# An Iterative Regularized Incremental Projected Subgradient Method for a Class of Bilevel Optimization Problems 

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#### Abstract

We study a class of bilevel convex optimization problems where the goal is to find the minimizer of an objective function in the upper level, among the set of all optimal solutions of an optimization problem in the lower level. A wide range of problems in convex optimization can be formulated using this class. An important example is the case where an optimization problem is ill-posed. In this paper, our interest lies in addressing the bilevel problems, where the lower level objective is given as a finite sum of separate nondifferentiable convex component functions. This is the case in a variety of applications in distributed optimization, such as large-scale data processing in machine learning and neural networks. To the best of our knowledge, this class of bilevel problems, with a finite sum in the lower level, has not been addressed before. Motivated by this gap, we develop an iterative regularized incremental subgradient method, where the agents update their iterates in a cyclic manner using a regularized subgradient. Under a suitable choice of the regularization parameter sequence, we establish the convergence of the proposed algorithm and derive a rate of $\mathscr{O}\left(1 / k^{0.5-\varepsilon}\right)$ in terms of the lower level objective function for an arbitrary small $\varepsilon>0$. We present the performance of the algorithm on a binary text classification problem.


## I. Introduction

In this paper, we consider a class of bilevel optimization problems as follows

$$
\begin{array}{cl}
\operatorname{minimize} & h(x)  \tag{f}\\
\text { subject to } & x \in X^{*} \triangleq \arg \min _{y \in X} f(y) .
\end{array}
$$

where $f, h: \mathbb{R}^{n} \rightarrow \mathbb{R}$ denote the lower and upper level objective functions, respectively, and $X \subseteq \mathbb{R}^{n}$ is a constraint set. This is called the selection problem ([8], [22]) as we are selecting among optimal solutions of a lower level problem, one that minimizes the objective function $h$. In particular, we consider the case where the lower level objective function is given as $f(x) \triangleq \sum_{i=1}^{m} f_{i}(x)$, where $f_{i}: \mathbb{R}^{n} \rightarrow \mathbb{R}$ is the $i$ th component function for $i=1, \cdots, m$.

We make the following basic assumptions.
Assumption 1 (Problem properties):
(a) The set $X \subset \mathbb{R}^{n}$ is nonempty, compact and convex; also $X \subseteq \operatorname{int}(\operatorname{dom}(f) \cap \operatorname{dom}(h))$.
(b) The functions $f_{i}(x)$ for $i=1, \cdots, m$ are proper, closed, convex, and possibly nondifferentiable.
(c) The function $h$ is strongly convex with parameter $\mu_{h}>0$ and possibly nondifferentiable.
Next, we present two instances of the applications of formulation $\left\langle P_{f}^{h}\right\rangle$.

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## A. Example problems

(i) Constrained nonlinear optimization: Consider a constrained convex optimization problem given as

$$
\begin{aligned}
\operatorname{minimize} & h(x) \\
\text { subject to } & q_{i}(x) \leq 0, \text { for } i=1, \cdots, m \\
& x \in X
\end{aligned}
$$

where $X \subseteq \mathbb{R}^{n}$ is an easy-to-project constraint set, $h, q_{i}: \mathbb{R}^{n} \rightarrow$ $\mathbb{R}$ for all $i=1, \cdots, m$ are convex (and possibly nonlinear) funcitons. This problem can be reformulated as $\left.P_{f}^{h}\right\rangle$ by setting (cf. [23])

$$
f(x) \triangleq \sum_{i=1}^{m} f_{i}(x)=\sum_{i=1}^{m} \max \left\{0, q_{i}(x)\right\} .
$$

(ii) Ill-posed distributed optimization: An optimization problem is called ill-posed when it has multiple optimal solutions or it is very sensitive to data perturbations [25]. For instance, in applications arising in machine learning, consider the empirical risk minimization problem where the goal is to minimize the total loss $\sum_{i=1}^{m} \mathfrak{L}\left(a_{i} x, b_{i}\right)$, where $a_{i}$ is the input, $b_{i}$ is the output of $i$ th observed datum and $\mathfrak{L}$ is the loss function. For example, in logisitic loss regression, $\mathfrak{L}$ is merely convex. In these cases, another criterion such as sparsity may be taken into account for the optimal solution. So, to induce sparsity, a secondary objective function $h$ is considered in the given problem. For instance, the wellknown elastic net regularization can be used as function $h$. Hence, to address ill-posedness, the following bilevel optimization model is considered [8], [22]:

$$
\begin{array}{ll}
\text { minimize } & \|x\|_{1}+\mu\|x\|_{2}^{2} \\
\text { subject to } & x \in \arg \min _{y \in X} \sum_{i=1}^{m} \mathfrak{L}\left(a_{i}^{T} y, b_{i}\right),
\end{array}
$$

where $\mu>0$ regulates the trade-off between $\ell_{1}$ and $\ell_{2}$ norms.

## B. Existing methods

Problem $\left.\sqrt{P_{f}^{h}}\right)$, that is also referred to as hierarchical optimization, is a particular case of mathematical program with generalized equation (or equilibrium) constraint [13], [15]. There has been a few approaches to tackle this problem. Note that in all approaches the following minimization problem and its minimizer have been extensively utilized.

Definition 1: Given a parameter $\lambda>0$, the regularized problem corresponding to $\left(P_{f}^{h}\right)$, is defined as

$$
\begin{array}{cl}
\operatorname{minimize} & f_{\lambda}(x) \triangleq f(x)+\lambda h(x)  \tag{1}\\
\text { subject to } & x \in X .
\end{array}
$$

TABLE I: Comparison of Methods for Solving the Bilevel Optimization Problem $P$ Ph.

| Paper | Assumptions on lower level obj. function | Assumptions on upper level obj. function | Methodology | Metric | Convergence |
| :---: | :---: | :---: | :---: | :---: | :---: |
| [24] | convex, locally Lipschitz | convex, smooth | iter. regu. | $h_{k}-h^{*}$ | asymptotic |
|  |  |  |  | $f_{k}-f^{*}$ |  |
| [23] | convex, nonsmooth | convex, nonsmooth | iter. regu. | $h_{k}-h^{*}$ | asymptotic |
|  |  |  |  | $f_{k}-f^{*}$ |  |
| [2] | convex, Lipschitz | strongly convex, continuously differentiable | MNG | $h_{k}-h^{*}$ | asymptotic |
|  |  |  |  | $f_{k}-f^{*}$ | $\mathscr{O}(1 / \sqrt{k})$ |
| [22] | convex, continuously differentiable, Lipschitz | strongly convex, smooth | SAM | $h_{k}-h^{*}$ | asymptotic |
|  |  |  |  | $f_{k}-f^{*}$ | $\mathscr{O}(1 / k)$ |
| [29] | convex, differentiable | strongly convex, differentiable | iter. regu. | $h_{k}-h^{*}$ | $\mathscr{O}\left(1 / k^{1 / 6-\varepsilon}\right)$ |
|  |  |  |  | $f_{k}-f^{*}$ | asymptotic |
| [9] | convex, continuously differentiable | strongly convex, continuously differentiable | iter. regu. | $h_{k}-h^{*}$ | $\mathscr{O}(1 / k)$ |
|  |  |  |  | $f_{k}-f^{*}$ | asymptotic |
| This work | convex, nondifferentiable (finite sum form) | strongly convex, nondifferentiable | incremental iter. regu. | $h_{k}-h^{*}$ | asymptotic |
|  |  |  |  | $f_{k}-f^{*}$ | $\mathscr{O}\left(1 / k^{0.5-\varepsilon}\right)$ |

