A Convex Cycle-based Degradation Model for Battery Energy Storage Planning and Operation

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Abstract

A vital aspect in energy storage planning and operation is to accurately model its operational cost, which mainly comes from the battery cell degradation. Battery degradation can be viewed as a complex material fatigue process that based on stress cycles. Rainflow algorithm is a popular way for cycle identification in material fatigue process, and has been extensively used in battery degradation assessment. However, the rainflow algorithm does not have a closed form, which makes the major difficulty to include it in optimization. In this paper, we prove the rainflow cycle-based cost is convex. Convexity enables the proposed degradation model to be incorporated in different battery optimization problems and guarantees the solution quality. We provide a subgradient algorithm to solve the problem. A case study on PJM regulation market demonstrates the effectiveness of the proposed degradation model in maximizing the battery operating profits as well as extending its lifetime.

I. Introduction

Battery energy storage (BES) is becoming an essential resource in energy systems with high renewable penetrations. Applications of BES include peak shaving [1], frequency regulation [2], demand response [3], [4], renewable integration [5], grid transmission and distribution support [6], and many others. Although each application has different objectives and constraints, a common theme among them is that accurately accounting for the *operational* cost of batteries is of critical importance.

Battery operating cost is mainly caused by the degradation effect of repeated charging and discharging [7]. Battery cells typically reach end-of-life (EoL) if their capacity degrades beyond a certain minimum threshold (for example, 80% of original capacity) [8], after which cells can no longer perform as expected. The resulting cell replacement cost represents the predominate operating cost of batteries, because of high manufacturing price for most electrochemical battery cells [9]. The goal of this paper is to present an *electrochemically accurate* and *tractable* battery degradation model that can be used in multiple applications.

To account for battery degradation, two main classes of models have been considered. The first kind of degradation model is based on battery charging/discharging *power* [3], [2], [10]. For example, [2] used a linear degradation cost while considering peak shaving and regulation services, and [3] adopts a general convex cost for battery demand response. This power-based model decouples the degradation cost in time, since the costs at each of the time steps are independent. It is easy to be incorporated in battery optimization. However, the major concern of the power-based

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model is about its accuracy in capturing the actual degradation cost. For example, a Lithium Nickel Manganese Cobalt Oxide (NMC) battery has *ten* times more degradation when operated at near 100% cycle depth of discharge (DoD) compared to operated at 10% DoD for the *same* amount of charged power [11]. Power-based degradation model fails to capture such cumulative effect of battery cell aging and may severely deviate the operation from optimal.

Therefore, we consider a more accurate model to account for battery degradation, which is called *cycle-based* model. This model is based on fundamental battery aging mechanism. It views the battery degradation process as a complex material fatigue process [12] that based on stress cycles. Battery aging at each time is not independent, but closely related to accumulation of previous charging and discharging.

To accurately identify cycles in an irregular battery state of charge (SoC) profile, we adopt the rainflow cycle counting method. Rainflow algorithm has been extensively used for cycle identification in material fatigue analysis [13], [14], [15] as well as battery degradation model [16], [17], [18]. Fig. 1 gives an example of rainflow cycle identification results of a battery SoC profile. The total battery degradation cost is modeled as the sum of degradation from all cycles.

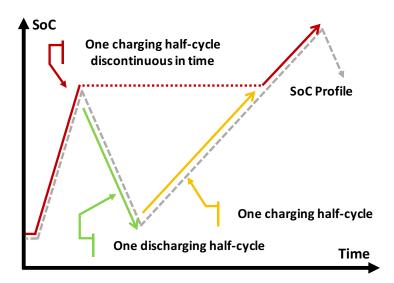


Fig. 1: Rainflow cycle counting example. There are three half cycles, one charging half-cycle (the red one) is discontinuous in time. The green discharging half-cycle and yellow charging half-cycle are of the same cycle depth.

Cycle-based degradation model can accurately model the fundamental battery aging. However, it is considerably more cumbersome and computationally difficult than the power-based cost model. The rainflow algorithm, despite its wide array of applications, is a procedure that does not have a closed form. This makes it difficult to be incorporated in optimization problems. Most previous works (see [12] and references within) apply the rainflow algorithm for a posterior battery degradation assessment with fixed operating strategies.

Instead of posterior degradation assessment, we want to incorporate rainflow algorithm in battery optimization. Several efforts have been made by simplifying the algorithm procedure, either simplifying the rainflow counting procedure [19], [20], or using more trivial cycle definition [21], [22]. These approaches improve the problem solvability, however the simplifications introduce additional errors and sacrifices the optimality of solutions.

The main contribution of our paper is two-folded:

- 1) We prove the rainflow cycle-based degradation model is *convex*. Convexity enables the proposed cost model to be used in various battery optimization problems and guarantees the solution quality.
- 2) We provide a subgradient algorithm to solve the battery optimization problems.

The proposed degradation model and subgradient solver algorithm have a broad application scope for battery operation optimization. We implement it for a case study on PJM regulation market, and verify this model accurately captures the battery cell aging, significantly improve the operational revenue (up to 30%) and almost doubled the expected BES lifetime compared with previous degradation models.

The rest of the paper is organized as follows. Section II describes the proposed rainflow cycle-based degradation model. Section III sketches the convexity proof. Section IV gives a subgradient algorithm. We provide a case study in Section V using real data from PJM regulation market, and demonstrate the effectiveness of the proposed degradation model in maximizing the BES operating profits as well as extending battery lifetime. Finally, Section VII concludes the paper and outlines directions for future work.

II. MODEL

A. Battery degradation model

We consider a battery operating in power markets and providing certain profitable service. We use c(t) to denote battery charging and d(t) for discharging at time t, where t = 1, 2, ..., T. The battery SoC is $s \in \mathbb{R}^T$, which is an accumulation of charged/discharged power. In Section IV, we will describe in detail how we model the battery operation and constraints. Here, we focus on degradation model.

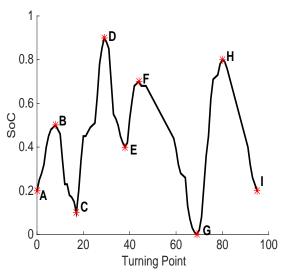
Suppose the operation revenue from battery application is $R(\cdot)$, which is a function of battery power output. The battery degradation is captured by rainflow cycle-based degradation model. Therefore, a utility function $U(\cdot)$ that quantifies the profits BES owner obtains over $t \in [1, T]$ is,

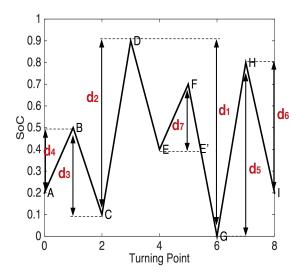
$$U = R(\mathbf{c}, \mathbf{d}) - \lambda^{r} \sum_{i=1}^{N} \Phi(d_i), \qquad (1)$$

where $[d_1, d_2, ..., d_N]$ are cycle depths. We use rainflow algorithm for cycle identification, which is defined as below,

$$[d_1, d_2, ..., d_N] = \mathbf{Rainflow}(s(1), s(2), ..., s(T)),$$
 (2)

The second term in Eq. (1) together with Eq. (2) constitute the proposed rainflow cycle-based degradation model, where $[d_1, d_2, ..., d_N]$ are cycle depths identified by rainflow algorithm. $\Phi(\cdot)$ is the DoD stress function, which defines the degradation of one cycle under reference condition. $\sum_{i=1}^{N} \Phi(d_i)$ calculates the total degradation by summing over all cycles, and $\lambda^{\rm r}$ is the battery cell replacement price. It's worth to point out that DoD stress function can be fitted with test data of different types of batteries, which offers the proposed degradation model strong generalization ability.





- (a) Battery SoC time history. Red stars mark the local maximum and local minimum points
- (b) Rainflow cycle counting results, based on extracted local maximum and minimum points

Fig. 2: Rainflow cycle counting algorithm procedures

Path	A-B	B-C	C-D	D-G	E-F-E'	G-H	H-I
SoC range	0.3	0.4	0.8	0.9	0.3	0.8	0.6
Cycle	half	half	half	half	full	half	half

TABLE I: Rainflow cycle counting results of the SoC profile in Fig. 2a. The "SoC range" is calculated as the absolute difference between the cycle starting and ending SoC.

B. Rainflow cycle counting algorithm

The process of reducing a strain/time history into a number of smaller cycles is termed as cycle counting. Rainflow algorithm is the most popular cycle counting algorithm used for analyzing fatigue data [13], [14], [15]. When used for battery life assessment, it takes a time series of battery's state of charge (SoC) as input, and identifies the depth of all cycles contained in this series.

Fig. 2 gives an example on the rainflow algorithm implementation for cycle counting of a battery SoC profile, and the cycle analysis results are reported in Table I. The standard rainflow algorithm procedures based on paper [13] are given in Algorithm 1.

Following the procedures in Algorithm 1, we count cycles in Fig. 2a. First, all local extremes points of the SoC profile are extracted and arranged as Fig. 2b. Then the deepest half cycle D-G is identified. Next, half cycles before D are identified, which are C-D(d_2), B-C (d_3), A-B (d_4); also half cycles after G are counted, which are G-H (d_5) and H-I (d_6). Finally, there is one remaining full cycle E-F-E' with depth d_7 , which could also be viewed as one charging half cycle plus one discharging half cycle with equal depth.

Algorithm 1: Rainflow Cycle Counting Algorithm

Input: Battery SoC profiles, $\mathbf{s} \in \mathbb{R}^T$

Output: Cycle counting results: $d_1, d_2, ... d_N$

- 1 Reduce the time history to a sequence of turning points local maximum and local minimum.
- 2 Find the global maximum and global minimum, counted as a half cycle.
- 3 If the global maximum happens first:
 - a. Half cycles are counted between the global maximum and the most negative minimum occurs before it, the most positive maximum occurring prior to this minimum, and so on to the beginning of the history.
 - b. Half cycles are also counted after the global minimum in the history and terminates at the most positive maximum occurring subsequently, the most negative minimum occurring after this maximum, and so on to the end of the history.
 - c. Remains are full cycles.
- 4 If the global minimum happens first:

Adjust step 3a, 3b, 3c accordingly to pair up sequential most positive maximum and most negative minimum that happen prior to the global minimum, or after the global maximum. Remains are small full cycles.

