)\pi n}{N} \\ &= \frac{2}{N^2}\,. \end{aligned}$$

■

Once we obtain the biharmonic distance between any vertices $j$ and $k$, we can derive the other related indices. From (47), we derive the biharmonic Kirchhoff index for a complete graph with $N$ vertices.

$$D_B^2(\mathcal{G}) = \frac{N(N-1)}{2} \cdot \frac{2}{N^2} = \frac{N-1}{N}\,.$$

We also derive the biharmonic vertex index and biharmonic centrality for a complete graph,

$$\begin{aligned} D_B^2(j) &= (N-1) \cdot \frac{2}{N^2} = \frac{2(N-1)}{N^2}\,, \\ C_B(j) &= \frac{N^3}{2(N-1)}\,. \end{aligned}$$

Finally, we use the biharmonic distance and Theorems IV.2, IV.4, and IV.6 to determine closed-form solutions for the three performance measures defined in Section II-C.

**Theorem VI.2.** *For a complete graph $\mathcal{G}$ with $N$ vertices, where the system dynamics are as given in (1),*

$$\begin{aligned} H_{\mathrm{SO}}(j,k) &= \frac{1}{N^2}\,, \qquad j,k \in V,\ j \neq k\,; \\ H_{\mathrm{SO}}(j) &= \frac{N-1}{2N^3}\,, \qquad j \in V\,; \\ H_{\mathrm{SO}}(\mathcal{G}) &= \frac{N-1}{2N^2}\,. \end{aligned}$$

We recall that in a complete graph, the total variance in a system with first-order noisy consensus dynamics (36) is $H_{\mathrm{FO}}(\mathcal{G}) \in O(1)$ [8]. This is in contrast with $H_{\mathrm{SO}}(\mathcal{G})$ which is in $O(1/N)$.

### *B. Star Graph*

We consider a star graph of order $N$, which consists of one hub and $N-1$ leaves. Its Laplacian matrix is

$$L_N^{star} = \begin{pmatrix} N-1 & -1 & \cdots & -1 & -1 \\ -1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & 0 & \cdots & 1 & 0 \\ -1 & 0 & \cdots & 0 & 1 \end{pmatrix}. \tag{48}$$

Its eigenvalues and corresponding orthonormal eigenvectors are [20],

$$\lambda_0 = 0 \tag{49}$$

$$\lambda_n = 1\,, \quad n = 1,2,\cdots,N-2\,, \tag{50}$$

$$\lambda_{N-1} = N\,, \tag{51}$$

and

$$u_0 = \frac{1}{\sqrt{N}}(1,1,1,\cdots,1,1,1)^\top \tag{52}$$

$$u_n = \frac{1}{\sqrt{n(n+1)}}(0,\underbrace{-1,\cdots,-1}_{n},n,0,0,\cdots,0)^\top, \quad n = 1,2,\cdots,N-2\,, \tag{53}$$

$$u_{N-1} = \frac{1}{\sqrt{N(N-1)}}(1-N,1,\cdots,1,1)^\top\,. \tag{54}$$

We use these eigenvalues and eigenvectors to find the biharmonic distances between vertices in a star graph.

**Proposition VI.3.** *In a star network $\mathcal{G} = (\mathcal{V},\mathcal{E})$ with vertex $0$ being the hub with degree $N-1$, and the remaining $N-1$ vertices as leaves, the biharmonic distance between the hub and a leaf is given by*

$$d_B(0,j) = \sqrt{\frac{N-1}{N}}, \qquad j = 1,2,\cdots,N-1, \tag{55}$$

*and the biharmonic distance between any two leaves is*

$$d_B(j,k) = \sqrt{2}, \qquad j,k = 1,2,\cdots,N-1; \quad j \neq k\,. \tag{56}$$

*Proof:* The biharmonic distance between any two vertices $j,k \in \mathcal{V}$, $j \neq k$ is given by

$$d_B^2(j,k) = \sum_{n=1}^{N-2} \frac{(u_{nj}-u_{nk})^2}{1^2} + \frac{(u_{N-1,j}-u_{N-1,k})^2}{N^2}. \tag{57}$$

Substituting (49) - (51) and (52) - (54) into (57) yields the theorem. ■

With these biharmonic distances, we easily obtain the biharmonic Kirchhoff index,

$$\begin{aligned} D_B^2(\mathcal{G}) &= (N-1)\frac{N-1}{N} + \frac{(N-1)(N-2)}{2} \cdot 2 \\ &= N^2 - 2N + \frac{1}{N}\,. \end{aligned}$$

The expressions for biharmonic vertex index and biharmonic centrality also immediately follow from the proposition, For the central vertex in a star graph,

$$\begin{aligned} D_B^2(0) &= (N-1) \cdot \frac{N-1}{N} = \frac{(N-1)^2}{N}\,, \\ C_B(0) &= \frac{N^2}{(N-1)^2}\,, \end{aligned}$$

and any leaf vertex $j$,

$$\begin{aligned} D_B^2(j) &= \frac{N-1}{N} + (N-2)\cdot 2 = \frac{2N^2-3N-1}{N}\,, \\ C_B(j) &= \frac{N^2}{2N^2-3N-1}\,. \end{aligned}$$

Applying Proposition VI.3 and Theorems IV.2, IV.4, and IV.6, we obtain closed-form solutions for the three steady-state variance performance measures.