Also, let $x_{\lambda}^{*}$ denote the unique minimizer of this problem. We may categorize the existing algorithms as follows.
(i) Exact regularization: The regularization technique has been highly used in some applications such as signal processing with $h(x)=\|x\|_{2}^{2}$ or $h(x)=\|x\|_{1}$ [25], [3]. This technique needs a proper parameter $\lambda$ which is difficult to determine in most of cases. To address this issue, Mangasarian et al. [11], [17] introduced exact regularization. A solution of problem (1) is called exact when it is in the set $X^{*}$. The main drawback of this approach is that the threshold below which for any $\lambda$ the regularization (1) is exact, is very difficult to determine a priori (see [8]).
(ii) Iterative regularization: In this approach, the idea is to develop a single-loop scheme where the regularization parameter is updated iteratively during the algorithm. In [24], an explicit descent algorithm is proposed, where problem (1) is solved as a single-level unconstrained problem iteratively. In the smooth case, the convergence is shown when $\sum_{k=1}^{\infty} \lambda_{k}=\infty$ and $\lim _{k \rightarrow \infty} \lambda_{k}=0$. For nonsmooth cases, a bundle method was proposed, which has a descent step for the weighted combination of objective functions in the lower and upper levels [23]. Another algorithm called hybrid steepest descent method (HSDM) and its extensions were developed in [28], [19]. The main drawback of all these works is that no complexity analysis was provided for the underlying algrotihm. In [29], the rate of $\mathscr{O}\left(1 / k^{1 / 6-\varepsilon}\right)$ is derived for a special case of problem $\left.\sqrt{P_{f}^{h}}\right\rangle$, where the objective function is $\|\cdot\|_{2}$ and the constraint is solutions of a stochastic variational inequality problem. Recently, a primal-dual algorithm in [9] was offered that uses the idea of iterative regularization. Despite a sublinear rate in terms of $h$, the algorithm can only be applied to continuously differentiable and small-scale (i.e., $m=1$ ) regimes.
(iii) Minimal norm gradient: In [2], the minimal norm gradient (MNG) method was developed for solving problem $\sqrt{P_{f}^{h}}$ with $m=1$. The rate of $\mathscr{O}(1 / \sqrt{k})$ was derived for the convergence with respect to lower level problem. The
main disadvantage of the MNG method is that it is a twoloop scheme where at each iteration a minimization problem should be solved.
(iv) Sequential averaging: The sequential averaging method (SAM), developed in [27], was employed in [22] for solving the problem in a more general setting. The proposed method is proved to have the rate of convergence of $\mathscr{O}(1 / k)$ in terms of the function $f$. The method is called Big-SAM. Despite that it is a single-loop scheme, sequential averaging schemes require smoothness properties of the problem and seem not to lend themselves to distributed implementations.

## C. Main Contributions

For more details on the main distinctions between our work and the existing methods see Table 1. In none of the existing methods, the finite sum form for the lower level problem is considered. The sum structure is very rampant in practice when we have separate objective functions related to different agents in a distributed setting. This is the case for example in machine learning for very large datasets [6], where each $f_{i}$ represents an agent that is cooperating with others. When the complete information of all the agents, i.e. summation of all (sub)gradients is not available, these agents can be treated distinctly. Due to its wide range of applications in distributed optimization, finite sum problem has been extensively studied. Among popular methods are incremental (sub)gradient (IG) [4], [18] and incremental aggregated (sub)gradients (IAG) [5], [26] for deterministic and stochastic average gradient (SAG) [21], SAGA [7] and MISO methods [16] for stochastic regimes. These algorithms have faster convergence and are computationally efficient in large-scale optimization since a very less amount of memory is required at each step in order to store only one agent's information and subsequently update the iterate based on that [10]. Despite the widespread use of these first-order methods, they do not address the bilevel problem $\left(P_{f}^{h}\right)$.

Motivated by the existing lack in the literature and inspired by the advantages of incremental approaches, in this paper,
we let the lower level objective function to be a summation of $m$ components. Then we use the idea of incremental subgradient optimization to address problem $\left.\sqrt{P_{f}^{h}}\right)$. We let functions in both levels to be nondifferentiable. We then prove the convergence of our proposed algorithm as well as the $\mathscr{O}\left(1 / k^{0.5-\varepsilon}\right)$ rate of convergence.

Remark 1: An interesting research question that is remained as a future direction to our research is if we can establish the convergence of iterative regularized IAG method in solving problem $\left\langle P_{f}^{h}\right\rangle$ or similarly SAG, SAGA and MISO in stochastic regimes.

Notation The inner product of two vectors $x, y \in \mathbb{R}^{n}$, is shown as $x^{T} y$. Also, $\|\cdot\|$ denotes Euclidean norm known as $\|\cdot\|_{2}$. For a convex function $f$ with the domain $\operatorname{dom}(f)$, any vector $g_{f}$ with $f(x)+g_{f}^{T}(y-x) \leq f(y)$ for all $x, y \in$ $\operatorname{dom}(f)$, is called a subgradient of $f$ at $x$. We let $\partial f(x)$ and $\partial h(x)$ denote the set of all subgradients of functions $f$ and $h$ at $x$. Let $f^{*}$ be the optimal value and $X^{*}$ represent the set of all optimal solutions of the lower level problem in $P_{f}^{h}$ ) and $x^{*}$ shows any element of this set. Likewise, $x_{h}^{*}$ denotes the optimal solution of problem $\left(P_{f}^{h}\right)$, and $x_{\lambda}^{*}$ denotes the optimal solution of problem (1). Also, we let $\mathscr{P}_{X}(x)$ denote the Euclidean projection of vector $x$ onto the set $X$.

The rest of this paper is organized as follows. In Section III. we present the algorithm outline. Then, we discuss the convergence analysis in Section III, and derive the convergence rate in Section IV. We present the numerical results in Section V, and conclude in Section VI

## II. Algorithm outline

In this section, we introduce the iterative regularized incremental projected (sub)gradient (IR-IG) for generating a sequence that converges to the unique optimal solution of $P_{f}^{h}$. See Algorithm 1 . IR-IG method includes two main

```
Algorithm 1 IR-IG
    initialization: Set an arbitrary initial point \(x_{0} \in X, \bar{x}_{0}=x_{0}\),
    and \(S_{0}=\gamma_{0}^{r}\) and pick \(r<1\).
    for \(k=0,1, \ldots, N-1\) do
        Set \(x_{k, 0}=x_{k}\) and pick \(\gamma_{k}>0\) and \(\lambda_{k}>0\).
        for \(i=0,1, \ldots, m-1\) do
            Pick \(g_{f_{i+1}}\left(x_{k, i}\right) \in \partial f_{i+1}\left(x_{k, i}\right)\) and \(g_{h}\left(x_{k, i}\right) \in \partial h\left(x_{k, i}\right)\).
            Update \(x_{k, i}\) using the following relation:
\[
\begin{equation*}
x_{k, i+1}:=\mathscr{P}_{X}\left(x_{k, i}-\gamma_{k}\left(g_{f_{i+1}}\left(x_{k, i}\right)+\frac{\lambda_{k}}{m} g_{h}\left(x_{k, i}\right)\right)\right) . \tag{2}
\end{equation*}
\]
```


## end for

Set $x_{k+1}:=x_{k, m}$.
Update $S_{k}$ and $\bar{x}_{k}$ using:

$$
\begin{equation*}
S_{k+1}:=S_{k}+\gamma_{k+1}^{r}, \quad \bar{x}_{k+1}:=\frac{S_{k} \bar{x}_{k}+\gamma_{k+1}^{r} x_{k+1}}{S_{k+1}} \tag{3}
\end{equation*}
$$

end for
return $\bar{x}_{N}$.
steps. First, the agents update their iterates in an incremental
fashion similar to the standard IG method. This step takes a circle around the all components of function $f$ to update the iterate. However, The main difference lies in the secondary objective function $h$, which is added by a vanishing multiplier $\lambda_{k}$. Second, we do averaging in order to accelerate the convergence speed of the algorithm. For this, we consider a weighted average sequence $\left\{\bar{x}_{k}\right\}$ defined as below:

$$
\begin{equation*}
\bar{x}_{k+1}:=\sum_{t=0}^{k} \psi_{t, k} x_{t}, \text { where } \psi_{t, k} \triangleq \frac{\gamma_{t}^{r}}{\sum_{i=0}^{k} \gamma_{i}^{r}} \tag{4}
\end{equation*}
$$

in which $r<1$ is a constant, controlling the weights. Note that (3) in Algorithm 1 follows from the relation (4) by applying induction, (see e.g., Proposition 3 in [30]).

## III. Convergence analysis

In this section, our goal is to show that the generated sequence $\left\{\bar{x}_{k}\right\}$ by Algorithm 1 converges to the unique optimal solution of problem $\left.\sqrt{P_{f}^{n}}\right\}$ (see Theorem 1 .