C. DoD stress function

DoD stress function $\Phi(\cdot)$ is a critical part of the degradation model since it captures the cell aging caused by one cycle under reference conditions [12]. Different types of batteries may have different stress function forms. Some commonly used DoD stress functions are given below.

(1) Linear DoD stress model [2],

$$\Phi(d) = k_1 d$$
,

This linear DoD stress function is equivalent to the linear power-based degradation model. It is simple and suitable under some conditions. However, lab tests show that cycle DoD has a highly nonlinear impact on degradation under most conditions. The following two types of nonlinear DoD stress models are commonly used in literature.

(2) Exponential DoD stress model [23],

$$\Phi(d) = k_2 d e^{k_3 d},$$

(3) Polynomial DoD stress model [21], where

$$\Phi(d) = k_4 d^{k_5} \,, \tag{3}$$

where all k's are model coefficients, which could be estimated by fitting battery cycling aging test data. In paper, the DoD stress function is assumed to be a convex function, where a higher cycle DoD leads to a more severe damage.

III. CONVEXITY

The major difficulty of incorporating the rainflow cycle-based degradation model to optimization is that the rainflow algorithm does not have a closed form. It could be solved by some computer numerical methods, however the computational complexity is high and there is no guarantee for the solution quality. We prove that the rainflow

cycle-based degradation cost f(s), given a convex DoD stress function, is *convex* in terms of s. It helps to overcome the difficulty of incorporating the rainflow algorithm to battery optimization. We provide a sketch of the convexity proof below. A detailed version of the proof is given in Section VI.

Theorem 1. Rainflow cycle cost is convex

The rainflow cycle-based battery degradation model,

$$f(\mathbf{s}) = \sum_{i=1}^{N} \Phi(d_i), \quad \text{where } [d_1, d_2, ..., d_N] = \mathbf{Rainflow}(\mathbf{s}),$$

is convex. That is, $\forall \mathbf{s}_1, \mathbf{s}_2 \in \mathbb{R}^T$,

$$f(\lambda \mathbf{s_1} + (1 - \lambda)\mathbf{s_2}) \le \lambda f(\mathbf{s_1}) + (1 - \lambda)f(\mathbf{s_2}), \forall \lambda \in [0, 1]$$

$$\tag{4}$$

This above theorem is intuitively pleasing. Consider two SoC series s_1 and s_2 , if they change in different directions, the two signals can cancel each other out partially so that the cost of left side is less than the right side. When s_1 changes in the same direction as s_2 for all time steps, the equality holds.

We prove Theorem 1 by induction. It contains three steps, 1) unit step function decompostion of battery SoC, 2) the initial condition proof and 3) the induction relation proof from K to K+1 step. Here we sketch a proof for the conveixty.

A. Unit step function decompostion

First, we introduce the step function decompostion of SoC signal. We notice that, any SoC series s, could be written out as a finite sum of step functions, where

$$\mathbf{s} = \sum_{i=1}^{T} P_i u(t-i), \qquad (5)$$

where u(t-i) is a unit step function with a jump at time i, and P_i is the jump amplitude.

$$u(t-i) = \begin{cases} 1 & t \ge i \\ 0 & \text{otherwise} \end{cases} \quad \forall t = 1, 2, ..., T,$$
 (6)

For notation convenience, we use U_i to denote u(t-i). Fig. 3 gives an example of step function decomposition results of s_1 .

B. Initial condition proof

Since all SoC profiles can be written as the sum of step functions. We first need to prove that f(s) is convex up to one step function as the base case. As shown in Fig. 4, we want to prove that,

$$f(\lambda \mathbf{s_1} + (1 - \lambda)P_iU_i) \le \lambda f(\mathbf{s_1}) + (1 - \lambda)f(P_iU_i), \lambda \in [0, 1],$$

where $\mathbf{s_1} \in \mathbb{R}^T$, and P_iU_i is a step function with a jump happens at time i with amplitude P_i .

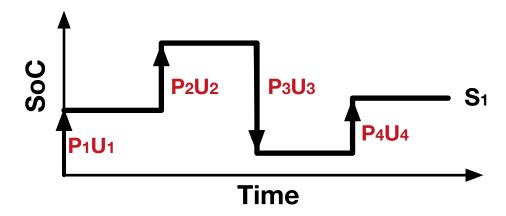


Fig. 3: Step function decomposition of SoC. s_1 is decomposed to four step functions P_1U_1 , P_2U_2 , P_3U_3 and P_4U_4 .

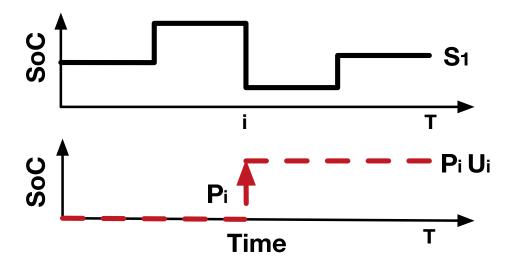


Fig. 4: Single step convexity. The upper plot is a SoC profile $\mathbf{s_1} \in \mathbb{R}^T$, the bottom plot is a step function with a amplitude P_i jump at time i.

C. Induction relation

Having proved the single step convexity, we know that Theorem 1 is true if both $s_1, s_2 \in \mathbb{R}^1$. Now, let's assume that, f(s) is convex up to the sum of K step changes (arranged by time index),

$$f(\lambda \mathbf{s_1} + (1 - \lambda)\mathbf{s_2}) \le \lambda f(\mathbf{s_1}) + (1 - \lambda)f(\mathbf{s_2}), \lambda \in [0, 1]$$

where $s_1, s_1 \in \mathbb{R}^K$. Considering proof by induction, we need to show f(s) is convex up to the sum of K+1 step changes (see Fig. 5),

$$f(\lambda \mathbf{s_1} + (1 - \lambda)\mathbf{s_2}) \le \lambda f(\mathbf{s_1}) + (1 - \lambda)f(\mathbf{s_2}), \lambda \in [0, 1]$$

where $\mathbf{s_1}, \mathbf{s_2} \in \mathbb{R}^{K+1}$.

The above three-step sketch builds the overall framework for convexity proof. For a more detailed proof step by

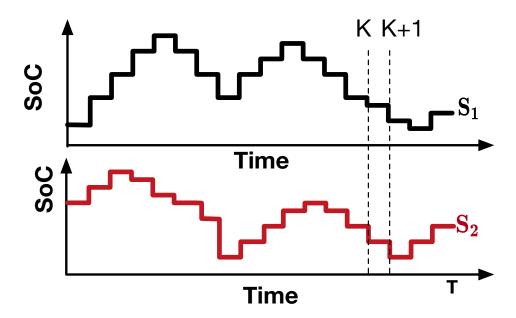


Fig. 5: Induction step from K to K+1. Both \mathbf{S}_1 and \mathbf{S}_2 are decomposed as the sum of T step functions.

step, interested readers could refer to Section VI.

IV. SUBGRADIENT ALGORITHM

In this section, we formulate a general optimization problem of battery operation in power market. We use the rainflow cycle-based model for battery degradation calculation. The optimization problem is convex, but the objective function is not differentiable at some points (cycle junction points). Therefore, we provide a subgradient algorithm. With proper step size, the subgradient algorithm is guaranteed to converge to the optimal solution with a user-defined precision level.

A. Battery operation in markets

Assume we have a battery operating in power markets and providing some profitable service (eg. regulation service, demand response), the battery operation is defined over a finite discrete time intervals, where t = 1, 2, ..., T with time resolution t_s . Following the same formulation in Section II-A, we aim to maximize the utility of battery

operation over a period of time $t \in [1, T]$.

$$\max_{\mathbf{c}, \mathbf{d}} U(\cdot) = R(\mathbf{c}, \mathbf{d}) - \lambda^{\mathrm{r}} \left[\sum_{i \in N^{\mathrm{ch}}} \Phi(d_i^{\mathrm{ch}}) + \sum_{j \in N^{\mathrm{dc}}} \Phi(d_j^{\mathrm{dc}}) \right], \tag{7a}$$

s.t.
$$[d^{\operatorname{ch}} d^{\operatorname{dc}}]^T = \operatorname{\mathbf{Rainflow}}(s(1), \dots, s(T)),$$
 (7b)

$$s(1) = s^0, (7c)$$

$$s(t+1) = s(t) + [c(t)\eta_{c} - d(t)/\eta_{d}]t_{s},$$
 (7d)

$$s^{\min} < s(t) < s^{\max}, \tag{7e}$$

$$0 \le c(t) \le P^{\max}, \tag{7f}$$

$$0 \le d(t) \le P^{\max}. \tag{7g}$$

where in Eq. (7a), the first term represents the operation revenue and the second term captures the battery degradation cost by rainflow cycle model. Eq. (7c) and (7d) describes the evolution of battery SoC, where s^0 is the given initial state. Battery SoC is limited within $[s^{\min}, s^{\max}]$, in general, $s^{\min} = 0$ (empty) and $s^{\max} = 1$ (full charge). We include the battery charging/discharging efficiency η_c and η_d in the optimization model. The battery output power is subject to power rating in (7f) - (7g).

Note, different from previous notations, we seperate cycles into charging half cycles (CHC) d^{ch} and discharging half cycles (DHC) d^{dc} in Eq. (7b) for optimization convenience¹. Assume we have N^{ch} number of CHCs, each one is indexed by i with cycle depth d_i^{ch} ; and N^{dc} number of DHCs, each is indexed by j with depth d_i^{dc} .