**Theorem VI.4.** *For a star graph $\mathcal{G}$ with $N$ vertices, where the system dynamics are as given in (1), and where vertex $0$ is the hub,*

$$\begin{aligned} H_{\mathrm{SO}}(0,j) &= \frac{N-1}{2N}\,, \qquad j \neq 0\,; \\ H_{\mathrm{SO}}(j,k) &= 1\,, \qquad j \neq k;\ j,k \neq 0\,; \\ H_{\mathrm{SO}}(0) &= \frac{N-1}{2N^3}\,; \\ H_{\mathrm{SO}}(j) &= \frac{N^3-N^2-N-1}{2N^3}\,, \qquad j \neq 0\,; \\ H_{\mathrm{SO}}(\mathcal{G}) &= \frac{N}{2} - 1 + \frac{1}{2N^2}\,. \end{aligned}$$

We recall that in an $N$-node star graph, the total variance for a system with first-order noisy consensus dynamics is $H_{\mathrm{FO}}(\mathcal{G}) \in O(N)$ [8], and interestingly, in second order systems, the total variance is also in $O(N)$.

### C. Cycle

The Laplacian of a cycle $C_N$ with $N$ vertices is given by

$$L_N^{cyc}=\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & -1\\ -1 & 2 & -1 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & -1 & 2 & -1\\ -1 & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

$L_N^{cyc}$ is a circulant matrix. Therefore, its spectrum is given by a discrete Fourier transform. Let $\phi_n=\frac{n\pi}{N}$; the eigenvalues and eigenvectors of $L_N^{cyc}$ are

$$\lambda_n = 2(1-\cos 2\phi_n), n=0,1,2,\cdots,N-1 \tag{58}$$

$$u_{nm}=\frac{1}{\sqrt{N}}e^{i2m\phi_n}, n,m=0,1,\cdots,N-1. \tag{59}$$

We use these eigenvalues and eigenvectors to determine the biharmonic distance.

**Proposition VI.5.** *In a cycle graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$, let $j,k\in\mathcal{V}$, $k\le j$ and $j-k=l$. Then, the biharmonic distance between $j$ and $k$ is*

$$d_B(j,k)=\sqrt{\frac{l^4}{12N}-\frac{l^3}{6}+\frac{l^2N}{12}-\frac{l^2}{6N}+\frac{l}{6}}. \tag{60}$$

The proof of the proposition is given in Appendix B.

Next, we calculate the derived indices using biharmonic distances. For a cycle $C_N$ with $N$ nodes, the biharmonic Kirchhoff index is

$$D_B^2(\mathcal{G})=\frac{1}{720}(N^5+10N^3-11N).$$

For any vertex $j$ in a cycle, its biharmonic vertex index and biharmonic centrality are

$$D_B^2(j)=\frac{1}{360}(N^4+10N^2-11),$$
$$C_B(j)=\frac{360N}{N^4+10N^2-11}.$$

By applying Theorems IV.2, IV.4, and IV.6, along with Proposition VI.5, we obtain closed-form solutions for the steady-state variance performance measures.

**Theorem VI.6.** *For a cycle graph $\mathcal{G}$ with $N$ vertices where the dynamics are given by (1),*

$$H_{\mathrm{SO}}(j,k)=\frac{l^4}{24N}-\frac{l^3}{12}+\frac{l^2N}{24}-\frac{l^2}{12N}+\frac{l}{12}, \quad \text{For } j,k\in\mathcal{V},\quad k\le j \text{ and } j-k=l; \tag{61}$$

$$H_{\mathrm{SO}}(j)=\frac{1}{1440}\left(N^3+10N-\frac{11}{N}\right), \quad j\in\mathcal{V}; \tag{62}$$

$$H_{\mathrm{SO}}(\mathcal{G})=\frac{1}{1440}\left(N^4+10N^2-11\right). \tag{63}$$

To give some examples for $H_{\mathrm{SO}}(j,k)$ in a cycle of $N$ vertices, it holds that $H_{\mathrm{SO}}(0,1)=\frac{1}{24}(N-1/N)$. For a even $N$, $H_{\mathrm{SO}}(0,N/2)=\frac{1}{384}N(N^2+8)$.

To compare with the first-order consensus dynamics, we recall that in a cycle graph with $N$ vertices, $H_{\mathrm{FO}}(\mathcal{G})\in O(N^2)$ [7], whereas in second-order systems $H_{\mathrm{SO}}(\mathcal{G})\in O(N^4)$.