Remark 2: (a) Note that from Theorem 3.16, pg. 42 of [1], Assumption 1 implies that there exist constants $C_{f}, C_{h} \in$ $\mathbb{R}$ such that $\left\|g_{f_{i}}(x)\right\| \leq C_{f}$ and $\left\|g_{h}(x)\right\| \leq C_{h}$ for all $i=$ $1, \cdots, m$ and $x \in X$, where $g_{f_{i}}(x) \in \partial f_{i}(x)$ and $g_{h}(x) \in \partial h(x)$. (b) From Theorem 3.61, pg. 71 of [1], functions $f_{i}$ and $h$ are Lipschitz over $X$ with parameters $C_{f}$ and $C_{h}$, respectively, i.e., for all $i=1, \cdots, m$ and $x, y \in X$

$$
\left|f_{i}(x)-f_{i}(y)\right| \leq C_{f}\|x-y\|, \quad|h(x)-h(y)| \leq C_{h}\|x-y\|
$$

(c) Assumption 1 a,b) imply that the optimal solution set, $X^{*}$, is nonempty.

Here, we start with a lemma which helps bound the error of optimal solutions of the problem (1) for two different values of $\lambda$. We will make use of this lemma in the convergence analysis. The proof for this lemma can be done in a same fashion to that of Proposition 1 in [29].

Lemma 1: Let Assumption 1 hold. Suppose $\left\{x_{\lambda_{k}}^{*}\right\}$ be the sequence of the optimal solutions of problem (1) with parameter $\lambda:=\lambda_{k}$. Then,
(a) $\left\|x_{\lambda_{k}}^{*}-x_{\lambda_{k-1}}^{*}\right\| \leq \frac{C_{h}}{\mu_{h}}\left|1-\frac{\lambda_{k-1}}{\lambda_{k}}\right|$.
(b) If $\lambda_{k} \rightarrow 0$, then the sequence $\left\{x_{\lambda_{k}}^{*}\right\}$ converges to the unique optimal solution of problem $\left\langle P_{f}^{h}\right\rangle$, i.e., $x_{h}^{*}$.
To get started, we also need a recursive upper bound on the term $\left\|x_{k+1}-x_{\lambda_{k}}^{*}\right\|$. This is provided by the following lemma and will be used in Proposition 1 to prove the convergence of sequence $\left\{x_{k}\right\}$ generated by the algorithm to $x_{h}^{*}$.

Lemma 2 (A recursive error bound): Let Assumption 1 hold and $0<\mu_{k} \lambda_{k} \mu_{h} \leq 2 m$. Then, for the sequence $\left\{x_{k}\right\}$ generated by Algorithm 1 and for all $k>0$ we have

$$
\begin{aligned}
\left\|x_{k+1}-x_{\lambda_{k}}^{*}\right\|^{2} & \leq\left(1-\frac{\gamma_{k} \lambda_{k} \mu_{h}}{2 m}\right)\left\|x_{k}-x_{\lambda_{k-1}}^{*}\right\|^{2} \\
& +\frac{3 m C_{h}^{2}}{\gamma_{k} \lambda_{k} \mu_{h}^{3}}\left|1-\frac{\lambda_{k-1}}{\lambda_{k}}\right|^{2}+6 m^{2} \gamma_{k}^{2}\left(C_{f}^{2}+\lambda_{k}^{2} C_{h}^{2}\right)
\end{aligned}
$$

where $x_{\lambda_{k}}^{*}$ is the unique optimal solution of problem (1) with $\lambda:=\lambda_{k}$.

Proof: Using (2) and the nonexpansiveness property of projection, we have
$\left\|x_{k, i+1}-x_{\lambda_{k}}^{*}\right\|^{2}$
$=\left\|\mathscr{P}_{X}\left(x_{k, i}-\gamma_{k}\left(g_{f_{i+1}}\left(x_{k, i}\right)+\frac{\lambda_{k}}{m} g_{h}\left(x_{k, i}\right)\right)\right)-\mathscr{P}_{X}\left(x_{\lambda_{k}}^{*}\right)\right\|^{2}$
$\leq\left\|x_{k, i}-x_{\lambda_{k}}^{*}\right\|^{2}+\gamma_{k}^{2}\left\|g_{f_{i+1}}\left(x_{k, i}\right)+\frac{\lambda_{k}}{m} g_{h}\left(x_{k, i}\right)\right\|^{2}$
$-2 \gamma_{k}\left(g_{f_{i+1}}\left(x_{k, i}\right)+\frac{\lambda_{k}}{m} g_{h}\left(x_{k, i}\right)\right)^{T}\left(x_{k, i}-x_{\lambda_{k}}^{*}\right)$.
By boundedness of subgradients from Remark 2(a), the definition of subgradient for $f_{i+1}$, and the strong convexity of $h$, we obtain

$$
\begin{aligned}
& \left\|x_{k, i+1}-x_{\lambda_{k}}^{*}\right\|^{2} \\
& \leq\left\|x_{k, i}-x_{\lambda_{k}}^{*}\right\|^{2}+2 \gamma_{k}^{2}\left(C_{f}^{2}+\lambda_{k}^{2} C_{h}^{2}\right)-2 \gamma_{k}\left(f_{i+1}\left(x_{k, i}\right)-f_{i+1}\left(x_{\lambda_{k}}^{*}\right)\right) \\
& -\frac{2 \gamma_{k} \lambda_{k}}{m}\left(h\left(x_{k, i}\right)-h\left(x_{\lambda_{k}}^{*}\right)\right)-\frac{\gamma_{k} \lambda_{k} \mu_{h}}{m}\left\|x_{k, i}-x_{\lambda_{k}}^{*}\right\|^{2} \\
& =\left(1-\frac{\gamma_{k} \lambda_{k} \mu_{h}}{m}\right)\left\|x_{k, i}-x_{\lambda_{k}}^{*}\right\|^{2}-2 \gamma_{k}\left(f_{i+1}\left(x_{k, i}\right)+\frac{\lambda_{k}}{m} h\left(x_{k, i}\right)\right) \\
& +2 \gamma_{k}\left(f_{i+1}\left(x_{\lambda_{k}}^{*}\right)+\frac{\lambda_{k}}{m} h\left(x_{\lambda_{k}}^{*}\right)\right)+2 \gamma_{k}^{2}\left(C_{f}^{2}+\lambda_{k}^{2} C_{h}^{2}\right) .
\end{aligned}
$$

Taking summation from both sides over $i$, using $x_{k, 0}=$ $x_{k}, x_{k, m}=x_{k+1}$, and that $\gamma_{k} \lambda_{k} \mu_{h}>0$, we obtain

$$
\begin{align*}
& \sum_{i=0}^{m-1}\left\|x_{k, i+1}-x_{\lambda_{k}}^{*}\right\|^{2} \\
& \leq\left(1-\frac{\gamma_{k} \lambda_{k} \mu_{h}}{m}\right)\left\|x_{k}-x_{\lambda_{k}}^{*}\right\|^{2}+\sum_{i=1}^{m-1}\left\|x_{k, i}-x_{\lambda_{k}}^{*}\right\|^{2} \\
& +2 m \gamma_{k}^{2}\left(C_{f}^{2}+\lambda_{k}^{2} C_{h}^{2}\right)-2 \gamma_{k} \sum_{i=0}^{m-1}\left(f_{i+1}\left(x_{k, i}\right)+\frac{\lambda_{k}}{m} h\left(x_{k, i}\right)\right) \\
& +2 \gamma_{k}\left(f\left(x_{\lambda_{k}}^{*}\right)+\lambda_{k} h\left(x_{\lambda_{k}}^{*}\right)\right), \tag{5}
\end{align*}
$$

where we used the definition of function $f$ in the second inequality. Now by rearranging the terms and adding and subtracting $f\left(x_{k}\right)+\lambda_{k} h\left(x_{k}\right)$ we obtain

$$
\begin{aligned}
& \left\|x_{k+1}-x_{\lambda_{k}}^{*}\right\|^{2} \\
& \leq\left(1-\frac{\gamma_{k} \lambda_{k} \mu_{h}}{m}\right)\left\|x_{k}-x_{\lambda_{k}}^{*}\right\|^{2}+2 m \gamma_{k}^{2}\left(C_{f}^{2}+\lambda_{k}^{2} C_{h}^{2}\right) \\
& -2 \gamma_{k} \sum_{i=0}^{m-1}\left(\left(f_{i+1}\left(x_{k, i}\right)-f_{i+1}\left(x_{k}\right)\right)+\frac{\lambda_{k}}{m}\left(h\left(x_{k, i}\right)-h\left(x_{k}\right)\right)\right) \\
& +2 \gamma_{k} \underbrace{\left(f\left(x_{\lambda_{k}}^{*}\right)+\lambda_{k} h\left(x_{\lambda_{k}}^{*}\right)-f\left(x_{k}\right)-\lambda_{k} h\left(x_{k}\right)\right)}_{\text {Term } 1} \\
& \leq\left(1-\frac{\gamma_{k} \lambda_{k} \mu_{h}}{m}\right)\left\|x_{k}-x_{\lambda_{k}}^{*}\right\|^{2}+2 m \gamma_{k}^{2}\left(C_{f}^{2}+\lambda_{k}^{2} C_{h}^{2}\right) \\
& +2 \gamma_{k} \sum_{i=0}^{m-1}(\underbrace{\left|f_{i+1}\left(x_{k, i}\right)-f_{i+1}\left(x_{k}\right)\right|}_{\text {Term } 2}+\frac{\lambda_{k}}{m} \underbrace{\left|h\left(x_{k, i}\right)-h\left(x_{k}\right)\right|}_{\text {Term } 3})
\end{aligned}
$$