Battery SoC s is a linear function of battery power output c and d according to Eq. (7d). We have proved that the rainflow cycle cost is convex in terms of s. Therefore, the rainflow cycle cost is also convex in terms of c and d. If the revenue function $R(\cdot)$ is concave, the overall battery operation optimization problem formulated in (7a) - (7g) is a convex optimization problem.

B. Subgradient algorithm

To solve the convex battery operation problem in Eq. (7a) - (7g), we provide a subgradient algorithm. Using a log-barrier function [24], we re-write the constrained optimization problem to an unconstrained optimization

¹This is because charging and discharging response have different subgradient forms, which will be discussed in Section IV-B

problem.

$$\begin{split} & \min_{\mathbf{c}, \mathbf{d}} U(\cdot) := -R(\mathbf{c}, \mathbf{d}) + \lambda^{\mathrm{r}} \Big[\sum_{i \in N^{\mathrm{ch}}} \Phi(d_i^{\mathrm{ch}}) + \sum_{j \in N^{\mathrm{dc}}} \Phi(d_j^{\mathrm{dc}}) \Big] \\ & - \frac{1}{\lambda} \cdot \Big\{ \sum_{t=1}^{T} \log[s^{max} - s(t)] + \sum_{t=1}^{T} \log[s(t) - s^{min}] \\ & + \sum_{t=1}^{T} \log[P^{max} - c(t)] + \sum_{t=1}^{T} \log[c(t)] \\ & + \sum_{t=1}^{T} \log[P^{max} - d(t)] + \sum_{t=1}^{T} \log[d(t)] \Big\} \end{split} \tag{8}$$

when $\lambda \to +\infty$, the uncontrained problems (8) equals to the original constrained problem Eq. (7a) - (7g). Note here we change the original maximization problem to an equivalent minimization problem for standard form expression.

The major challenge of solving Eq. (8) lies in the second term. We need to find the mathematical relationship between charging cycle depth d_i^{ch} and charging power c(t), as well as the relationship between discharging cycle depth d_j^{dc} and discharging power d(t). Recall that the rainflow cycle counting algorithm introcuded in Section II-B, each time index is mapped to at least one charging half cycle (CHC) or at least one discharging half cycle (DHC). Some time steps sit on the *junction* of two cycles. For example, in Fig. 2b, E' lies on the junction of discharging cycle D-G and discharging cycle F-E'. No time step belongs to more than two cycles.

Let the time index mapped to the CHC i belong to the set T_i^{ch} , and let the time index mapped to the DHC j belong to the set T_j^{dc} , it follows that

$$T_1^{\operatorname{ch}} \cup \ldots \cup T_{N^{\operatorname{ch}}}^{\operatorname{ch}} \cup T_1^{\operatorname{dc}} \cup \ldots \cup T_{N^{\operatorname{ch}}}^{\operatorname{dc}} = T,$$

$$\tag{9}$$

$$T_i^{\text{ch}} \cap T_j^{\text{dc}} = \emptyset, \, \forall i, j \,.$$
 (10)

Eq. (10) shows there is no overlapping between a charging and a discharging interval. That is, each half-cycle is either charging or discharging. The cycle depth therefore equals to the sum of battery charging or battery discharging within the cycle time frame,

$$d_i^{\text{ch}} = \sum_{t \in T^{\text{ch}}} \frac{c(t)t_s\eta_c}{E} \,, \tag{11}$$

$$d_j^{\text{dc}} = \sum_{t \in T_j^{\text{dc}}} \frac{d(t)t_s/\eta_d}{E}, \qquad (12)$$

The rainflow cycle cost $\sum_{i \in N^{\mathrm{ch}}} \Phi(d_i^{\mathrm{ch}})$ is not continuously differentiable. At cycle junction points, it has more than one subgradient. Therefore, we use $\partial \sum_{i \in N^{\mathrm{ch}}} \Phi(d_i^{\mathrm{ch}})|_{c(t)}$ to denote a subgradient at c(t),

$$\partial \sum_{i \in N^{\text{ch}}} \Phi(d_i^{\text{ch}})|_{c(t)} = \Phi'(d_i^{\text{ch}}) \frac{t_s \cdot \eta_c}{E}, t \in T_i^{\text{ch}},$$

$$\tag{13}$$

where d_i^{ch} is the depth of cycle that c(t) belongs to. Note, at junction points, c(t) belongs to two cycles so that the subgradient is not unique. We can set d_i^{ch} to any value between d_{i1}^{ch} and d_{i2}^{ch} , where d_{i1}^{ch} and d_{i2}^{ch} are the depths of

two junction cycles c(t) belongs to.

Similarly for discharging cycle, a subgradient at d(t) is

$$\partial \sum_{j \in N^{\mathrm{dc}}} \Phi(d_j^{\mathrm{dc}})|_{d(t)} = \Phi'(d_j^{\mathrm{dc}}) \frac{t_s}{\eta_d \cdot E}, t \in T_j^{\mathrm{dc}}$$

$$\tag{14}$$

where d_j^{dc} is the depth of the cycle that d(t) belongs to. At the junction point, d_j^{dc} could be set to any value between d_{j1}^{dc} and d_{j2}^{dc} , which are the two junction cycles d(t) belongs to.

Therefore, we write the subgradient of $U(\cdot)$ with respect to c(t) and d(t) as $\partial U_{c(t)}$ and $\partial U_{d(t)}$, where

$$\partial U_{c(t)} = -\frac{\partial R}{\partial c(t)} + \lambda^{T} \Phi'(d_{i}^{ch}) \frac{t_{s} \eta_{c}}{E}$$

$$-\frac{1}{\lambda} \left\{ \sum_{\tau=t}^{T} \frac{1}{s(\tau) - s^{max}} (\frac{\eta_{c} t_{s}}{E}) + \sum_{\tau=t}^{T} \frac{1}{s(\tau) - s^{min}} (\frac{\eta_{c} t_{s}}{E}) + \frac{1}{c(t) - P^{max}} + \frac{1}{c(t)} \right\}, t \in T_{i}^{ch}$$

$$(15)$$

$$\partial U_{d(t)} = -\frac{\partial R}{\partial d(t)} + \lambda^T \Phi'(d_j^{dc}) \frac{t_s}{\eta_d \cdot E}$$

$$-\frac{1}{\lambda} \left\{ -\sum_{\tau=t}^T \frac{1}{s(\tau) - s^{max}} (\frac{t_s}{\eta_d E}) - \sum_{\tau=t}^T \frac{1}{s(\tau) - s^{min}} (\frac{t_s}{\eta_d E}) + \frac{1}{d(t) - P^{max}} + \frac{1}{d(t)} \right\}, t \in T_j^{dc}$$

$$(16)$$

The update rules for c(t) and d(t) at the k-th iteration are,

$$c_{(k)}(t) = c_{(k-1)}(t) - \alpha_k \cdot \partial U_{c(t)},$$

$$d_{(k)}(t) = d_{(k-1)}(t) - \alpha_k \cdot \partial U_{d(t)},$$

where α_k is the step length at k-th iteration. Since the subgradient method is not a decent method [25], it is common to keep track of the best point found so far, i.e., the one with smallest function value. At each step, we set

$$U_{(k)}^{best} = \min \left\{ U_{(k-1)}^{best}, U(\mathbf{c}_{(k)}, \mathbf{d}_{(k)}) \right\},$$

Since the $U(\cdot)$ is convex, for constant step length (α_k is constant), the subgradient algorithm is guaranteed to converge to the global optima within certain gap bound.

$$U_{(k)}^{best} - U^* o \frac{G^2 \alpha}{2}$$
,

where U^* as the global minimum of Eq. (8), α is the contant step size, and G is the upper bound of the 2-norm

of the following sub-gradient vectors,

$$\partial U_{\mathbf{c}} = [\partial U_{c(1)}, \partial U_{c(2)}, ..., \partial U_{c(T)}],$$
$$\partial U_{\mathbf{d}} = [\partial Y_{d(1)}, \partial U_{d(2)}, ..., \partial U_{d(T)}],$$

If we set the step size α to be small enough, we could guarantee the precision of the subgradient algorithm.

V. CASE STUDY

The proposed battery degradation model has a wide application scope in battery planning and operation problems. In this section, we provide a case study on PJM regulation market to demonstrate the effectiveness of the proposed degradation model in optimizing BES operation utilities and extending BES lifetime.

A. Frequency regulation market

We consider the optimal operation of a BES in frequency regulation market. In particular, we adopt a simplified version of the PJM fast regulation market [26]. For market rules, the grid operator pays a per-MW option fee (λ_c) to a battery storage with stand-by power capacity C that it can provide for the grid. While during the regulation procurement period, a regulation instruction signal r(t) is sent to the battery. Battery should respond to follow the instruction signal and is subjected to a per-MWh regulation mismatch penalty λ^p for the absolute error between the instructed dispatch Cr(t) and the resource's actual response b(t). Fig. 6 gives an example of the PJM fast regulation signal for 2 hours.

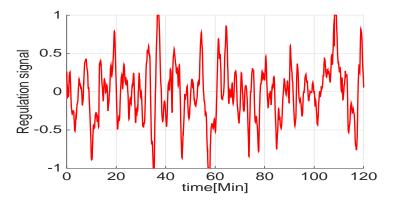


Fig. 6: PJM fast regulation signal for 2 hours

We decompose the regulation signal r(t) into charging and discharging parts, $r(t) = r_d(t) - r_c(t)$, where $r_c(t)$ is the charging instruction signal and $r_d(t)$ is the discharging signal. Thus, the regulation service revenue is,

$$R(\mathbf{c}, \mathbf{d}) = \lambda_c C \cdot T - \sum_{t \in T} \lambda^{P} t_s |(r_c(t) - c(t)) + (r_d(t) - d(t))|,$$
(17)

where the first part represents the regulation capacity payment and the second part is the mismatch penalty. In this work, we assume the regulation capacity bidding C is fixed, and focus on the optimization of battery response. The regulation revenue $R(\mathbf{c}, \mathbf{d})$ is concave with respect to \mathbf{c} , \mathbf{d} . Therefore, plugging (17) to the battery market operation

model in (7a) - (7g), we obtain a convex optimization problem for optimizing battery usage in frequency regulation market.