### D. Path

We consider a path graph $P_N$ with $N$ vertices. Let the vertices be numbered $0,1,\ldots,N-1$. The Laplacian matrix of $P_N$ assumes the form

$$L_N^{path}=\begin{pmatrix} 1 & -1 & 0 & \cdots & 0 & 0 & 0\\ -1 & 2 & -1 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & -1 & 2 & -1\\ 0 & 0 & 0 & \cdots & 0 & -1 & 1 \end{pmatrix}.$$

The eigenvalues and eigenvectors of $L_N^{path}$ are [21].

$$\lambda_n =2(1-\cos\phi_n), \qquad n=0,1,2,\cdots,N-1 \tag{64}$$

$$u_{0m} =\frac{1}{\sqrt{N}}, \qquad m=0,1,\cdots N-1 \tag{65}$$

$$u_{nm} =\sqrt{\frac{2}{N}}\cos(m+1/2\phi_n), \quad n=1,2,\cdots,N-1, m=0,1,\cdots N-1 \tag{66}$$

where $\phi_n=n\pi/N$.

We use (64) - (66) to determine the biharmonic distance between two vertices in a path.

**Proposition VI.7.** *In a path graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$ with $N$ vertices, the biharmonic distance between two vertices $j,k\in\{0,1,\ldots,N-1\}$, $k<j$, is*

$$\begin{aligned} d_B(j,k)=\Bigg(&\frac{j}{6}+\frac{j^2}{2}-\frac{j^2}{4N}+\frac{j^3}{3}-\frac{j^3}{2N}-\frac{j^4}{4N}\\ &-\frac{k}{6}-jk+\frac{jk}{2N}+\frac{j^2k}{2N}+\frac{k^2}{2}-\frac{k^2}{4N}\\ &-jk^2+\frac{jk^2}{2N}+\frac{j^2k^2}{2N}+\frac{2k^3}{3}-\frac{k^3}{2N}-\frac{k^4}{4N}\Bigg)^{\frac{1}{2}}. \end{aligned} \tag{67}$$

The proof of Proposition VI.7 is given in Appendix C.

We next use Proposition VI.7 to derive the biharmonic Kirchhoff index for a path with $N$ nodes,

$$D_B^2(\mathcal{G})=\frac{1}{180}(2N^5+5N^3-7N).$$

We can also derive the biharmonic vertex index and biharmonic centrality for a node $j$,

$$\begin{aligned} D_B^2(j)=&\frac{1}{30}(N^4-10j(j+1)N^2\\ &+10j(2j+1)(j+1)N-10j^2(j+1)^2-1),\\ C_B(j)=&\frac{30N}{N^4-10j(j+1)N^2+10j(2j+1)(j+1)N-10j^2(j+1)^2-1}. \end{aligned}$$

Finally, we present the following theorem that gives the steady-state variance performance measures for $P_N$. This theorem follows directly from Proposition VI.7 and Theorems IV.2, IV.4, and IV.6.

**Theorem VI.8.** *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a path graph with $\mathcal{V} = \{0, 1, \ldots, N-1\}$ and with the dynamics (1). Let $j, k \in \mathcal{V}$ with $k < j$. Then,*

$$\begin{aligned} H_{\mathrm{SO}}(j,k) = \frac{1}{2}\bigg(&\frac{j}{6} + \frac{j^2}{2} - \frac{j^2}{4N} + \frac{j^3}{3} - \frac{j^3}{2N} - \frac{j^4}{4N} \\ &- \frac{k}{6} - jk + \frac{jk}{2N} + \frac{j^2k}{2N} + \frac{k^2}{2} - \frac{k^2}{4N} \\ &- jk^2 + \frac{jk^2}{2N} + \frac{j^2k^2}{2N} + \frac{2k^3}{3} - \frac{k^3}{2N} - \frac{k^4}{4N}\bigg), \end{aligned} \tag{68}$$

$$\begin{aligned} H_{\mathrm{SO}}(j) = &\frac{1}{360N}(4N^4 - (60j^2 + 60j + 5)N^2 \\ &+ 60j(2j+1)(j+1)N - 60j^2(j+1)^2 + 1) \end{aligned} \tag{69}$$

$$H_{\mathrm{SO}}(\mathcal{G}) = \frac{1}{360}\left(2N^4 + 5N^2 - 7\right). \tag{70}$$

To give some examples for $H_{\mathrm{SO}}(j,k)$ and $H_{\mathrm{SO}}(j)$, we note that $H_{\mathrm{SO}}(0, N-1) = \frac{1}{24}N(N^2-1)$ and $H_{\mathrm{SO}}(0) = \frac{1}{90}N(N^2 - 5/4) + \frac{1}{360N}$. For a even $N$, $H_{\mathrm{SO}}(N/2, N-1) = \frac{5}{384}N(N-2/5)(N-2)$ and $H_{\mathrm{SO}}(N/2) = \frac{1}{1440}N(N^2+40) + \frac{1}{360N}$.

We recall that in a $N$-vertex path graph with first order noisy consensus dynamics, the total variance is $H_{\mathrm{FO}} = O(N^2)$ [8]. This is in contrast with the second order system, which has total variance in $O(N^4)$.

## VII. Numerical Examples

In this section, we give numerical examples of the biharmonic and resistance distances in several graphs.