where Term $1 \leq 0$ is used due to optimality of $x_{\lambda_{k}}^{*}$ for $f+\lambda_{k} h$. Also, from Remark 2 (b) we know that $\operatorname{Term} 2 \leq^{\lambda_{k}}\left\|x_{k, i}-x_{k}\right\|$
and Term $3 \leq C_{h}\left\|x_{k, i}-x_{k}\right\|$. So, We have

$$
\begin{align*}
& \left\|x_{k+1}-x_{\lambda_{k}}^{*}\right\|^{2} \leq\left(1-\frac{\gamma_{k} \lambda_{k} \mu_{h}}{m}\right)\left\|x_{k}-x_{\lambda_{k}}^{*}\right\|^{2} \\
& +2 m \gamma_{k}^{2}\left(C_{f}^{2}+\lambda_{k}^{2} C_{h}^{2}\right)+2\left(C_{f}+\lambda_{k} C_{h}\right) \gamma_{k} \sum_{i=0}^{m-1}\left\|x_{k, i}-x_{k}\right\| . \tag{6}
\end{align*}
$$

Next, we find an upper bound for $\left\|x_{k, i}-x_{k}\right\|$. We have $\left\|x_{k, 1}-x_{k}\right\|$
$=\left\|\mathscr{P}_{X}\left(x_{k, 0}-\gamma_{k}\left(g_{f_{1}}\left(x_{k, 0}\right)+\frac{\lambda_{k}}{m} g_{h}\left(x_{k, 0}\right)\right)\right)-\mathscr{P}_{X}\left(x_{k}\right)\right\|$
$\leq \gamma_{k}\left\|g_{f_{1}}\left(x_{k, 0}\right)+\frac{\lambda_{k}}{m} g_{h}\left(x_{k, 0}\right)\right\| \leq \gamma_{k}\left(C_{f}+\frac{\lambda_{k}}{m} C_{h}\right)$.
For $i>0$, in a similar way, we have

$$
\left\|x_{k, i+1}-x_{k}\right\| \leq\left\|x_{k, i}-x_{k}\right\|+\gamma_{k}\left(C_{f}+\frac{\lambda_{k}}{m} C_{h}\right)
$$

So for $i=0,1, \cdots, m-1$, we have

$$
\begin{align*}
\left\|x_{k, i+1}-x_{k}\right\| & \leq(i+1) \gamma_{k}\left(C_{f}+\frac{\lambda_{k}}{m} C_{h}\right) \\
& \leq(i+1) \gamma_{k}\left(C_{f}+\lambda_{k} C_{h}\right) \tag{7}
\end{align*}
$$

Combining this with (6), we will obtain

$$
\begin{align*}
\left\|x_{k+1}-x_{\lambda_{k}}^{*}\right\|^{2} & \leq\left(1-\frac{\gamma_{k} \lambda_{k} \mu_{h}}{m}\right)\left\|x_{k}-x_{\lambda_{k}}^{*}\right\|^{2} \\
& +6 m^{2} \gamma_{k}^{2}\left(C_{f}^{2}+\lambda_{k}^{2} C_{h}^{2}\right) . \tag{8}
\end{align*}
$$

Next, we relate $x_{k}$ to $x_{\lambda_{k-1}}^{*}$. We have

$$
\begin{aligned}
& \left\|x_{k}-x_{\lambda_{k}}^{*}\right\|^{2}=\left\|x_{k}-x_{\lambda_{k-1}}^{*}\right\|^{2}+\left\|x_{\lambda_{k}}^{*}-x_{\lambda_{k-1}}^{*}\right\|^{2} \\
& +\underbrace{2\left(x_{k}-x_{\lambda_{k-1}}^{*}\right)^{T}\left(x_{\lambda_{k-1}}^{*}-x_{\lambda_{k}}^{*}\right)}_{\text {Term } 4} .
\end{aligned}
$$

Applying the fact that $2 a^{T} b \leq\|a\|^{2} / \alpha+\alpha\|b\|^{2}$ for all $a, b \in$ $\mathbb{R}^{n}$ and $\alpha>0$ for Term 4 when $\alpha=2 m / \gamma_{k} \lambda_{k} \mu_{h}$, we obtain

$$
\begin{aligned}
& \left\|x_{k}-x_{\lambda_{k}}^{*}\right\|^{2}=\left\|x_{k}-x_{\lambda_{k-1}}^{*}\right\|^{2}+\left\|x_{\lambda_{k}}^{*}-x_{\lambda_{k-1}}^{*}\right\|^{2} \\
& +\frac{\gamma_{k} \lambda_{k} \mu_{h}}{2 m}\left\|x_{k}-x_{\lambda_{k-1}}^{*}\right\|^{2}+\frac{2 m}{\gamma_{k} \lambda_{k} \mu_{h}}\left\|x_{\lambda_{k}}^{*}-x_{\lambda_{k-1}}^{*}\right\|^{2} \\
& =\left(1+\frac{\gamma_{k} \lambda_{k} \mu_{h}}{2 m}\right)\left\|x_{k}-x_{\lambda_{k-1}}^{*}\right\|^{2}+\left(1+\frac{2 m}{\gamma_{k} \lambda_{k} \mu_{h}}\right)\left\|x_{\lambda_{k}}^{*}-x_{\lambda_{k-1}}^{*}\right\|^{2} .
\end{aligned}
$$

Using Lemma 1 (a), we obtain

$$
\begin{aligned}
& \left\|x_{k}-x_{\lambda_{k}}^{*}\right\|^{2} \leq\left(1+\frac{\gamma_{k} \lambda_{k} \mu_{h}}{2 m}\right)\left\|x_{k}-x_{\lambda_{k-1}}^{*}\right\|^{2} \\
& +\left(1+\frac{2 m}{\gamma_{k} \lambda_{k} \mu_{h}}\right) \frac{C_{h}^{2}}{\mu_{h}^{2}}\left|1-\frac{\lambda_{k-1}}{\lambda_{k}}\right|^{2} .
\end{aligned}
$$

Plugging this inequality into (8) we obtain

$$
\begin{aligned}
& \left\|x_{k+1}-x_{\lambda_{k}}^{*}\right\|^{2} \leq\left(1-\frac{\gamma_{k} \lambda_{k} \mu_{h}}{m}\right)\left(1+\frac{\gamma_{k} \lambda_{k} \mu_{h}}{2 m}\right)\left\|x_{k}-x_{\lambda_{k-1}}^{*}\right\|^{2} \\
& +\left(1-\frac{\gamma_{k} \lambda_{k} \mu_{h}}{m}\right)\left(1+\frac{2 m}{\gamma_{k} \lambda_{k} \mu_{h}}\right) \frac{C_{h}^{2}}{\mu_{h}^{2}}\left|1-\frac{\lambda_{k-1}}{\lambda_{k}}\right|^{2} \\
& +6 m^{2} \gamma_{k}^{2}\left(C_{f}^{2}+\lambda_{k}^{2} C_{h}^{2}\right) .
\end{aligned}
$$

The desired result is obtained from $0<\mu_{k} \lambda_{k} \mu_{h} \leq 2 m$. We will make use of the following result in Proposition 1

Lemma 3 (Lemma 10, pg. 49, [20]): Let $\left\{v_{k}\right\}$ be a sequence of nonnegative scalars and let $\left\{\alpha_{k}\right\}$ and $\left\{\beta_{k}\right\}$ be scalar sequences such that:

$$
\begin{aligned}
& v_{k+1} \leq\left(1-\alpha_{k}\right) v_{k}+\beta_{k} \quad \text { for all } k \geq 0 \\
& 0 \leq \alpha_{k} \leq 1, \quad \beta_{k} \geq 0, \sum_{k=0}^{\infty} \alpha_{k}=\infty, \sum_{k=0}^{\infty} \beta_{k}<\infty, \lim _{k \rightarrow \infty} \frac{\beta_{k}}{\alpha_{k}}=0 .
\end{aligned}
$$