B. Benchmark

To demonstrate the efficiency of the proposed battery degradation model in maximizing the BES operation utilities as well as extending BES lifetime, we describe two benchmark battery degradation models: assuming zero operating cost [27] and a linear power-based degradation cost [2].

Assume the battery optimization horizon is 2 hours and the time granularity t_s is 4s, so that T=1800. We adopted the regulation market price and linear battery cost model coefficients from paper [28], where the regulation capacity payment is 50\$/MWh and the mismatch penalty is 150\$/MWh. We fixed the regulation capacity bidding as 1MW, with the same value as the battery power capacity. The battery energy capacity E is 15 minute max power output, and cell replacement price is 0.6\$/Wh. Assume battery DoD stress model has a polynomial form as Eq. (3), where $k_4 = 4.5e - 4$ and $k_5 = 1.3$ [12]. What's more, we set both the charging/discharging efficiency η_c , η_d to 0.95.

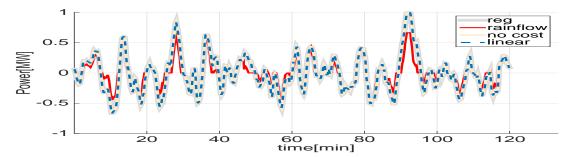


Fig. 7: Battery response to the regulation signal under three different battery cost models

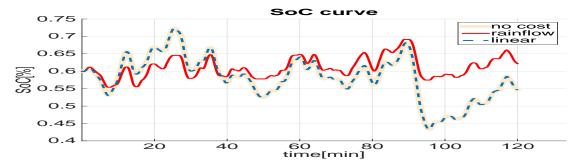


Fig. 8: Battery SoC under different battery cost models

Fig. 7 and Fig. 8 compare the power output and SoC evolution for the *same battery* optimized under three different degradation models: assuming no operating cost, linear power-based model and the proposed rainflow cycle-based model. The grey curve in Fig. 7 is the regulation instruction signal. We find that under no cost model (light yellow curve) and linear power-based degradation model (blue dashed line), the battery response is completely following

Annual cost	Rainflow	No cost	Linear
Total regulation utility (k\$)	176	137.9	137.9
Regulation service payment (k\$)	338.9	438	438
Modeled battery degradation (k\$)	162.9	0	207.2
Actual battery degradation (k\$)	162.9	300.1	300.1
Battery life expectancy (month)	11.1	6	6

TABLE II: Annual economics of BES operation under three battery cost models. "Modeled battery degradation" refers to the battery operation cost captured by the model in use, "actual degradation cost" is assessed by posterior rainflow algorithm, and "total regulation utility" is calculated by payment subtracting actual battery degradation cost.

the regulation instruction signal. However, for the rainflow cycle-based model, instead of the "blindly following" strategy, the battery response strives a good balance between mismatch penalty and degradation cost. In Fig. 8, we observed that the battery SoC is restricted to a moderate range and evolves smoothly for the rainflow cycle-based cost.

Table II summarizes the annual economics of the same battery optimized using three different degradation models. We find that, using the rainflow cycle-based degradation model, we have the highest operational utility and longest expected battery lifetime. The total regulation revenue increases by 27.6% and the battery lifetime almost doubles compared with the other two cases.

VI. PROOF

Section III provides a sketch of the convexity proof. Here we extends the sketch and explain the critical proof steps. A full version of the convexity proof could be found in the Appendix.

Proof. Firstly, we use step function decomposition method to write out s_1 , s_2 and $\lambda s_1 + (1 - \lambda)s_2$ as a finite sum of step functions, where

$$\mathbf{s_1} = \sum_{i=1}^{T} Q_i U_i, \mathbf{s_2} = \sum_{i=1}^{T} P_i U_i$$

$$\lambda \mathbf{s_1} + (1 - \lambda) \mathbf{s_1} = \sum_{i=1}^{T} X_i U_i, \tag{18}$$

We obtain the overall proof by induction. We first prove that f(s) is convex up to any *single* step change.

Lemma 1. The rainflow cycle-based cost function f(s) is convex up to one step change of the original SoC profile,

$$f(\lambda \mathbf{s_1} + (1 - \lambda)P_iU_i) \le \lambda f(\mathbf{s_1}) + (1 - \lambda)f(P_iU_i), \lambda \in [0, 1],$$

We need the following proposition to prove Lemma 1.

Proposition 1. Let $g(\cdot)$ be a convex function where g(0) = 0. Let $x_1, x_2, x_3, ..., x_N$ be real numbers, suppose $\sum_{i=1}^N x_i = D > 0$ and $|x_i| \leq D, \forall i \in \{1, 2, 3, ..., N\}$ Then,

$$g\left(\sum_{i=1}^{n} x_i\right) \ge \sum_{\{i: x_i \ge 0\}} g(x_i) - \sum_{\{i: x_i < 0\}} g(|x_i|),$$
(19)

Proposition 1 basically says that the function value of a large number D is larger than the sum of function values of N small numbers $x_1, ..., x_N$, where $\sum_{i=1}^N x_i = D$. Now, let's consider a step function added to λs_1 , where $s_1' = \lambda s_1 + (1 - \lambda) P_i U_i$. Suppose the rainflow cycle counting results are,

$$\begin{split} & \mathbf{Rainflow}(\lambda \mathbf{s_1}) : \underbrace{\lambda d_1, \lambda d_2, ..., \lambda d_M, 0, 0, ...}_{L}, \\ & \mathbf{Rainflow}(\mathbf{s_1}^{'}) : \underbrace{d_1^{'}, d_2^{'}, ..., d_N^{'}, 0, 0, ...}_{L}, \end{split}$$

where $[d_1, d_2, ..., d_M]$ are cycle counting results for s_1 . We add some cycles with 0 depth in the end to ensure that $\mathbf{Rainflow}(\lambda s_1)$ and $\mathbf{Rainflow}(\mathbf{s_1}')$ have the same number of cycles. Define Δd_i such that,

$$d'_{i} = \lambda d_{i} + (1 - \lambda) \Delta d_{i}, \forall i = 1, 2, ..., L,$$
(20)

We observe that,

$$\left| \sum_{i=1}^{L} \Delta d_i \right| \le |P_i| \text{ and } |\Delta d_i| \le |P_i| ,$$

When $\sum_{i=1}^{L} \Delta d_i = |P_i|$, Lemma 1 is directly proved by applying Proposition 1. When $-|P_i| \leq \sum_{i=1}^{L} \Delta d_i < |P_i|$, we first add some "virtual cycles" to make $\sum_{i=1}^{L} \Delta d_i = |P_i|$. Then we prove the original cost $f(\lambda \mathbf{s_1} + (1-\lambda)U_iP_i)$ is strictly less than the cost after adding the "virtual cycles".

Lemma 1 proved the convexity for base case. Next, we want to prove the induction relation from K to K+1 step. Assume that $f(\mathbf{s})$ is convex up to the sum of K step changes (arranged by time index), let's show $f(\mathbf{s})$ is convex up to the sum of K+1 step changes, where

$$f(\lambda \mathbf{s_1} + (1 - \lambda)\mathbf{s_1}) \le \lambda f(\mathbf{s_1}) + (1 - \lambda)f(\mathbf{s_1}), \lambda \in [0, 1],$$

where $\mathbf{s_1}, \mathbf{s_1} \in \mathbb{R}^{K+1}$. The following Proposition is needed for the proof.

Proposition 2.

$$f\left(\sum_{t=1}^{K} X_t U_t\right) \ge f\left(\sum_{t=1}^{i-1} X_t U_t + (X_i + X_{i+1}) U_i + \sum_{t=i+2}^{K} X_t U_t\right)$$

In other words, the cycle stress cost will reduce if combining adjacent unit changes.

Recall the step function decomposition results for s_1 , s_2 and $\lambda s_1 + (1 - \lambda)s_2$ in (18). There are three cases when we go from T = K to T = K + 1, classified by the value and sign of X_K , X_{K+1} .

Case 1: X_K and X_{K+1} are same direction.

If X_{K+1} and X_K are same direction, we could move X_{K+1} to the previous step without affecting the total cost $f(\mathbf{s_1} + (1 - \lambda)\mathbf{s_2})$. Then we prove the K+1 convexity by applying Proposition 2.

$$f(\lambda \mathbf{s_1} + (1 - \lambda)\mathbf{s_2})$$

$$= f(\lambda \mathbf{s_1}^K + (1 - \lambda)\mathbf{s_2}^K + X_{K+1}U_K)$$

$$= f\{\lambda \mathbf{s_1}^K + (1 - \lambda)\mathbf{s_2}^K + [\lambda Q_{K+1} + (1 - \lambda)P_{K+1}]U_K\}$$

$$\leq \lambda f(\mathbf{s_1}^K + Q_{K+1}U_K) + (1 - \lambda)f(\mathbf{s_2}^K + P_{K+1}U_K)$$

$$\leq \lambda f(\mathbf{s_1}) + (1 - \lambda)f(\mathbf{s_2}) \text{ (by Lemma 1)}$$

Case 2: X_K and X_{K+1} are different directions, with $|X_K| \ge |X_{K+1}|$. In this case, the last step X_{K+1} could be separated out from the previous SoC profile. Therefore

$$f(\lambda \mathbf{s_1} + (1 - \lambda)\mathbf{s_2})$$

$$= f\left(\lambda \mathbf{s_1}^K + (1 - \lambda)\mathbf{s_2}^K\right) + \Phi\left(X_{K+1}U_{K+1}\right)$$

$$\leq \lambda f(\mathbf{s_1}^K) + (1 - \lambda)f(\mathbf{s_2}^K) + \Phi\left[\lambda Q_{K+1}U_{K+1} + (1 - \lambda)P_{K+1}U_{K+1}\right]$$

$$\leq \lambda \left[f(\mathbf{s_1}^K) + \Phi(Q_{K+1}U_{K+1})\right] + (1 - \lambda)\left[f(\mathbf{s_2}^K) + \Phi(P_{K+1}U_{K+1})\right]$$

$$\leq \lambda f(\mathbf{s_1}) + (1 - \lambda)f(\mathbf{s_2})$$

Case 3: X_K and X_{K+1} are different directions, with $|X_K| < |X_{K+1}|$. In such condition, X_{K+1} is not easily separated out from previous SoC. It further contains three cases.