Figure 1 shows the square of biharmonic distance and the resistance distance in a cycle of 1000 vertices. Specifically we plot both the distances between vertices $j$ and $k$ where $k \le j$, as a function of $l = j - k$. The biharmonic distances are obtained using (60). The figure shows that the square of biharmonic distance and the resistance distance grow at different rates in a cycle, as a function of graph distance, while the vertices that have the largest graph distance have both the largest squared biharmonic distance and resistance distance.

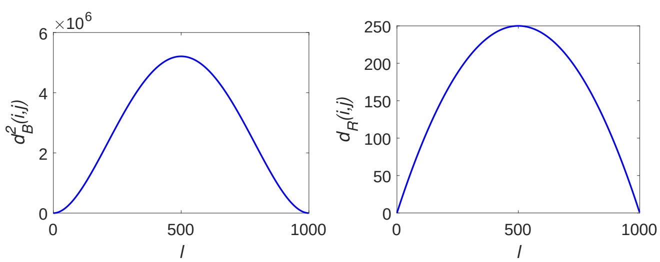
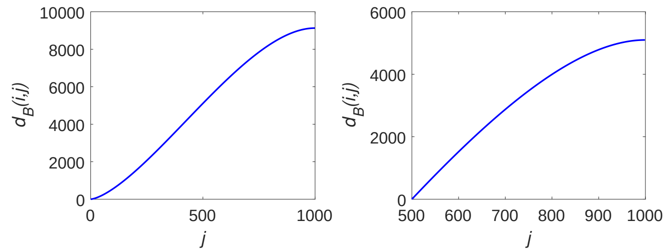



Fig. 1. The squared biharmonic distance $d_B^2(j,k)$ and resistance distance $d_R(j,k)$ between two vertices $j, k$ with $l = j-k$ in a cycle of 1000 vertices.

Figure 2 gives the biharmonic distances in a path graph. In particular, we show the biharmonic distances between vertices $j$ and $k$ where $k \le j$. We only show two cases, $k = 0$ and $k = 500$. The biharmonic distances are calculated using (67). For a given $k$, $d_B(j,k)$ grows slower near the ends of the path and faster around the middle of the path. In addition, since for even $N$, $d_B(0, N/2-1) = d_B(N/2, N-1)$; we observe that $d_B(0, N/2-1) + d_B(N/2-1, N/2) + d_B(N/2, N-1) > d_B(0, N-1)$ in this example. This is in contrast with resistance distance (and identically graph distance), where $d_R(0, N/2-1) + d_R(N/2-1, N/2) + d_R(N/2, N-1) = d_R(0, N-1)$.




(a) $k = 0$ (b) $k = 500$

Fig. 2. Biharmonic distance $d_B(j,k)$ between two vertices $j, k$ with $l = j-k$ in a path of 1000 vertices.

Figure 3 compares biharmonic centrality and information centrality in a path with 1000 vertices. Both curves are bell-like and the node in the middle has the largest centrality. The difference is that biharmonic distance distinguishes the center nodes better, as illustrated by the figure.

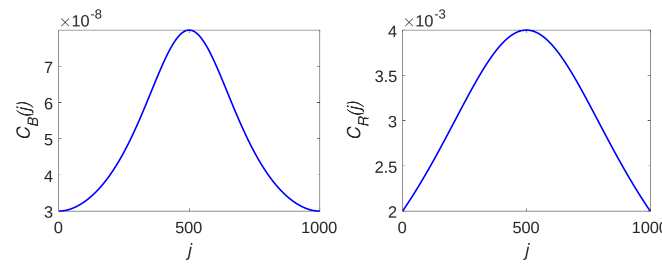



Fig. 3. Biharmonic centrality and information centrality in a path of 1000 vertices.

The next example is a starry-line graph, composed of two 20-vertex star graphs connected by a path of 5 vertices. Figure 4 shows the biharmonic centrality (above) and information centrality (below) in the graph. Vertices are colored according to their centrality in the network. Red vertices have the largest centrality and blue vertices have smallest centrality. The figure shows that the biharmonic centrality distinguishes the center of the line from other vertices on the line, while these vertices have comparable information centralities.

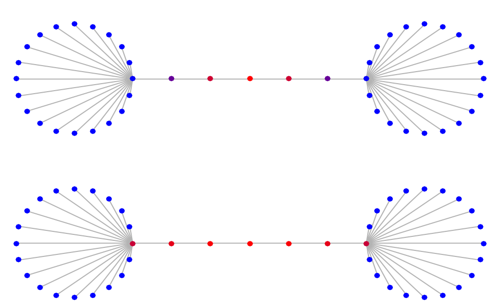
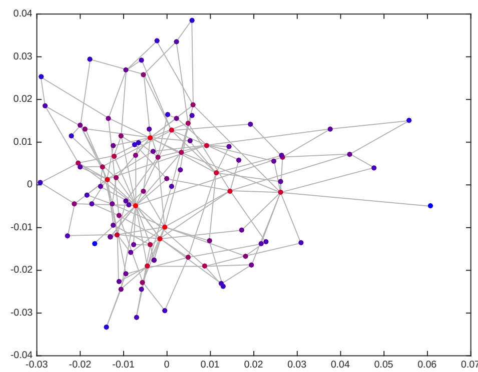
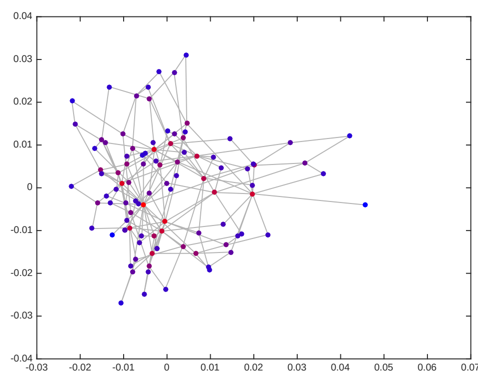


Fig. 4. Biharmonic centrality and information centrality in a starry-line graph.