Then, $\lim _{k \rightarrow \infty} v_{k}=0$.
Assumption 2: Assume that for all $k \geq 0$ we have
(a) $\left\{\gamma_{k}\right\}$ and $\left\{\lambda_{k}\right\}$ are non-increasing positive sequences with $\gamma_{0} \lambda_{0} \leq \frac{2 m}{\mu_{h}}$.
(b) $\sum_{k=0}^{\infty} \gamma_{k} \lambda_{k}=\infty$.
(c) $\sum_{k=0}^{\infty} \frac{1}{\gamma_{k} \lambda_{k}}\left(\frac{\lambda_{k-1}}{\lambda_{k}}-1\right)^{2}<\infty$.
(d) $\sum_{k=0}^{\infty} \gamma_{k}^{2}<\infty$.
(e) $\lim _{k \rightarrow \infty} \frac{1}{\gamma_{k}^{2} \lambda_{k}^{2}}\left(\frac{\lambda_{k-1}}{\lambda_{k}}-1\right)^{2}=0$.
(f) $\lim _{k \rightarrow \infty} \frac{\gamma_{k}}{\lambda_{k}}=0$.

Proposition 1 (Convergence of $\left\{\mathbf{x}_{\mathbf{k}}\right\}$ ): Consider
problem $\left.\sqrt{P_{f}^{h}}\right)$. Let Assumption 1 and 2 hold and $\left\{x_{k}\right\}$ be generated by Algorithm 1. Then,
(a) $\lim _{k \rightarrow \infty}\left\|x_{k}-x_{\lambda_{k-1}}^{*}\right\|^{2}=0$.
(b) If $\lim _{k \rightarrow \infty} \lambda_{k}=0, x_{k}$ converges to the unique optimal solution of problem $\left\langle P_{f}^{h}\right\rangle$, i.e., $x_{h}^{*}$.
Proof: (a) Consider the result from Lemma 2. We let

$$
\begin{aligned}
& v_{k} \triangleq\left\|x_{k}-x_{\lambda_{k-1}}^{*}\right\|^{2}, \alpha_{k} \triangleq \frac{\gamma_{k} \lambda_{k} \mu_{h}}{2 m} \\
& \beta_{k} \triangleq \frac{3 m C_{h}^{2}}{\gamma_{k} \lambda_{k} \mu_{h}^{3}}\left(1-\frac{\lambda_{k-1}}{\lambda_{k}}\right)^{2}+6 m^{2} \gamma_{k}^{2}\left(C_{f}^{2}+\lambda_{k}^{2} C_{h}^{2}\right)
\end{aligned}
$$

From Assumption $2(\mathrm{a}, \mathrm{b}, \mathrm{c})$, since $\left\{\gamma_{k}\right\}$ and $\left\{\lambda_{k}\right\}$ are positive and $\gamma_{0} \lambda_{0} \leq 2 m / \mu_{h}$, we have $0 \leq \alpha_{k} \leq 1$ and $\beta_{k} \geq 0$ and also $\sum_{k=1}^{\infty} \alpha_{k}=\infty$ and $\sum_{k=1}^{\infty} \beta_{k}<\infty$. To show that all the necessary assumptions for Lemma 3 are satisfied, we have

$$
\begin{aligned}
\lim _{k \rightarrow \infty} \frac{\beta_{k}}{\alpha_{k}} & =\frac{6 m^{2} C_{h}^{2}}{\mu_{h}^{4}} \lim _{k \rightarrow \infty} \frac{1}{\gamma_{k}^{2} \lambda_{k}^{2}}\left(\frac{\lambda_{k-1}}{\lambda_{k}}-1\right)^{2}+\frac{12 m^{3} C_{f}^{2}}{\mu_{h}} \lim _{k \rightarrow \infty} \frac{\gamma_{k}}{\lambda_{k}} \\
& +\frac{12 m^{3} C_{h}^{2}}{\mu_{h}} \lim _{k \rightarrow \infty} \gamma_{k} \lambda_{k}
\end{aligned}
$$

Considering Assumption 2(e,f), we only need to show that $\lim _{k \rightarrow \infty} \gamma_{k} \lambda_{k}=0$. Since $\left\{\lambda_{k}\right\}$ is non-increasing for all $k \geq 0$, we have $\lambda_{0}^{2} \gamma_{k} / \lambda_{k} \geq \gamma_{k} \lambda_{k}$. So by Assumption 2(f), $\lim _{k \rightarrow \infty} \gamma_{k} \lambda_{k}=0$. Consequently $\lim _{k \rightarrow \infty} \frac{\beta_{k}}{\alpha_{k}}=0$. Now Lemma 3 can be applied. We have

$$
\lim _{k \rightarrow \infty} v_{k}=\lim _{k \rightarrow \infty}\left\|x_{k}-x_{\lambda_{k-1}}^{*}\right\|^{2}=0
$$

(b) Applying the triangular inequality, we obtain
$\left\|x_{k}-x_{h}^{*}\right\|^{2} \leq 2\left\|x_{k}-x_{\lambda_{k-1}}^{*}\right\|^{2}+2\left\|x_{\lambda_{k-1}}^{*}-x_{h}^{*}\right\|^{2}$, for all $k \geq 0$.
From part (a), $\left\|x_{k}-x_{h}^{*}\right\|^{2}$ converges to zero. Also, from Lemma 1 (b), we know that when $\lambda_{k} \rightarrow 0$ the sequence $\left\{x_{\lambda_{k}}^{*}\right\}$ converges to the unique optimal solution of problem $\left\{P_{f}^{h}\right\}$, i.e., $x_{h}^{*}$. Therefore the result holds.

To have previous proposition work, we require that sequences $\left\{\gamma_{k}\right\}$ and $\left\{\lambda_{k}\right\}$ satisfy Assumption 2. Below, we provide a
set of feasible sequences for this assumption. The proof is analogous to that of Lemma 5 in [29].

Lemma 4: Assume $\left\{\gamma_{k}\right\}$ and $\left\{\lambda_{k}\right\}$ are sequences such that $\gamma_{k}=\frac{\gamma_{0}}{(k+1)^{a}}$ and $\lambda_{k}=\frac{\lambda_{0}}{(k+1)^{b}}$ where $a, b, \gamma_{0}$ and $\lambda_{0}$ are positive scalars and $\gamma_{0} \lambda_{0} \leq \frac{2 m}{\mu_{h}}$. If $a>b, a>0.5$ and $a+b<1$, then the sequences $\left\{\gamma_{k}\right\}$ and $\left\{\lambda_{k}\right\}$ satisfy Assumption 2 .
The following is a useful lemma in proving convergence that we will apply in Theorem 1.
Lemma 5 (Theorem 6, pg. 75 of [12]): Let $\left\{u_{t}\right\} \subset \mathbb{R}^{n}$ be a convergent sequence with the limit point $\hat{u} \in \mathbb{R}^{n}$ and let $\left\{\alpha_{k}\right\}$ be a sequence of positive numbers where $\sum_{k=0}^{\infty} \alpha_{k}=\infty$. Suppose $\left\{v_{k}\right\}$ is given by $v_{k} \triangleq\left(\sum_{t=0}^{k-1} \alpha_{t} u_{t}\right) / \sum_{t=0}^{k-1} \alpha_{t}$ for all $k \geq 1$. Then, $\lim _{k \rightarrow \infty} v_{k}=\hat{u}$.
Now, we can illustrate our ultimate goal in this section which is showing the convergence of the sequence $\left\{\bar{x}_{k}\right\}$ generated by Algorithm 1 to $x_{h}^{*}$

Theorem 1 (Convergence of $\left\{\overline{\mathbf{x}}_{\mathbf{k}}\right\}$ ): Consider problem $\left(P_{f}^{h}\right.$. Let Assumption 1 hold. Also assume $\left\{\gamma_{k}\right\}$ and $\left\{\lambda_{k}\right\}$ are sequences such that $\gamma_{k}=\frac{\gamma_{0}}{(k+1)^{a}}$ and $\lambda_{k}=\frac{\lambda_{0}}{(k+1)^{b}}$ where $a, b, \gamma_{0}$ and $\lambda_{0}$ are positive scalars and $\gamma_{0} \lambda_{0} \leq \frac{2 m}{\mu_{h}}$. Let $\left\{\bar{x}_{k}\right\}$ be generated by Algorithm 1. If $a>b, a>0.5, a+b<1$ and $a r \leq 1$, then $\left\{\bar{x}_{k}\right\}$ converges to $x_{h}^{*}$.