- X_{K-1} and X_K are the same direction. We could use the same "trick" in Case 1 to combine step K − 1 and
 K. Proof is trivial for this case.
- X_{K-1} and X_K are different directions, X_K and X_{K+1} form a cycle that is *separate* from the rest of the signal (eg. it is the deepest cycle). We can separate X_{K+1} out, and proof will be similar to Case 2.
- X_{K-1} and X_K are different directions, X_K and X_{K+1} do not form a separate cycle. This condition is the most complicated case, since it's hard to move X_{K+1} to the previous step, or separate it out. Therefore, we need to look into Q_K , Q_{K+1} , P_K , P_{K+1} in order to show the K+1 step convexity. It further contains four sub-cases as given in Fig. 9, showing convexity for each sub-case finishes the overall convexity proof.

VII. CONCLUSION

Battery operational cost modeling is very important for BES planning and operation. An ideal cost model needs to both capture the fundamental battery degradation, and be easy to incorporate in optimization and computational tractable to solve. Cycle-based degradation models, mainly using the rainflow algorithm for cycle counting is electrochemically accurate. However, it does not have a closed form thus hard to be optimized. In this paper, we prove the rainflow cycle-based degradation cost is convex (with respect to charging/discharging power). Convexity enables the degradation model easy to be incorporated and guarantee the solution quality. We provide a subgradient solver algorithm. The proposed degradation model and subgradient algorithm have a broad application scope in

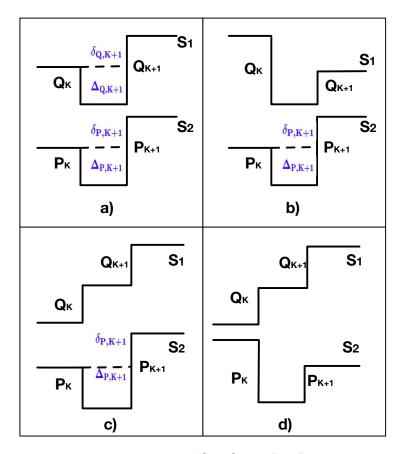


Fig. 9: Four cases of Q_K , Q_{K+1} , P_K , P_{P+1}

various battery planning and operation optimization problems. From the case study in frequency regulation market, we verified the proposed degradation model can significantly improve the operational utility and extend battery lifetime. In future, we will apply the proposed degradation model to more BES applications besides frequency regulation.

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VIII. APPENDIX

Here we provide a detailed proof that the rainflow cycle life loss model is convex. We use s to denote the battery SoC profile, and the battery degradation cost based on the rainflow method is denoted by f, which is defined by,

$$f(\mathbf{s}) = \sum_{i=1}^{N} \Phi(d_i), \qquad (21)$$

where Φ is a convex depth of discharge (DoD) stress function, N is the number charging/discharging cycles, and d_i is the depth of the ith charging/discharging cycle.

Theorem 1. Cost Model Convexity

The rainflow cycle-based battery life loss model $f(\mathbf{s}) = \sum_{i=1}^{N} \Phi(d_i)$, where $[d_1, d_2, ..., d_N] = \mathbf{Rainflow}(\mathbf{s})$ is convex given function $\Phi(\cdot)$ is convex. In mathematical form, $\forall \mathbf{s}_1, \mathbf{s}_2 \in \mathbb{R}^T$:

$$f(\lambda \mathbf{s}_1 + (1 - \lambda)\mathbf{s}_2) \le \lambda f(\mathbf{s}_1) + (1 - \lambda)f(\mathbf{s}_2), \forall \lambda \in [0, 1]$$
(22)

A. Step function decomposition

First, we introduce the step function decompostion of SOC signal. Notice that any SoC series S, could be written out as a finite sum of step functions:

$$\mathbf{s} = \sum_{i=1}^{T} P_i u(t-i),$$
 (23)

where u(t-i) is the unit step function and P_i is the signal amplitude.

$$u(t-i) = \begin{cases} 1 & t \ge i \\ 0 & \text{otherwise} \end{cases}, \quad \forall t = 1, 2, ..., T$$
 (24)

For notation convenience, we use U_i to denote u(t-i). Fig. 10 gives an example of the step function decomposition method. An SoC signal s_1 is decomposed into 4 step functions P_1U_1 , P_2U_2 , P_3U_3 and P_4U_4 .

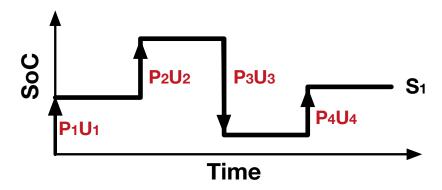


Fig. 10: Step function decomposition of S_1

B. Single step change convexity

Since all SoC profile can be written as the sum of step functions, by induction method, we first need to prove that f(s) is convex up to a step function as base case.

Lemma 1. Rainflow Convexity for single step

The rainflow cycle-based degradation cost function is convex up to one step change of the original SoC profile, where

$$f(\lambda \mathbf{s}_1 + (1 - \lambda)P_iU_i) \le \lambda f(\mathbf{s}_1) + (1 - \lambda)f(P_iU_i), \lambda \in [0, 1]$$

In order to prove the Lemma 1, we need the following propositions.

Proposition 1. Let $g(\cdot)$ be a convex function where g(0) = 0. Let x_1 , x_2 be positive real numbers. Then

$$g(x_1 + x_2) \ge g(x_1) + g(x_2)$$
,

Proof. By convexity of g, we have

$$\frac{x_1}{x_1 + x_2}g(x_1 + x_2) + \frac{x_2}{x_1 + x_2}g(0) \ge g(x_1),$$

and

$$\frac{x_2}{x_1 + x_2} g(x_1 + x_2) + \frac{x_1}{x_1 + x_2} g(0) \ge g(x_2),$$

Adding the two equations finish the proof.

Proposition 2. Let $g(\cdot)$ be a convex function where g(0) = 0. Let x_1 , x_2 be positive real numbers, and $x_1 \ge x_2$. Then

$$g(x_1 - x_2) \le g(x_1) - g(x_2),$$

Proof. By Proposition 1,

$$g(x+y) \ge g(x) + g(y), \forall x, y > 0,$$

Let $x = x_1 - x_2 > 0$, $y = x_2 > 0$, so that

$$q(x_1 - x_2 + x_2) > q(x_1 - x_2) + q(x_2)$$
,

Q.E.D.

Proposition 3. Let $g(\cdot)$ be a convex function where g(0) = 0. Let $x_1 \ge x_2 > 0$ be positive real numbers. Then

$$g(\frac{1}{2}x_1 - \frac{1}{2}x_2) \le \frac{1}{2}g(x_1) - \frac{1}{2}g(x_2),$$

Proof. From Proposition 2,

$$g(\frac{1}{2}x_1 - \frac{1}{2}x_2) \le g(\frac{1}{2}x_1) - g(\frac{1}{2}x_2), \forall x_1 \ge x_2 > 0$$

Therefore, it suffices to show,

$$g(\frac{1}{2}x_1) - g(\frac{1}{2}x_2) \le \frac{1}{2}g(x_1) - \frac{1}{2}g(x_2),$$

Define $h(z) = g(\frac{1}{2}z) - \frac{1}{2}g(z)$,

$$h'(z) = \frac{1}{2}g'(\frac{1}{2}z) - \frac{1}{2}g(z) = \frac{1}{2}[g'(\frac{1}{2}z) - g'(z)] < 0,$$

 $h(\cdot)$ is a monotone decreasing function. For $x_1 \geq x_2 > 0$,

$$h(x_1) \le h(x_2)$$

$$g(\frac{1}{2}x_1) - \frac{1}{2}g(x_1) \le g(\frac{1}{2}x_2) - \frac{1}{2}g(x_2),$$

$$g(\frac{1}{2}x_1) - g(\frac{1}{2}x_2) \le \frac{1}{2}g(x_1) - \frac{1}{2}g(x_2)$$

If $g(\cdot)$ is continous, we can generalize the midpoint property to a more broad λ ,

$$g(\lambda x_1 - (1 - \lambda)x_2) \le \lambda g(x_1) - (1 - \lambda)g(x_2), \forall \lambda x_1 \ge (1 - \lambda)x_2 > 0$$

Q.E.D.

Proposition 4. Let $g(\cdot)$ be a convex function where g(0) = 0. Let x_1, x_2, x_3 be positive real numbers, which satisfy that $x_1 + x_2 - x_3 \ge 0$, and $x_i \le x_1 + x_2 - x_3$, $\forall i \in [1, 2, 3]$. Then

$$g(x_1 + x_2 - x_3) \ge g(x_1) + g(x_2) - g(x_3)$$
,

Proof. From $x_1 \le x_1 + x_2 - x_3$ we have $x_2 \ge x_3$. From $x_2 \le x_1 + x_2 - x_3$, we have $x_1 \ge x_3$ Let's further assume $x_1 \ge x_2$,

$$g(x_1 + x_2 - x_3) - g(x_1) = (x_2 - x_3) \cdot g'(\theta_1), \theta_1 \in [x_1, x_1 + x_2 - x_3],$$
$$g(x_2) - g(x_3) = (x_2 - x_3) \cdot g'(\theta_2), \theta_2 \in [x_3, x_2],$$

Since $g(\cdot)$ is a convex function, for $\theta_2 \le x_2 \le x_1 \le \theta_1$, we have $g'(\theta_2) \le g'(\theta_1)$. Therefore,

$$g(x_2) - g(x_3) \le g(x_1 + x_2 - x_3) - g(x_1)$$
,

$$g(x_1 + x_2 - x_3) \ge g(x_1) + g(x_2) - g(x_3)$$

If $x_1 < x_2$, similarly we have

$$g(x_1) - g(x_3) \le g(x_1 + x_2 - x_3) - g(x_2)$$
,

$$g(x_1 + x_2 - x_3) \ge g(x_1) + g(x_2) - g(x_3)$$

Q.E.D

Proposition 5. Let $g(\cdot)$ be a convex function where g(0) = 0. Let $x_1, x_2, x_3, ..., x_n$ be real numbers, suppose

- $\sum_{i=1}^{n} x_i = D > 0$
- $|x_i| \le D, \forall i \in \{1, 2, 3, ..., n\}$

Then,

$$g(\sum_{i=1}^{n} x_i) \ge \sum_{\{i: x_i \ge 0\}} g(x_i) - \sum_{\{i: x_i < 0\}} g(|x_i|),$$

Proof. (1) If all x_n 's are positive, it is trivial to show $g(\sum_{i=1}^n x_i) \ge \sum_i g(x_i)$ by Proposition 1.