Figure 5 shows the first two principle components of the biharmonic embedding as well as the biharmonic embedding of a Barabási-Albert network with $100$ nodes. We observe that the biharmonic embedding stretches the edges out a bit more than the resistance embedding. In fact, by reviewing their definitions, we observe that the normalized components in PCA for these two embeddings are the same; the differences are the variances of the components.




(a) Biharmonic embedding and biharmonic centrality.




(b) Resistance embedding and information centrality.

Fig. 5. Embeddings and centralities of a 100-vertex BA network

## VIII. Conclusion

We have investigated the performance of undirected networks with second-order consensus dynamics with stochastic disturbances. We have established the connection between second-order network performance measures and the biharmornic distances in the communication graph. We introduced the notions of a Kirchhoff index and vertex centrality based on biharmonic distance to further help us describe the behavior of second-order consensus dynamics, and we derived closed-form expressions for the performance measures for complete graphs, star graphs, cycles, and paths. Future work should include the study of additional properties of biharmonic distances, as well as analysis of the steady-state variance performance measures in more general networks, including random networks and real-world networks.

## Appendix A
## Trigonometric Identities

We use the notation $\phi_n = \frac{n\pi}{N}$. We next introduce the following identities.

$$\begin{aligned} G_N(1) &= \frac{1}{N}\sum_{n=1}^{N-1}\frac{1-\cos 2\phi_n}{(1-\cos 2\phi_n)^2} \\ &= \frac{1}{2N}\sum_{n=1}^{N-1}\frac{1}{\sin^2\phi_n} = \frac{N}{6} - \frac{1}{6N} \end{aligned} \tag{71}$$

$$\begin{aligned} G_N(2) &= \frac{1}{N}\sum_{n=1}^{N-1}\frac{1-\cos 4\phi_n}{(1-\cos 2\phi_n)^2} \\ &= \frac{1}{2N}\sum_{n=1}^{N-1}\frac{\sin^2 2\phi_n}{\sin^4\phi_n} = \frac{2}{N}\sum_{n=1}^{N-1}\frac{\cos^2\phi_n}{\sin^2\phi_n} \\ &= \frac{2N}{3} - 2 + \frac{4}{3N} \end{aligned} \tag{72}$$

$$\begin{aligned} F_N(1) &= \frac{1}{N}\sum_{n=1}^{N-1}\frac{1-\cos\phi_n}{(1-\cos\phi_n)^2} \\ &= \frac{1}{2N}\sum_{n=1}^{N-1}\frac{1}{\sin^2\phi_n/2} = \frac{N}{3} - \frac{1}{3N} \end{aligned} \tag{73}$$

$$\begin{aligned} F_N(2) &= \frac{1}{N}\sum_{n=1}^{N-1}\frac{1-\cos 2\phi_n}{(1-\cos\phi_n)^2} \\ &= \frac{1}{2N}\sum_{n=1}^{N-1}\frac{\sin^2\phi_n}{\sin^4\phi_n/2} = \frac{2}{N}\sum_{n=1}^{N-1}\frac{\cos^2\phi_n/2}{\sin^2\phi_n/2} \\ &= \frac{4N}{3} - 2 + \frac{2}{3N} \end{aligned} \tag{74}$$

## Appendix B
## Proof of Proposition VI.5

*Proof:* We note that $i$ denotes the imaginary unit in this proof.

Substituting (58) and (59) into Definition III.1, we obtain

$$d_B^2(j,k) = \frac{1}{N}\sum_{n=1}^{N-1}\frac{|e^{i2j\phi_n} - e^{i2k\phi_n}|^2}{4(1-\cos 2\phi_n)^2} = \frac{1}{2}G_N(j-k), \tag{75}$$

where

$$G_N(l) = \frac{1}{N}\sum_{n=1}^{N-1}\frac{1-\cos(2l\phi_n)}{(1-\cos 2\phi_n)^2}.$$

Without loss of generality, we assume $0 \leqslant l \leqslant 2N$.