Proof: Considering the given assumptions, by Lemma 4 we can see that Assumption 2 holds. We have

$$
\left\|\bar{x}_{k+1}-x_{h}^{*}\right\|=\left\|\sum_{t=0}^{k} \psi_{t, k} x_{t}-\sum_{t=0}^{k} \psi_{t, k} x_{h}^{*}\right\| \leq \sum_{t=0}^{k} \psi_{t, k}\left\|x_{t}-x_{h}^{*}\right\|
$$

applying $\sum_{t=0}^{k} \psi_{t, k}=1$ from (4) and the triangular inequality. Now consider the definition of $\psi_{k}$ and let $\alpha_{t} \triangleq \gamma_{t}^{r}, u_{t} \triangleq \| x_{t}-$ $x_{h}^{*} \|$ and $v_{k+1} \triangleq \sum_{t=0}^{k} \psi_{t, k}\left\|x_{t}-x_{h}^{*}\right\|$. Since ar $\leq 1$ we have $\sum_{t=0}^{\infty} \alpha_{t}=\sum_{t=0}^{\infty} \gamma_{t}^{r}=\sum_{t=0}^{\infty}(t+1)^{-a r}=\infty$. The sequence $\left\{\lambda_{t}\right\}$ is decreasing to zero due to $b>0$, So, from Proposition 1(b), $u_{t}=\left\|x_{t}-x_{h}^{*}\right\|$ converges to zero. Therefore, for $\hat{u}=0$ we can apply Lemma 5 and thus, $\left\|\bar{x}_{k+1}-x_{h}^{*}\right\|$ converges to zero.

## IV. Rate analysis

In this section, we first find an error bound with respect to the optimal values of the lower level function $f$ which indeed shows the feasibility of the problem $\left(P_{f}^{h}\right)$. Then, we apply it to derive a convergence rate for the algorithm.

Lemma 6: Consider the sequence $\left\{\bar{x}_{N}\right\}$ generated by Algorithm 1 Let Assumption 1 hold and $\left\{\gamma_{k}\right\}$ and $\left\{\lambda_{k}\right\}$ be positive and non-increasing sequences. Then, for all $N \geq 1$ and $z \in X$ we have

$$
\begin{aligned}
& f\left(\bar{x}_{N}\right)-f^{*} \leq\left(\sum_{k=0}^{N-1} \gamma_{k}^{r}\right)^{-1}\left(m \sum_{k=0}^{N-1} \gamma_{k}^{r+1}\left(C_{f}^{2}+\lambda_{k}^{2} C_{h}^{2}\right)\right. \\
& \left.+m^{2} C_{f} \sum_{k=0}^{N-1} \gamma_{k}^{r+1}\left(C_{f}+\lambda_{k} C_{h}\right)+2 M_{h} \sum_{k=0}^{N-1} \gamma_{k}^{r} \lambda_{k}+2 M^{2} \gamma_{N-1}^{r-1}\right),
\end{aligned}
$$

where $M_{h}, M$ are scalars such that $\|h(x)\| \leq M_{h},\|x\| \leq M$ for all $x \in X$.

Proof: Similar to relation (5), we can have

$$
\begin{aligned}
& \left\|x_{k+1}-x^{*}\right\|^{2} \leq\left\|x_{k}-x^{*}\right\|^{2}+2 m \gamma_{k}^{2}\left(C_{f}^{2}+\lambda_{k}^{2} C_{h}^{2}\right) \\
& -2 \gamma_{k} \sum_{i=0}^{m-1}\left(f_{i+1}\left(x_{k, i}\right)+\frac{\lambda_{k}}{m} h\left(x_{k, i}\right)\right)+2 \gamma_{k}\left(f^{*}+\lambda_{k} h\left(x^{*}\right)\right) \\
& \leq\left\|x_{k}-x^{*}\right\|^{2}+2 m \gamma_{k}^{2}\left(C_{f}^{2}+\lambda_{k}^{2} C_{h}^{2}\right) \\
& -2 \gamma_{k} \sum_{i=0}^{m-1} f_{i+1}\left(x_{k, i}\right)+2 \gamma_{k} f^{*}+4 \gamma_{k} \lambda_{k} M_{h}
\end{aligned}
$$

Adding and subtracting $f\left(x_{k}\right)$, we obtain

$$
\begin{aligned}
& \left\|x_{k+1}-x^{*}\right\|^{2} \leq\left\|x_{k}-x^{*}\right\|^{2}+2 m \gamma_{k}^{2}\left(C_{f}^{2}+\lambda_{k}^{2} C_{h}^{2}\right) \\
& -2 \gamma_{k} \sum_{i=0}^{m-1}\left(f_{i+1}\left(x_{k, i}\right)-f_{i}\left(x_{k}\right)\right)+2 \gamma_{k}\left(f^{*}-f\left(x_{k}\right)\right)+4 \gamma_{k} \lambda_{k} M_{h} \\
& \leq\left\|x_{k}-x^{*}\right\|^{2}+2 m \gamma_{k}^{2}\left(C_{f}^{2}+\lambda_{k}^{2} C_{h}^{2}\right) \\
& +2 \gamma_{k} \sum_{i=0}^{m-1}\left|f_{i+1}\left(x_{k, i}\right)-f_{i}\left(x_{k}\right)\right|+2 \gamma_{k}\left(f^{*}-f\left(x_{k}\right)\right)+4 \gamma_{k} \lambda_{k} M_{h}
\end{aligned}
$$

Applying Remark 2(b), $\left|f_{i+1}\left(x_{k, i}\right)-f_{i}\left(x_{k}\right)\right| \leq C_{f}\left\|x_{k, i}-x_{k}\right\|$, we have

$$
\begin{aligned}
& \left\|x_{k+1}-x^{*}\right\|^{2} \leq\left\|x_{k}-x^{*}\right\|^{2}+2 m \gamma_{k}^{2}\left(C_{f}^{2}+\lambda_{k}^{2} C_{h}^{2}\right) \\
& +2 C_{f} \gamma_{k} \sum_{i=0}^{m-1}\left\|x_{k, i}-x_{k}\right\|+2 \gamma_{k}\left(f^{*}-f\left(x_{k}\right)\right)+4 \gamma_{k} \lambda_{k} M_{h}
\end{aligned}
$$

Using the inequality (7), we obtain

$$
\begin{align*}
& \left\|x_{k+1}-x^{*}\right\|^{2}-\left\|x_{k}-x^{*}\right\|^{2} \leq 2 \gamma_{k}\left(f^{*}-f\left(x_{k}\right)\right) \\
& +2 m \gamma_{k}^{2}\left(C_{f}^{2}+\lambda_{k}^{2} C_{h}^{2}\right)+2 m^{2} C_{f} \gamma_{k}^{2}\left(C_{f}+\lambda_{k} C_{h}\right)+4 \gamma_{k} \lambda_{k} M_{h} \tag{9}
\end{align*}
$$

Multiplying both sides by $\gamma_{k}^{r-1}$ and adding and subtracting $\gamma_{k-1}^{r-1}\left\|x_{k}-x^{*}\right\|^{2}$ to the left hand side, we have
$\gamma_{k}^{r-1}\left\|x_{k+1}-x^{*}\right\|^{2}-\gamma_{k-1}^{r-1}\left\|x_{k}-x^{*}\right\|^{2}+\left(\gamma_{k-1}^{r-1}-\gamma_{k}^{r-1}\right)\left\|x_{k}-x^{*}\right\|^{2}$
$\leq 2 \gamma_{k}^{r}\left(f^{*}-f\left(x_{k}\right)\right)+2 m \gamma_{k}^{r+1}\left(C_{f}^{2}+\lambda_{k}^{2} C_{h}^{2}\right)$
$+2 m^{2} C_{f} \gamma_{k}^{r+1}\left(C_{f}+\lambda_{k} C_{h}\right)+4 \gamma_{k}^{r} \lambda_{k} M_{h}$.
Since $\left\{\gamma_{k}\right\}$ in non-increasing and $r<1$ we have $\gamma_{k-1}^{r-1} \leq$ $\gamma_{k}^{r-1}$. Also, by the triangle inequality $\left\|x_{k}-x^{*}\right\|^{2} \leq 2\left\|x_{k}\right\|^{2}+$ $2\left\|x^{*}\right\|^{2} \leq 4 M^{2}$. So, we obtain

$$
\begin{aligned}
& \gamma_{k}^{r-1}\left\|x_{k+1}-x^{*}\right\|^{2}-\gamma_{k-1}^{r-1}\left\|x_{k}-x^{*}\right\|^{2}+4 M^{2}\left(\gamma_{k-1}^{r-1}-\gamma_{k}^{r-1}\right) \\
& \leq 2 \gamma_{k}^{r}\left(f^{*}-f\left(x_{k}\right)\right)+2 m \gamma_{k}^{r+1}\left(C_{f}^{2}+\lambda_{k}^{2} C_{h}^{2}\right) \\
& +2 m^{2} C_{f} \gamma_{k}^{r+1}\left(C_{f}+\lambda_{k} C_{h}\right)+4 \gamma_{k}^{r} \lambda_{k} M_{h} .
\end{aligned}
$$