(2) If x_n contains both positive and negative numbers, we order them in an ascending order and renumber them as,

$$x_1 \le x_2 \le \dots \le 0 \le \dots \le x_n$$

Pick x_1 (the most negative number), and find some positive x_i , x_{i+1} such that

$$x_{i+1} \ge x_i \ge |x_1| > 0$$
,

Applying Proposition 4, we have

$$g(x_{i+1} + x_i + x_1) = g(x_{i+1} + x_i - |x_1|) \ge g(x_{i+1}) + g(x_i) - g(|x_1|),$$

Note, if we can not find such x_i , x_{i+1} , eg. $x_n < |x_1|$. We could group a bunch of *postive* x_i 's to form two new variables $y_1 = \sum_{i \in N_1} x_i$, $y_2 = \sum_{i \in N_2} x_i$ where $N_1 \cap N_2 = \emptyset$. For sure there exists such $y_1 \ge y_2 \ge |x_1|$, since

$$|\sum_{i=1}^{n} x_i| = |\sum_{i:x_i \ge 0} x_i + \sum_{j:x_j < 0, j \ne 1} x_j + x_1|$$

$$= \sum_{i:x_i \ge 0} x_i - |\sum_{j:x_j < 0, j \ne 1} x_j| - |x_1|$$

$$= D$$

$$\sum_{i:x_i \ge 0} x_i = |\sum_{j:x_j < 0, j \ne 1} x_j| + |x_1| + D \ge 2|x_1|$$

Applying Proposition 4,

$$g(y_1 + y_2 + x_1) = g(y_1 + y_2 - |x_1|)$$

$$\geq g(y_1) + g(y_2) - g(|x_1|)$$

$$\geq \sum_{i \in N_1} g(x_i) + \sum_{j \in N_2} g(x_j) - g(|x_1|), N_1 \cap N_2 = \emptyset$$

Define $x_{1}^{'}=x_{i+1}+x_{i}+x_{1}>0$ and re-order $x_{1}^{'},x_{2},x_{3},...,x_{i-1},x_{i+2},...,x_{n}.$

Or define $x_1^{'}=y_1+y_2+x_1>0$, re-order $\left\{x_i:i\neq 1, i\notin N_1\cup N_2\right\}, x_1^{'}$

Repeat the above steps till all x_i are postive and finish the proof.

Proposition 6. Consider a step change added to \mathbf{s}_1 , where $\mathbf{s}_1'(t) = \mathbf{s}_1(t) + P_iU_i$, $t \in [0, T]$. Suppose P_i is positive P_i , the rainflow cycle decomposition results for \mathbf{s}_1 and \mathbf{s}_1' are,

$$\mathbf{S}_1: d_1, d_2, ..., d_m, ..., d_M$$

$$\mathbf{S}_{1}':d_{1}',d_{2}',...,d_{n}',...,d_{N}',$$

Define L = max(M, N), we could re-write the cycles in \mathbf{s}_1 and \mathbf{s}_1' as,

$$\mathbf{s}_1: \underbrace{d_1, d_2, ..., d_M, 0, 0, ...}_{t},$$

$$\mathbf{s_{1}}^{'}:\underbrace{d_{1}^{'},d_{2}^{'},...,d_{N}^{'},0,0,...}_{L},$$

Define Δd_i such that,

$$d_{i}' = d_{i} + \Delta d_{i}, \forall i = 1, 2, ..., L$$

The following relations always holds,

$$\left| \sum_{i=1}^{L} \Delta d_i \right| \le P_i \,, \tag{25}$$

$$|\Delta d_i| < P_i \,, \tag{26}$$

Proof. There exists a small enough ΔP such that only one cycle depth d_i will change.

$$|\Delta d_i| \leq \Delta P$$
,

²The proof for negative P_i is the same, just change P_i to $|P_i|$

$$-\Delta P \leq \Delta d_i \leq \Delta P$$
,

Consider P_i as a cumulation of small ΔP , by the principle of integration, we have

$$-\int \Delta P dp \le \sum_{i=1}^{L} \Delta d_i \le \int \Delta P dp,$$

Such that,

$$\left| \sum_{i=1}^{L} \Delta d_i \right| \le P_i$$

 $|\Delta d_i| \leq P_i$ holds for the worst case where all cycle depth changes happen at one certain cycle. Therefore, it is trivial to show that $|\Delta d_i| \leq P_i$ hold in all conditions.

Now, by Proposition 1-6, we finish the following proof of Lemma 1.

Proof. Let's consider $\mathbf{s}'(t) = \lambda \mathbf{s}_1(t) + (1 - \lambda)P_iU_i, t \in [0, T]$. Then the rainflow cycle decomposition results for $\lambda \mathbf{s}_1$ and \mathbf{s}' are

$$\lambda s1 : \underbrace{\lambda d_{1}, \lambda d_{2}, ..., \lambda d_{M}, 0, 0, ...}_{L}$$

$$s_{1}^{'} : \underbrace{d_{1}^{'}, d_{2}^{'}, ..., d_{N}^{'}, 0, 0, ...}_{L}$$

Define Δd_i such that,

$$d_{i}' = \lambda d_{i} + (1 - \lambda) \Delta d_{i}, \forall i = 1, 2, ..., L$$

$$f(\lambda \mathbf{s}_{1} + (1 - \lambda)P_{i}U_{i})$$

$$= \sum_{i=1}^{L} \Phi(\lambda d_{i} + (1 - \lambda)\Delta d_{i})$$

$$= \sum_{i=1}^{l^{+}} \underbrace{\Phi(\lambda d_{i} + (1 - \lambda)\Delta d_{i})}_{\Delta d_{i} \geq 0} + \sum_{i=1}^{l^{-}} \underbrace{\Phi(\lambda d_{i} - (1 - \lambda)|\Delta d_{i}|)}_{\Delta d_{i} < 0}$$

$$\leq \sum_{i=1}^{l^{+}} [\lambda \Phi(d_{i}) + (1 - \lambda)\Phi(\Delta d_{i})] + \sum_{i=1}^{l^{-}} [\lambda \Phi(d_{i}) - (1 - \lambda)\Phi(|\Delta d_{i}|)]$$

$$\leq \lambda \sum_{i=1}^{L} \Phi(d_{i}) + (1 - \lambda) \left[\sum_{i=1}^{l^{+}} \Phi(\Delta d_{i}) - \sum_{i=1}^{l^{-}} \Phi(|\Delta d_{i}|) \right]$$

$$(27a)$$

By Proposition 6, we have

$$|\sum_{i=1}^{L} \Delta d_i| \le P_i,$$
$$|\Delta d_i| \le P_i,$$

(1) Assume $\sum_{i=1}^{L} \Delta d_i = P_i$, $|\Delta d_i| \leq P_i$. By Proposition 5,

$$f(\lambda \mathbf{s}_1 + (1 - \lambda)P_iU_i)$$

$$\leq \lambda \sum_{i=1}^{L} \Phi(d_i) + (1 - \lambda) \left[\sum_{i=1}^{l^+} \Phi(\Delta d_i) - \sum_{i=1}^{l^-} \Phi(|\Delta d_i|) \right]$$

$$\leq \lambda \sum_{i=1}^{L} \Phi(d_i) + (1 - \lambda)\Phi(\sum_{i=1}^{L} \Delta d_i)$$

$$= \lambda \sum_{i=1}^{L} \Phi(d_i) + (1 - \lambda)\Phi(P_i)$$

(2) Assume $-P_i \leq \sum_{i=1}^{L} \Delta d_i < P_i, |\Delta d_i| \leq P_i$.

Add some "virtual cycles" $d_{L+1}', d_{L+2}', ..., d_{L+K}'$ at the end of \mathbf{s}_1 , each d_{L+i}' is positive and satisfies that $|d_{L+i}'| \leq P_i$. So that $\sum_{i=1}^{L+K} \Delta d_i = P_i, |\Delta d_i| \leq P_i, \forall i \in [1,2,...,L+K]$. Write 0 at the end of $\lambda \mathbf{s}_1$ to achieve the same cycle number.

$$\lambda \mathbf{s1} : \underbrace{\lambda d_{1}, \lambda d_{2}, ..., \lambda d_{M}, 0, 0, 0, ..., 0}_{L+K} \\ \mathbf{s_{1}^{'}} : \underbrace{d_{1}^{'}, d_{2}^{'}, ..., d_{N}^{'}, 0, 0, ..., 0, d_{L+1}^{'}, d_{L+2}^{'}, ..., d_{L+K}^{'}}_{L+K}$$

$$f(\lambda \mathbf{s}_{1} + (1 - \lambda)P_{i}U_{i})$$

$$\leq \lambda \sum_{i=1}^{L} \Phi(d_{i}) + (1 - \lambda) \left[\sum_{i=1}^{l^{+}} \Phi(\Delta d_{i}) - \sum_{i=1}^{l^{-}} \Phi(|\Delta d_{i}|) \right]$$

$$<\lambda \sum_{i=1}^{L} \Phi(d_{i}) + (1 - \lambda) \left[\sum_{i=1}^{l^{+}} \Phi(\Delta d_{i}) + \sum_{i=L+1}^{L+K} \Phi(\Delta d_{i}) - \sum_{i=1}^{l^{-}} \Phi(|\Delta d_{i}|) \right]$$

$$\leq \lambda \sum_{i=1}^{L} \Phi(d_{i}) + (1 - \lambda) \Phi(\sum_{i=1}^{L+K} \Delta d_{i})$$

$$=\lambda \sum_{i=1}^{L} \Phi(d_{i}) + (1 - \lambda) \Phi(P_{i})$$

To sum up,

$$f(\lambda \mathbf{s}_1 + (1 - \lambda)P_i U_i) \le \lambda \sum_{i=1}^{L} \Phi(d_i) + (1 - \lambda)\Phi(P_i)$$
$$= \lambda f(\mathbf{s}_1) + (1 - \lambda)f(P_i U_i), \tag{28}$$

where $\lambda \in [0,1]$. Thus, f(s) is convex up to every step change in s.