In order to simplify $G_N(l)$, we give two equivalent expressions for the real part of the following sum

$$H_N(l) = \frac{1}{N}\sum_{n=1}^{N-1}\frac{1-e^{2il\phi_n}}{(1-e^{2i\phi_n})^2}. \tag{76}$$

The first expression is

$$\begin{aligned}
&\mathrm{Re}\big(H_N(l)\big)\\
&= \frac{1}{4N}\sum_{n=1}^{N-1}\bigg(\frac{1-\cos 2l\phi_n}{(1-\cos\phi_n)^2} - \frac{2-2\cos 2(l-1)\phi_n}{(1-\cos\phi_n)^2}\\
&\quad + \frac{1-\cos 2(l-2)\phi_n}{(1-\cos\phi_n)^2} - \frac{1-\cos 4\phi_n}{(1-\cos\phi_n)^2} + \frac{2-2\cos 2\phi_n}{(1-\cos\phi_n)^2}\bigg)\\
&= \frac{1}{4}\bigg(G_N(l) - 2G_N(l-1)\\
&\qquad + G_N(l-2) - G_N(2) + 2G_N(1)\bigg). \qquad (77)
\end{aligned}$$

We note that $G_N(0)=0$. Let $K_N(l) = G_N(l) - G_N(l-1)$. We rewrite (77) for the sake of conciseness in future derivation as

$$\begin{aligned}
&\mathrm{Re}(H_N(l))\\
&= \frac{1}{4}\Big(\ \big[\big(G_N(l)-G_N(l-1)\big) - \big(G_N(l-1)-G_N(l-2)\big)\big]\\
&\quad - \big[(G_N(2)-G_N(1)) - (G_N(1)-G_N(0))\big]\Big)\\
&= \frac{1}{4}\left[(K_N(l)-K_N(l-1)) - (K_N(2)-K_N(1))\right] \qquad (78)
\end{aligned}$$

Next, we use the summation formula $\sum_{j=0}^{n-1} x^j = \frac{1-x^n}{1-x}$ to expand (76)

$$\begin{aligned}
H_N(l) =& \frac{1}{N}\sum_{n=1}^{N-1}\frac{1-e^{2il\phi_n}}{1-e^{2i\phi_n}}\frac{1}{1-e^{2i\phi_n}}\\
=& \frac{1}{N}\sum_{n=1}^{N-1}\sum_{l'=0}^{l-1}\left(\frac{e^{2il'\phi_n}}{1-e^{2i\phi_n}} - \frac{1}{1-e^{2i\phi_n}} + \frac{1}{1-e^{2i\phi_n}}\right)\\
=& \frac{1}{N}\sum_{n=1}^{N-1}\left(\sum_{l'=2}^{l-1}\frac{e^{2il'\phi_n}-1}{1-e^{2i\phi_n}} - 1\right) + \frac{1}{N}\sum_{n=1}^{N-1}\sum_{l'=0}^{l-1}\frac{1}{1-e^{2i\phi_n}}\\
=& -\frac{1}{N}\sum_{n=1}^{N-1}\sum_{l'=2}^{l-1}\sum_{l''=1}^{l'-1} e^{2il''\phi_n} - \frac{1}{N}\sum_{n=1}^{N-1} 1\\
& -\frac{1}{N}\sum_{n=1}^{N-1}\sum_{l'=2}^{l-1} 1 + \frac{1}{N}\sum_{n=1}^{N-1}\sum_{l'=0}^{l-1}\frac{1}{1-e^{2i\phi_n}}\,. \qquad (79)
\end{aligned}$$

The triple summation in the last equality can be simplified by carrying out the summation over $n$ first,

$$\begin{aligned}
E_1 &\equiv -\frac{1}{N}\sum_{n=1}^{N-1}\sum_{l'=2}^{l-1}\sum_{l''=1}^{l'-1} e^{2il''\phi_n}\\
&= -\frac{1}{N}\sum_{l'=2}^{l-1}\sum_{l''=1}^{l'-1}\left(\frac{1-e^{i2\pi l''}}{1-e^{i\pi y''/N}} - 1\right)\\
&= \frac{1}{N}\sum_{l'=2}^{l-1}\sum_{l''=1}^{l'-1} 1,
\end{aligned}$$

where last equality is obtained by applying $e^{i2\pi l''} = 1$ for $l'' \in \mathbb{Z}$.

Using the fact that $\mathrm{Re}\left(1/(1-e^{i\theta})\right) = 1/2$, $0<\theta<2\pi$, the real part of the fourth term in (79) is

$$\mathrm{Re}(E_4) = \mathrm{Re}\left(\frac{1}{N}\sum_{n=1}^{N-1}\sum_{l'=0}^{l-1}\frac{1}{1-e^{i\phi_n}}\right) = \frac{(N-1)l}{2N}\,.$$

Therefore,

$$\mathrm{Re}(H_N(l)) = \frac{l^2}{2N} - \frac{l}{N} - \frac{l}{2} + 1\,. \qquad (80)$$