Taking summation over $k=1,2, \cdots, N-1$, we obtain

$$
\begin{aligned}
& \gamma_{N-1}^{r-1}\left\|x_{N}-x^{*}\right\|^{2}-\gamma_{0}^{r-1}\left\|x_{1}-x^{*}\right\|^{2}+4 M^{2}\left(\gamma_{0}^{r-1}-\gamma_{N-1}^{r-1}\right) \\
& \leq 2 \sum_{k=1}^{N-1} \gamma_{k}^{r}\left(f^{*}-f\left(x_{k}\right)\right)+2 m \sum_{k=1}^{N-1} \gamma_{k}^{r+1}\left(C_{f}^{2}+\lambda_{k}^{2} C_{h}^{2}\right) \\
& +2 m^{2} C_{f} \sum_{k=1}^{N-1} \gamma_{k}^{r+1}\left(C_{f}+\lambda_{k} C_{h}\right)+4 M_{h} \sum_{k=1}^{N-1} \gamma_{k}^{r} \lambda_{k}
\end{aligned}
$$

Removing non-negative terms from the left-hand side of the preceding inequality, we have

$$
\begin{align*}
& -\gamma_{0}^{r-1}\left\|x_{1}-x^{*}\right\|^{2}-4 M^{2} \gamma_{N-1}^{r-1} \leq 2 \sum_{k=1}^{N-1} \gamma_{k}^{r}\left(f^{*}-f\left(x_{k}\right)\right) \\
& +2 m \sum_{k=1}^{N-1} \gamma_{k}^{r+1}\left(C_{f}^{2}+\lambda_{k}^{2} C_{h}^{2}\right)+2 m^{2} C_{f} \sum_{k=1}^{N-1} \gamma_{k}^{r+1}\left(C_{f}+\lambda_{k} C_{h}\right) \\
& +4 M_{h} \sum_{k=1}^{N-1} \gamma_{k}^{r} \lambda_{k} . \tag{10}
\end{align*}
$$

From (9) for $k=0$, we obtain

$$
\begin{aligned}
\left\|x_{1}-x^{*}\right\|^{2} & \leq 2 \gamma_{0}\left(f^{*}-f\left(x_{0}\right)\right)+2 m \gamma_{0}^{2}\left(C_{f}^{2}+\lambda_{0}^{2} C_{h}^{2}\right) \\
& +2 m^{2} C_{f} \gamma_{0}^{2}\left(C_{f}+\lambda_{0} C_{h}\right)+4 \gamma_{0} \lambda_{0} M_{h}+4 M^{2}
\end{aligned}
$$

By multiplying both sides of the preceding inequality by $\gamma_{0}^{r-1}$ and summing it with the relation (10), we obtain

$$
\begin{aligned}
& -4 M^{2} \gamma_{N-1}^{r-1} \leq 2 \sum_{k=0}^{N-1} \gamma_{k}^{r}\left(f^{*}-f\left(x_{k}\right)\right)+4 M_{h} \sum_{k=0}^{N-1} \gamma_{k}^{r} \lambda_{k} \\
& +2 m \sum_{k=0}^{N-1} \gamma_{k}^{r+1}\left(C_{f}^{2}+\lambda_{k}^{2} C_{h}^{2}\right)+2 m^{2} C_{f} \sum_{k=0}^{N-1} \gamma_{k}^{r+1}\left(C_{f}+\lambda_{k} C_{h}\right)
\end{aligned}
$$

Rearranging the terms we have

$$
\begin{aligned}
& 2 \sum_{k=0}^{N-1} \gamma_{k}^{r}\left(f\left(x_{k}\right)-f^{*}\right) \leq 2 m \sum_{k=0}^{N-1} \gamma_{k}^{r+1}\left(C_{f}^{2}+\lambda_{k}^{2} C_{h}^{2}\right) \\
& +2 m^{2} C_{f} \sum_{k=0}^{N-1} \gamma_{k}^{r+1}\left(C_{f}+\lambda_{k} C_{h}\right)+4 M_{h} \sum_{k=0}^{N-1} \gamma_{k}^{r} \lambda_{k}+4 M^{2} \gamma_{N-1}^{r-1} .
\end{aligned}
$$

Now we divide both sides by $2 \sum_{k=0}^{N-1} \gamma_{k}^{r}$ and use the definition of $\psi_{k, N-1}$ in (4),

$$
\begin{aligned}
& \sum_{k=0}^{N-1} \psi_{k, N-1}\left(f\left(x_{k}\right)-f^{*}\right) \\
& \leq\left(\sum_{k=0}^{N-1} \gamma_{k}^{r}\right)^{-1}\left(m \sum_{k=0}^{N-1} \gamma_{k}^{r+1}\left(C_{f}^{2}+\lambda_{k}^{2} C_{h}^{2}\right)\right. \\
& \left.+m^{2} C_{f} \sum_{k=0}^{N-1} \gamma_{k}^{r+1}\left(C_{f}+\lambda_{k} C_{h}\right)+2 M_{h} \sum_{k=0}^{N-1} \gamma_{k}^{r} \lambda_{k}+2 M^{2} \gamma_{N-1}^{r-1}\right) .
\end{aligned}
$$

We know $\sum_{k=0}^{N-1} \psi_{k, N-1}=1$ also $f\left(\bar{x}_{N}\right) \leq \sum_{k=0}^{N-1} \psi_{k, N-1} f\left(x_{k}\right)$ because of convexity of $f$. So, we obtain the desired result.

The following lemma, will be used to find a convergence rate statement in Theorem 2
Lemma 7 (Lemma 9, page 418 in [29]): For any scalar $\alpha \neq-1$ and integers $\ell$ and $N$ where $0 \leq \ell \leq N-1$, we have

$$
\begin{aligned}
\frac{N^{\alpha+1}-(\ell+1)^{\alpha+1}}{\alpha+1} & \leq \sum_{k=\ell}^{N-1}(k+1)^{\alpha} \\
& \leq(\ell+1)^{\alpha}+\frac{(N+1)^{\alpha+1}-(\ell+1)^{\alpha+1}}{\alpha+1}
\end{aligned}
$$

In the following theorem, we present a rate statement for Algorithm 1.

Theorem 2 (A rate statement for Algorithm (1):
Assume $\left\{\bar{x}_{N}\right\}$ is generated by Algorithm 1 to solve problem $\left(P_{f}^{h}\right)$. Let Assumption 1 and 2 hold and also $0<\varepsilon<0.5$ and $r<1$ be arbitrary constants. Assume for $0<\varepsilon<0.5$, $\left\{\gamma_{k}\right\}$ and $\left\{\lambda_{k}\right\}$ are sequences defined as

$$
\gamma_{k}=\frac{\gamma_{0}}{(k+1)^{0.5+0.5 \varepsilon}} \text { and } \lambda_{k}=\frac{\lambda_{0}}{(k+1)^{0.5-\varepsilon}}
$$

such that $\gamma_{0}$ and $\lambda_{0}$ are positive scalars and $\gamma_{0} \lambda_{0} \mu_{h} \leq 2 m$. Then,
(a) The sequence $\left\{\bar{x}_{N}\right\}$ converges to the unique optimal solution of problem $\left\langle P_{f}^{h}\right\rangle$, i.e., $x_{h}^{*}$.
(b) $f\left(\bar{x}_{N}\right)$ converges to $f^{*}$ with the rate $\mathscr{O}\left(1 / N^{0.5-\varepsilon}\right)$.