Lemma 1 shows that f(s) is convex up to every step change in s. Next, we will prove the general rainflow convexity by induction method.

C. General rainflow cycle life loss convexity

We will prove the general rainflow convexity by induction.

I). Initial condiction: By Lemma 1, K = 1

$$f(\lambda \mathbf{s}_1 + (1 - \lambda)\mathbf{s}_2) \le \lambda f(\mathbf{s}_1) + (1 - \lambda)f(\mathbf{s}_2), \lambda \in [0, 1]$$

II). Suppose that, f(s) is convex up to the sum of K step changes (arranged by time index)

$$f(\lambda \mathbf{s}_1 + (1 - \lambda)\mathbf{s}_2) \le \lambda f(\mathbf{s}_1) + (1 - \lambda)f(\mathbf{s}_2), \lambda \in [0, 1], \mathbf{s}_1, \mathbf{s}_2 \in \mathbb{R}^K$$

Then we prove f(s) is convex up to the sum of K+1 step changes (see Fig. 5),

$$f(\lambda \mathbf{s}_1 + (1 - \lambda)\mathbf{s}_2) \le \lambda f(\mathbf{s}_1) + (1 - \lambda)f(\mathbf{s}_2), \lambda \in [0, 1], \mathbf{s}_1, \mathbf{s}_2 \in \mathbb{R}^{K+1}$$

The following proposition is needed for the proof.

Proposition 7.

$$f(\sum_{t=1}^{K} X_t U_t) \ge f(\sum_{t=1}^{i-1} X_t U_t + (X_i + X_{i+1}) U_i + \sum_{t=i+2}^{K} X_t U_t),$$
(29)

In other words, the cycle stress cost will reduce if combining adjacent unit changes.

Proof. Rainflow cycle counting algorithm only considers local extreme points.

- I) If X_i and X_{i+1} are the same direction, combining them doesn't affect the value of local extreme points. Therefore the left side cost equals right side cost.
- II) If X_i and X_{i+1} are different directions, suppose X_i is negative and X_{i+1} positive (otherwise the same). Time t = i makes a local minimum point.
 - Case a: If $|X_{i+1}| \le |X_i|$, combining them will raise the value of local minimum point i, thus reducing the depth of cycles which contains i. Therefore, the cost after combining is less than the original cost.
 - Case b: If $|X_{i+1}| > |X_i|$, combining them will lead to the removal of local minimum point i. In one case, if X_{i-1} and X_i are the same direction, time t = i - 1 will make a local minimum point taking the place of time t = i. Therefore, the magnitude of the local minimum point decreases, similar to case a, the total cost after combining is less than the original cost.

In the other case, if X_{i-1} and X_i are different directions, we lose a full cycle with depth $|X_i|$ fater combining. So the cost after combining is also less than the original.

To sum up, the cycle stress cost will reduce if combining adjacent unit changes.

Recall the step function decomposition results for s_1 , s_2 and $\lambda s_1 + (1 - \lambda)s_2$.

$$\mathbf{s}_1 = \sum_{i=1}^T Q_i U_i,$$

$$\mathbf{s}_2 = \sum_{i=1}^T P_i U_i,$$

$$\lambda \mathbf{s}_1 + (1 - \lambda) \mathbf{s}_2 = \sum_{i=1}^T X_i U_i,$$

There are three cases when we go from T = K to T = K + 1, classfied by the value and symbols of X_K , X_{K+1} . Case 1: X_K and X_{K+1} are same direction.

Proof. If X_{K+1} and X_K are same direction, we could move X_{K+1} to the previous step without affecting the total cost $f(\mathbf{s}_1 + (1 - \lambda)\mathbf{s}_2)$. Then we prove the K+1 convexity by applying Proposition 7.

$$\begin{split} &f\left(\lambda\mathbf{s_{1}}+(1-\lambda)\mathbf{s_{2}}\right)\\ &=f\left(\lambda\mathbf{s_{1}}^{K}+(1-\lambda)\mathbf{s_{2}}^{K}+X_{K+1}U_{K}\right)\\ &=f\left\{\lambda\mathbf{s_{1}}^{K}+(1-\lambda)\mathbf{s_{2}}^{K}+[\lambda Q_{K+1}+(1-\lambda)P_{K+1}]U_{K}\right\}\\ &\leq \lambda f\left(\mathbf{s_{1}}^{K}+Q_{K+1}U_{K}\right)+(1-\lambda)f\left(\mathbf{s_{2}}^{K}+P_{K+1}U_{K}\right)\\ &\leq \lambda f(\mathbf{s_{1}})+(1-\lambda)f(\mathbf{s_{2}}) \text{ (by Lemma 1)} \end{split}$$

Case 2: X_K and X_{K+1} are different directions, with $|X_K| \ge |X_{K+1}|$.

Proof. In this case, the last step X_{K+1} could be separated out from the previous SoC profile. Therefore

$$f(\lambda \mathbf{s_1} + (1 - \lambda)\mathbf{s_2})$$

$$= f(\lambda \mathbf{s_1} + (1 - \lambda)\mathbf{s_2}^K) + \Phi(X_{K+1}U_{K+1})$$

$$\leq \lambda f(\mathbf{s_1}^K) + (1 - \lambda)f(\mathbf{s_2}^K) + \Phi[\lambda Q_{K+1}U_{K+1} + (1 - \lambda)P_{K+1}U_{K+1}]$$

$$\leq \lambda [f(\mathbf{s_1}^K) + \Phi(Q_{K+1}U_{K+1})] + (1 - \lambda)[f(\mathbf{s_2}^K) + \Phi(P_{K+1}U_{K+1})]$$

$$\leq \lambda f(\mathbf{s_1}) + (1 - \lambda)f(\mathbf{s_2})$$

Case 3: X_K and X_{K+1} are different directions, with $|X_K| < |X_{K+1}|$.

Proof. In such condition, X_{K+1} may not easily separated out from previous SoC. It further contains three cases.

• X_{K-1} and X_K are the same direction. We could use the same "trick" in Case 1 to combine step K-1 and K. Proof is trivial for this case.

- X_{K-1} and X_K are diffrent directions, X_K and X_{K+1} form a cycle that is *separate* from the rest of the signal (eg. it is the deepest cycle). We can separate X_{K+1} out, and proof will be similar to Case 2.
- X_{K-1} and X_K are diffrent directions, X_K and X_{K+1} do not form a separate cycle. This condition is the most complicated case, since it's hard to move X_{K+1} to the previous step, or separate it out. For example, in Fig. 11, δ_{K+1} was counted as a part of the previous charging cycle while Δ_{K+1} forms a closed full cycle together with X_K by Rainflow counting algorithm. Therefore, we need to look into Q_K , Q_{K+1} , P_K , P_{K+1} in order to show the K+1 step convexity.

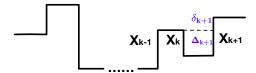


Fig. 11: X_K and X_{K+1} are different directions, with $|X_K| < |X_{K+1}|$

There are further four sub-cases (Fig. 12), for simplicity we only consider the cost of charging cycles. Showing convexity for each sub-case finishes the overall convexity proof.

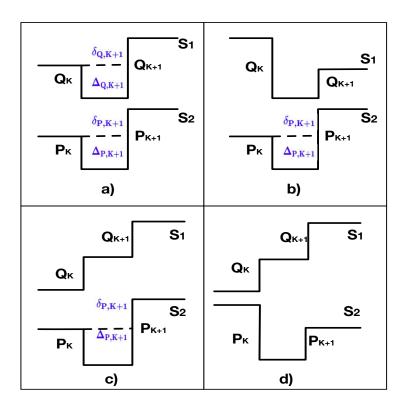


Fig. 12: Four cases of Q_K , Q_{K+1} , P_K , P_{P+1}

Case a)

$$f(\lambda \mathbf{s}_{1} + (1 - \lambda)\mathbf{s}_{2})$$

$$= f(\lambda \mathbf{s}_{1}^{K} + (1 - \lambda)\mathbf{s}_{2}^{K} + X_{K+1}U_{K+1})$$

$$= f(\lambda \mathbf{s}_{1}^{K} + (1 - \lambda)\mathbf{s}_{2}^{K} + X_{K+1}U_{K}) + \Phi(\Delta_{K+1})$$

$$\leq \lambda f(\mathbf{s}_{1}^{K} + Q_{K+1}U_{K}) + (1 - \lambda)f(\mathbf{s}_{2}^{K} + P_{K+1}U_{K})$$

$$+ \Phi(\lambda \Delta_{Q,K+1} + (1 - \lambda)\Delta_{P,K+1})$$

$$\leq \lambda [f(\mathbf{s}_{1}^{K} + Q_{K+1}U_{K}) + \Phi(\Delta_{Q,K+1})]$$

$$+ (1 - \lambda)[f(\mathbf{s}_{2}^{K} + P_{K+1}U_{K}) + \Phi(\Delta_{P,K+1})]$$

$$= \lambda f(\mathbf{s}_{1}) + (1 - \lambda)f(\mathbf{s}_{2})$$

Case c)