Let $X_N(l) = 4\mathrm{Re}(H_N(l))$. From the equivalence of (78) and (80), we derive

$$X_N(l) = \Big(K_N(l) - K_N(l-1)\Big) - \Big(K_N(2) - K_N(1)\Big).$$

This recursive equation can be solved to give

$$\begin{aligned}
K_N(l) &= G_N(l) - G_N(l-1)\\
&= Y_N(l) + (l-1)G_N(2) - (2l-3)G_N(1),
\end{aligned}$$

and

$$G_N(l) = Z_N(l) + \left(\frac{l^2}{2} - \frac{l}{2}\right)G_N(2) - (l^2-2l)G_N(1), \quad (81)$$

where

$$Y_N(l) = \sum_{j=2}^{l} X_N(j) \quad \text{and}$$

$$Z_N(l) = \sum_{j=2}^{l} Y_N(j) = \sum_{j=2}^{l}\sum_{k=2}^{j} X_N(k)\,.$$

Substituting (71), (72), and (80) into (81), we finally obtain the result for $G_N(l)$ as

$$G_N(l) = \frac{l^4}{6N} - \frac{l^3}{3} + \frac{l^2 N}{6} - \frac{l^2}{3N} + \frac{l}{3}\,.$$

Plugging this value into (75) generates the result in Proposition VI.5. ■

## APPENDIX C
## PROOF OF PROPOSITION VI.7

*Proof:* We note that $i$ denotes the imaginary unit in this proof.

By definition, the biharmonic distance between $j$ and $k$, $j \le k$ is

$$\begin{aligned}
d_B^2(j,k) =& \frac{1}{N}\sum_{n=1}^{N-1}\frac{[\cos(j+\frac{1}{2})\phi_n - \cos(k+\frac{1}{2})\phi_n]^2}{2(1-\cos\phi_n)^2}\\
=& \frac{1}{2}\Big(F_N(j+k+1) + F_N(j-k)\\
& - \frac{1}{2}F_N(2j+1) - \frac{1}{2}F_N(2k+1)\Big) \qquad (82)
\end{aligned}$$

where

$$F_N(l) = \frac{1}{N}\sum_{n=1}^{N-1}\frac{1-\cos l\phi_n}{(1-\cos\phi_n)^2}\,.$$

Next, we calculate the real part of the following sum in two different ways

$$T_N(l) = \frac{1}{N}\sum_{n=1}^{N-1}\frac{1-e^{il\phi_n}}{(1-e^{i\phi})^2}. \qquad (83)$$

First, let $E_N(l) = F_N(l) - F_N(l-1)$. We obtain

$$\begin{aligned}
\mathrm{Re}\big(T_N(l)\big) &= \frac{1}{4}\Big(F_N(l) - 2F_N(l-1) + F_N(l-2)\\
&\qquad - F_N(2) + 2F_N(1)\Big)\\
&= \frac{1}{4}\Big[\big(E_N(l) - E_N(l-1)\big) - \big(E_N(2) - E_N(1)\big)\Big] \qquad (84)
\end{aligned}$$

Second, we use the summation formula $\sum_{j=0}^{n-1} x^j = \frac{1-x^n}{1-x}$ and derive

$$\begin{aligned}T_N(l) =& \frac{1}{N}\sum_{n=1}^{N-1}\frac{1-e^{il\phi_n}}{1-e^{i\phi_n}}\frac{1}{1-e^{i\phi_n}}\\ =&\frac{1}{N}\sum_{n=1}^{N-1}\sum_{l'=0}^{l-1}\left(\frac{e^{il'\phi_n}}{1-e^{i\phi_n}}-\frac{1}{1-e^{i\phi_n}}+\frac{1}{1-e^{i\phi_n}}\right)\\ =&\frac{1}{N}\sum_{n=1}^{N-1}\left(\sum_{l'=2}^{l-1}\frac{e^{il'\phi_n}-1}{1-e^{i\phi_n}}-1\right)+\frac{1}{N}\sum_{n=1}^{N-1}\sum_{l'=0}^{l-1}\frac{1}{1-e^{i\phi_n}}\\ =&-\frac{1}{N}\sum_{n=1}^{N-1}\sum_{l'=2}^{l-1}\sum_{l''=1}^{l'-1}e^{il''\phi_n}-\frac{1}{N}\sum_{n=1}^{N-1}1\\ &-\frac{1}{N}\sum_{n=1}^{N-1}\sum_{l'=2}^{l-1}1+\frac{1}{N}\sum_{n=1}^{N-1}\sum_{l'=0}^{l-1}\frac{1}{1-e^{i\phi_n}}\,. \end{aligned} \tag{85}$$

Again, we change the order of summation over $n$, $l'$ and $l''$ to simplify the first term in (85),

$$E_1' \equiv -\frac{1}{N}\sum_{n=1}^{N-1}\sum_{l'=2}^{l-1}\sum_{l''=1}^{l'-1}e^{il''\phi_n} = -\frac{1}{N}\sum_{l'=2}^{l-1}\sum_{l''=1}^{l'-1}\left[\frac{1-(-1)^{l''}}{1-e^{i\pi y''/N}}-1\right].$$

The real part of $E_1'$ and the fourth term in (85), denoted $E_4'$, are

$$\mathrm{Re}(E_1') = \frac{1}{8N}\left[2l^2-8l+7+(-1)^l\right],$$

$$\mathrm{Re}(E_4') = \mathrm{Re}\left(\frac{1}{N}\sum_{n=1}^{N-1}\sum_{l'=0}^{l-1}\frac{1}{1-e^{i\phi_n}}\right) = \frac{(N-1)l}{2N}.$$