Proof: Throughout, we set $a:=0.5+0.5 \varepsilon, b:=0.5-\varepsilon$.
(a) From the values of $a$ and $b$, and that $r<1$ and $0<\varepsilon<0.5$, we have

$$
\begin{array}{r}
a>b>0, a>0.5, a+b=1-0.5 \varepsilon<1, \\
a r=0.5(1+\varepsilon) r<0.5(1.5)=0.75<1 .
\end{array}
$$

This implies that all conditions of Theorem 1 are satisfied. Therefore, $\left\{\bar{x}_{N}\right\}$ converges to $x_{h}^{*}$ almost surely.
(b) Since $\left\{\lambda_{k}\right\}$ is a non-increasing sequence from Lemma 6 we have

$$
\begin{aligned}
& f\left(\bar{x}_{N}\right)-f^{*} \leq\left(\sum_{k=0}^{N-1} \gamma_{k}^{r}\right)^{-1}\left(m\left(C_{f}^{2}+\lambda_{0}^{2} C_{h}^{2}\right) \sum_{k=0}^{N-1} \gamma_{k}^{r+1}\right. \\
& \left.+m^{2} C_{f}\left(C_{f}+\lambda_{0} C_{h}\right) \sum_{k=0}^{N-1} \gamma_{k}^{r+1}+2 M_{h} \sum_{k=0}^{N-1} \lambda_{k} \gamma_{k}^{r}+2 M^{2} \gamma_{N-1}^{r-1}\right)
\end{aligned}
$$

We have $\gamma_{k}=\gamma_{0} /(k+1)^{a}$ and $\lambda_{k}=\lambda_{0} /(k+1)^{b}$, thus

$$
\begin{aligned}
& f\left(\bar{x}_{N}\right)-f^{*} \\
& \leq\left(\sum_{k=0}^{N-1} \frac{\gamma_{0}^{r}}{(k+1)^{a r}}\right)^{-1}\left(m\left(C_{f}^{2}+\lambda_{0}^{2} C_{h}^{2}\right) \sum_{k=0}^{N-1} \frac{\gamma_{0}^{r+1}}{(k+1)^{a(r+1)}}\right. \\
& +m^{2} C_{f}\left(C_{f}+\lambda_{0} C_{h}\right) \sum_{k=0}^{N-1} \frac{\gamma_{0}^{r+1}}{(k+1)^{a(r+1)}}+2 M_{h} \sum_{k=0}^{N-1} \frac{\lambda_{0} \gamma_{0}^{r}}{(k+1)^{a r+b}}
\end{aligned}
$$

$$
\left.+2 M^{2} \gamma_{0}^{r-1} N^{a(1-r)}\right)
$$

Rearranging the terms, we have

$$
\begin{aligned}
& f\left(\bar{x}_{N}\right)-f^{*} \leq\left(\sum_{k=0}^{N-1} \frac{\gamma_{0}^{r}}{(k+1)^{a r}}\right)^{-1} \\
& \times\left(2 M_{h} \sum_{k=0}^{N-1} \frac{\lambda_{0} \gamma_{0}^{r}}{(k+1)^{a r+b}}+2 M^{2} \gamma_{0}^{r-1} N^{a(1-r)}\right. \\
& \left.+\left(m^{2} C_{f}\left(C_{f}+\lambda_{0} C_{h}\right)+m\left(C_{f}^{2}+\lambda_{0}^{2} C_{h}^{2}\right)\right) \sum_{k=0}^{N-1} \frac{\gamma_{0}^{r+1}}{(k+1)^{a(r+1)}}\right)
\end{aligned}
$$

Let us define

$$
\begin{aligned}
& \text { Term1 }=\left(\sum_{k=0}^{N-1} \frac{1}{(k+1)^{a r}}\right)^{-1} N^{a(1-r)} \\
& \text { Term2 }=\left(\sum_{k=0}^{N-1} \frac{1}{(k+1)^{a r}}\right)^{-1}\left(\sum_{k=0}^{N-1} \frac{1}{(k+1)^{a r+b}}\right) .
\end{aligned}
$$

We have

$$
\begin{aligned}
& \text { Term } 1 \leq \frac{N^{a(1-r)}}{\frac{N^{1-a r}-1}{1-a r}}=\frac{(1-a r) N^{a(1-r)}}{N^{1-a r}-1}=\mathscr{O}\left(N^{-(1-a)}\right) \\
& \text { Term } 2 \leq \frac{\frac{(N+1)^{1-a r-b}-1}{1-a r-b}+1}{\frac{N^{1-a r}-1}{1-a r}}=\frac{(1-a r)\left((N+1)^{1-a r-b}-1\right)}{(1-a r-b)\left(N^{1-a r}-1\right)} \\
& +\frac{1-a r}{N^{1-a r}-1}=\mathscr{O}\left(N^{-b}\right)+\mathscr{O}\left(N^{-(1-a r)}\right)
\end{aligned}
$$

So, we have

$$
f\left(\bar{x}_{N}\right)-f^{*} \leq \mathscr{O}\left(N^{-\min \{1-a r, 1-a, b\}}\right)=\mathscr{O}\left(N^{-\min \{1-a, b\}}\right),
$$

where we used $1-a \leq 1-a r$. Replacing $a$ and $b$ by their values, we have
$f\left(\bar{x}_{N}\right)-f^{*} \leq \mathscr{O}\left(N^{-\min \{0.5-0.5 \varepsilon, 0.5-\varepsilon\}}\right)=\mathscr{O}\left(N^{-(0.5-\varepsilon)}\right)$.


Fig. 1: IR-IG performane for binary text classification problem with respect to different initial point $x_{0}$, parameters $\left(\gamma_{0}, \lambda_{0}\right)$ and $r$. The vertical axis represents the $\log$ of average loss and the horizontal axis is the number of iterations.

## V. Numerical results

In this section, we apply the IR-IG method on a text classification problem. In this problem we assume to have a summation of hinge loss function, i.e., $\mathscr{L}(\langle x, a\rangle, b) \triangleq$ $\max \{0,1-b\langle x, a\rangle\}$ for any sample $a, b$ in the lower level of $\left(P_{f}^{h}\right)$. The set of observations $\left(a_{i}, b_{i}\right)$ are derived from the Reuters Corpus Volume I (RCV1) dataset (see [14]). This dataset has categorized Reuters articles, from 1996 to 1997, into four groups: Corporate/Industrial, Economics, Government/Social and Markets. In this application, we use a subset of this dataset with $N=50,000$ articles and 138,921 tokens to perform a binary classification of articles only with
respect to the Markets class. Each vector $a_{i}$ represents the existence of all tokens in article $i$, and $b_{i}$ shows whether the article belongs to the Markets class. To decide that a new article can be placed in the Markets class, the problem in the lower level should be solved such that the optimal solution is a weight vector of the tokens regarding the Markets class which minimizes the total loss. However, to make such a decision, an optimal solution with a large number of nonzero components are undesirable. To induce sparsity, we consider the following bilevel problem:

$$
\begin{equation*}
\operatorname{minimize} h(x) \triangleq \frac{\mu_{h}}{2}\|x\|_{2}^{2}+\|x\|_{1} \tag{11}
\end{equation*}
$$

subject to $x \in \arg \min _{y \in \mathbb{R}^{n}} \sum_{i=1}^{m} \sum_{j=1}^{N / m} \mathscr{L}\left(a_{(i-1) N / m+j}^{T} y, b_{(i-1) N / m+j}\right)$,
where, we let $\mu_{h}=0.1$ and we consider each batch of $N / m=$ 1000 articles to be one component function with the total number of component functions $m=50$. The function $h$ is strongly convex with parameter $\mu_{h}$. We let $\gamma_{k}$ and $\lambda_{k}$ be given by the rules in Theorem 2 We study the sensitivity of the method by changing $x_{0}, \gamma_{0}, \lambda_{0}$, and the averaging parameter $r<1$. We finally report the logarithm of average of the loss function $\mathscr{L}$. The plots in Fig. 1 show the convergence of the IR-IG method for the problem (11). The results show the convergece of Algorithm 1 with different initial values such as the starting point or parameters $\gamma_{0}, \lambda_{0}$ while when we pick a smaller $r$ the algorithm is faster in all the cases.

## VI. Concluding Remarks

Motivated by the applications of incremental gradient schemes in distributed optimization, especially in machine learning and large data training, we develop an iterative regularized incremental first-order method, called IR-IG, for solving a class of bilevel convex optimization. We prove the convergence of IR-IG and establish the corresponding rate in terms of the lower level objective function. We finally apply IR-IG to a binary text classification problem and demonstrate the performance of the proposed algorithm.

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