$$f(\lambda \mathbf{s}_{1} + (1 - \lambda)\mathbf{s}_{2})$$

$$= f(\lambda \mathbf{s}_{1}^{K} + (1 - \lambda)\mathbf{s}_{2}^{K} + X_{K+1}U_{K+1})$$

$$= f(\lambda \mathbf{s}_{1}^{K} + (1 - \lambda)\mathbf{s}_{2}^{K} + X_{K+1}U_{K}) + \Phi(\Delta_{K+1})$$

$$\leq \lambda f(\mathbf{s}_{1}^{K} + Q_{K+1}U_{K}) + (1 - \lambda)f(\mathbf{s}_{2}^{K} + P_{K+1}U_{K}) + \Phi(\Delta_{K+1})$$

$$\leq \lambda f(\mathbf{s}_{1}^{K} + Q_{K+1}U_{K}) + (1 - \lambda)f(\mathbf{s}_{2}^{K} + P_{K+1}U_{K})$$

$$+ \Phi[(1 - \lambda)\Delta_{P,K+1} - \lambda Q_{K+1}]$$

$$\leq \lambda f(\mathbf{s}_{1}^{K} + Q_{K+1}U_{K}) + (1 - \lambda)f(\mathbf{s}_{2}^{K} + P_{K+1}U_{K})$$

$$+ \Phi[(1 - \lambda)\Delta_{P,K+1}]$$

$$\leq \lambda f(\mathbf{s}_{1}) + (1 - \lambda)f(\mathbf{s}_{2})$$

Then we try to finish the proof for case b) and d). First, note that b) implies d). To show this, for case d), define $\hat{\mathbf{s}}_1 = \sum_{t=1}^{K+1} \hat{Q}_t U_t$ as a modified version of \mathbf{s}_1 , where $\hat{Q}_t = Q_t$ for t = 1, ..., K-1, $\hat{Q}_K = 0$, $\hat{Q}_{K+1} = Q_K + Q_{K+1}$. We have $f(\hat{\mathbf{s}}_1) = f(\mathbf{s}_1)$. We also have $f(\lambda \hat{\mathbf{s}}_1 + (1-\lambda)\mathbf{s}_2) \geq f(\lambda \mathbf{s}_1 + (1-\lambda)\mathbf{s}_2)$ because of the decreasing signal at K for \mathbf{s}_2 . Thus,

$$f(\lambda \mathbf{s}_1 + (1 - \lambda)\mathbf{s}_2) \le f(\lambda \hat{\mathbf{s}}_1 + (1 - \lambda)\mathbf{s}_2)$$

$$\stackrel{i)}{\le} \lambda f(\hat{\mathbf{s}}_1) + (1 - \lambda)f(\mathbf{s}_2)$$

$$= \lambda f(\mathbf{s}_1) + (1 - \lambda)f(\mathbf{s}_2)$$

where i) follows from assuming b) is true and letting $\Delta_{P,K+1} = 0$ and reversing the label of P and Q. Therefore we only need to prove case b).

Case b) contains two different circumstances in terms of P_K and P_{K+1} . We need the following proposition

for the two cases. To not use too many negative signs, we denote $\bar{Q}_t = -Q_t$, $\bar{P}_t = -P_t$.

Proposition 8. Let g be a convex increasing function and given numbers a > b > 0. Then $g(a) + g(b) \ge g(a - \delta) + g(b + \delta)$ if $\delta > 0$ and $b + \delta < a$,

Proof. Define $h(x) = g(x) - g(x - \delta)$ and $x, x - \delta \ge 0$. We have

$$h'(x) = g'(x) - g'(x - \delta) \ge 0$$
,

because g is convex and $x > x - \delta$. Therefore, $h(\cdot)$ is an increasing function, $\forall a > b + \delta$,

$$h(a) \ge h(b+\delta)$$

$$g(a) - g(a-\delta) \ge g(b+\delta) - g(b)$$

Moving g(b) to the left side and $g(a-\delta)$ to the right side of the inequation finish the proof.

Case b) \bigcirc P_K , P_{K+1} do not form a cycle that is separate from the rest of S_2 .

$$f(\lambda \mathbf{s}_{1} + (1 - \lambda)\mathbf{s}_{2})$$

$$= f(\lambda \mathbf{s}_{1}^{K} + (1 - \lambda)\mathbf{s}_{2}^{K} + X_{K+1}U_{K}) + \Phi(\Delta_{K+1})$$

$$= f(\lambda \mathbf{s}_{1}^{K} + (1 - \lambda)\mathbf{s}_{2}^{K} + \lambda Q_{K+1}U_{K} + (1 - \lambda)P_{K+1}U_{K}) + f(\lambda \bar{Q}_{K}U_{K+1} + (1 - \lambda)\bar{P}_{K}U_{K+1})$$

$$= f(\lambda \mathbf{s}_{1}^{K} + \lambda \bar{Q}_{K}U_{K} - \lambda \bar{Q}_{K}U_{K} + (1 - \lambda)\mathbf{s}_{2}^{K} + \lambda Q_{K+1}U_{K} + (1 - \lambda)P_{K+1}U_{K})$$

$$+ f(\lambda Q_{K+1}U_{K+1} - \lambda Q_{K+1}U_{K+1} + \lambda \bar{Q}_{K}U_{K+1} + (1 - \lambda)\bar{P}_{K}U_{K+1})$$

$$= f(\lambda \mathbf{s}_{1}^{K} + \lambda \bar{Q}_{K}U_{K} + (1 - \lambda)(\mathbf{s}_{2}^{K} + P_{K+1}U_{K} + \frac{\lambda}{1 - \lambda}(Q_{K+1}U_{K} - \bar{Q}_{K}U_{K})))$$

$$+ f(\lambda Q_{K+1}U_{K+1} + (1 - \lambda)(\bar{P}_{K}U_{K+1} + \frac{\lambda}{1 - \lambda}(\bar{Q}_{K}U_{K+1} - Q_{K+1}U_{K+1})))$$

$$\leq \lambda f(\mathbf{s}_{1}^{K} + \bar{Q}_{K}U_{K}) + \lambda f(Q_{K+1}U_{K+1})$$

$$+ (1 - \lambda)f(\mathbf{s}_{2}^{K} + (P_{K+1} - \frac{\lambda}{1 - \lambda}(\bar{Q}_{K} - Q_{K+1}))U_{K})$$

$$+ (1 - \lambda)f((\bar{P}_{K} + \frac{\lambda}{1 - \lambda}(\bar{Q}_{K} - Q_{K+1}))U_{K+1})$$

Only considering charging cycles, the first line is the cost of $\lambda f(s_1)$. Now we show,

$$f\left(\mathbf{s}_{2}^{K} + \left(P_{K+1} - \frac{\lambda}{1-\lambda}(\bar{Q}_{K} - Q_{K+1})\right)U_{K}\right) + f\left(\left(\bar{P}_{K} + \frac{\lambda}{1-\lambda}(\bar{Q}_{K} - Q_{K+1})\right)U_{K+1}\right) \leq f(\mathbf{s}_{2}),$$

We can write out the cost of charging cycles in s_2 as $f(s_2)$,

$$f(\mathbf{s}_2) = \sum_{i}^{N-1} \Phi(Z_i) + \Phi(\bar{P}_K),$$

where Z_{N-1} is the cycle that $s_2(K+1)$ is assigned to.

By assumption that P_K and P_{K+1} do not form a separate cycle, so that $Z_{N-1} \geq P_{K+1}$. By assumption, $\frac{\lambda}{1-\lambda}(\bar{Q}_K-Q_{K+1})>0$. Let $\delta=\frac{\lambda}{1-\lambda}(\bar{Q}_K-Q_{K+1})$, and since $P_{K+1}+Q_{K+1}\geq \bar{P}_K+\bar{Q}_K$ in case b), $\bar{P}_K+\delta\leq P_{K+1}\leq Z_{N-1}$. Therefore applying Proposition 8, $a=Z_{N-1}$ and $b=\bar{P}_K$, $\delta=\frac{\lambda}{1-\lambda}(\bar{Q}_K-Q_{K+1})$, we have the desired result.

Case b) ② The other case is that P_K , P_{K+1} form a cycle that is separate from the rest of s_2 . Similar as ①, we need to show

$$f\left(\mathbf{s}_{2}^{K} + \left(P_{K+1} - \frac{\lambda}{1-\lambda}(\bar{Q}_{K} - Q_{K+1})\right)U_{K}\right) + f\left(\left(\bar{P}_{K} + \frac{\lambda}{1-\lambda}(\bar{Q}_{K} - Q_{K+1})\right)U_{K+1}\right) \leq f(\mathbf{s}_{2})$$

Since $P_{K+1} = \bar{P}_K + \delta_{P,K+1}$, we re-write the above inequation as,

$$f(\mathbf{s}_{2}^{K} + (\bar{P}_{K} + \delta_{P,K+1} - \frac{\lambda}{1-\lambda}(\bar{Q}_{K} - Q_{K+1}))U_{K}) + f((P_{K+1} - \delta_{P,K+1} + \frac{\lambda}{1-\lambda}(\bar{Q}_{K} - Q_{K+1}))U_{K+1}) \leq f(\mathbf{s}_{2}),$$

Denote $\delta = \delta_{P,K+1} - \frac{\lambda}{1-\lambda}(\bar{Q}_K - Q_{K+1}) > 0$

$$f(\mathbf{s}_2) = \sum_{i}^{N-2} \Phi(Z_i) + \Phi(Z_{N-1}) + \Phi(P_{K+1}),$$

 Z_{N-1} is the deepest charging cycle with ending SoC equals the P_K 's starting SoC, and $Z_{N-1} \leq P_{K+1}$ since P_{K+1} forms a separate cycle. Applying Proposition 8 by setting $a = P_{K+1}$, $b = Z_{N-1}$, we have the desired results.