Hence,

$$\mathrm{Re}(T_N(l)) = \frac{-4N(l-2)+2l^2-4l+(-1)^l-1}{8N}. \tag{86}$$

Equating (84) and (86) leads to

$$\begin{aligned}F_N(l) =&\sum_{i=2}^{l}\sum_{j=2}^{i}4\mathrm{Re}(T_N(l))+\left(\frac{l^2}{2}-\frac{l}{2}\right)F_N(2)\\ &-(l^2-2l)F_N(1)\,.\end{aligned} \tag{87}$$

Next, we evaluate $F_N(1)$ and $F_N(2)$

$$F_N(1) = \frac{1}{N}\sum_{n=1}^{N-1}\frac{1-\cos\phi_n}{(1-\cos\phi_n)^2} = \frac{1}{2N}\sum_{n=1}^{N-1}\frac{1}{\sin^2\phi_n/2}\,, \tag{88}$$

$$F_N(2) = \frac{1}{N}\sum_{n=1}^{N-1}\frac{1-\cos 2\phi_n}{(1-\cos\phi_n)^2} = \frac{2}{N}\sum_{n=1}^{N-1}\frac{\cos^2\phi_n/2}{\sin^2\phi_n/2}\,. \tag{89}$$

For $F_N(1)$, we start by expanding the expression $\sum_{n=1}^{2N-1} 1/(\sin^2\frac{n\pi}{2N})$. Since $\sum_{n=1}^{N-1} 1/(\sin^2\frac{n\pi}{N}) = \frac{N^2}{3}-\frac{1}{3}$, we derive

$$\begin{aligned}&\sum_{n=1}^{2N-1}\frac{1}{\sin^2\frac{n\pi}{2N}} = \frac{4N^2}{3}-\frac{1}{3}\\ &=\frac{1}{\sin^2\frac{\pi}{2N}}+\frac{1}{\sin^2\frac{2\pi}{2N}}+\cdots+\frac{1}{\sin^2\frac{(N-1)\pi}{2N}}+\frac{1}{\sin^2\frac{N\pi}{2N}}\\ &\quad+\frac{1}{\sin^2\frac{(N+1)\pi}{2N}}+\cdots+\frac{1}{\sin^2\frac{(2N-2)\pi}{2N}}+\frac{1}{\sin^2\frac{(2N-1)\pi}{2N}}\,.\end{aligned}$$

For $\sin x = \sin(\pi - x)$, $0 \le x \le 2\pi$ and $1/(\sin^2\frac{N\pi}{2N}) = 1$,

$$\sum_{n=1}^{2N-1}\frac{1}{\sin^2\frac{n\pi}{2N}} = 2\sum_{n=1}^{N-1}\frac{1}{\sin^2\frac{n\pi}{2N}}+1\,.$$

Thus, we obtain identity (73); that is,

$$F_N(1) = \frac{1}{2N}\sum_{n=1}^{N-1}\frac{1}{\sin^2\frac{n\pi}{2N}} = \frac{1}{4N}\left(\frac{4N^2}{3}-\frac{1}{3}-1\right) = \frac{N}{3}-\frac{1}{3N}\,.$$

Similarly, we expand (5) by noting that $\cos^2\frac{N\pi}{2N} = 0$,

$$\begin{aligned}\sum_{n=1}^{2N-1}\frac{\cos^2\frac{n\pi}{2N}}{\sin^2\frac{n\pi}{2N}} &= \sum_{n=1}^{N-1}\frac{\cos^2\frac{n\pi}{2N}}{\sin^2\frac{n\pi}{2N}}+\frac{\cos^2\frac{N\pi}{2N}}{\sin^2\frac{N\pi}{2N}}+\sum_{n=N+1}^{2N-1}\frac{\cos^2\frac{n\pi}{2N}}{\sin^2\frac{n\pi}{2N}}\\ &=2\sum_{n=1}^{N-1}\frac{\cos^2\frac{n\pi}{2N}}{\sin^2\frac{n\pi}{2N}}\,.\end{aligned}$$

For $\sum_{n=1}^{N-1}\frac{\cos^2\phi_n}{\sin^2\phi_n} = \frac{N^2}{3}-N+\frac{2}{3}$, we have $\sum_{n=1}^{2N-1}\frac{\cos^2\frac{n\pi}{2N}}{\sin^2\frac{n\pi}{2N}} = \frac{4N^2}{3}-2N+\frac{2}{3}$. Therefore, we obtain identity (74); that is,

$$F_N(2) = \frac{2}{N}\sum_{n=1}^{N-1}\frac{\cos^2\phi_n/2}{\sin^2\phi_n/2} = \frac{4N}{3}-2+\frac{2}{3N}\,.$$

By substituting (73), (74), and (86) into (87), we derive the following closed formula for $F_N(l)$

$$F_N(l) = \frac{l^4}{12N}-\frac{l^3}{3}+\frac{l^2N}{3}-\frac{l^2}{6N}+\frac{(-1)^l}{8N}+\frac{l}{3}-\frac{1}{8N}.$$

By plugging $F_N(l)$ into (82), we obtain the result in Proposition VI.7. ∎

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