An Approach to Duality in Nonlinear Filtering

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Abstract— This paper revisits the question of duality between minimum variance estimation and optimal control first described for the linear Gaussian case in the celebrated paper of Kalman and Bucy. A duality result is established for nonlinear filtering, mirroring closely the original Kalman-Bucy duality of control and estimation for linear systems. The result for the finite state-space continuous time Markov chain is presented. It's solution is used to derive the classical Wonham filter.

I. INTRODUCTION

In Kalman's celebrated paper with Bucy, it is shown that the problem of optimal estimation is dual to an optimal control problem [1]. A striking example of the dual relationship is that, with the time arrow reversed, the dynamic Riccati equation (DRE) of the optimal control is the same as the covariance update equation of the Kalman filter. The relationship is useful, e.g., to derive results on asymptotic stability of the linear filter based on asymptotic properties of the solution of the DRE [2].

A nonlinear extension of the minimum variance estimator has been considered to be a harder problem. In literature, it has been noted that: i) the dual relationship between the DRE of the LQ optimal control and the covariance update equation of the Kalman filter is *not* consistent with the interpretation of the negative log-posterior as a value function; and ii) some of the linear algebraic operations, e.g., the use of matrix transpose to define the dual system, are not applicable to nonlinear systems [3], [4]. For these reasons, the original duality of Kalman-Bucy is seen as an LQG artifact that does not generalize [3].

In this paper, a nonlinear extension of the minimum variance estimation is presented for the special case of a Markov process in continuous time, on a finite state-space. The dual system is a backward ordinary differential equation. An optimal control objective is formulated whose solution yields the minimum variance estimator. Using the elementary method of change of control, the formula for the optimal control is obtained and used to derive the classical Wonham filter.

The outline of the paper is as follows: classical duality is reviewed in Sec. II, and the new dual optimal control problem for the finite case is described in Sec. III. Its solution leading to the Wonham filter is presented in Sec. IV.

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II. BACKGROUND ON CLASSICAL DUALITY

Linear Gaussian filtering model: Specified by the linear stochastic differential equation (SDE):

Signal	$\mathrm{d}X_t = A^\top X_t \mathrm{d}t + \mathrm{d}B_t$
Observation	$\mathrm{d}Z_t = H^\top X_t \mathrm{d}t + \mathrm{d}W_t$

where $X_t \in \mathbb{R}^d$ is the state at time $t, Z_t \in \mathbb{R}^m$ is the observation, *A*, *H* are matrices of appropriate dimension, and *B*, *W* are mutually independent Wiener processes (w.p.) taking values in \mathbb{R}^d and \mathbb{R}^m , respectively. The covariance matrices associated with *B* and *W* are denoted by *Q* and *R*, respectively. The initial condition X_0 is drawn from a Gaussian distribution $\mathcal{N}(\hat{x}_0, \Sigma_0)$, independent of *B* or *W*. It is assumed that the noise covariance matrix is non-singular, R > 0.

Minimum-variance estimator: Consider the problem of constructing a minimum variance estimator for the random variable $f^{\top}X_T$, at some fixed time T, where $f \in \mathbb{R}^d$ is an arbitrary, known vector.

Given the observations $\{Z_t : t \in [0,T]\}$, the following linear structure for the optimal estimator is assumed:

$$S_T = y_0^{\mathsf{T}} \hat{x}_0 - \int_0^T u_t^{\mathsf{T}} \, \mathrm{d}Z_t$$

where $y_0 \in \mathbb{R}^d$ is constructed below, and the input $u = \{u_t : t \in [0,T]\}$ is chosen to solve the optimization problem,

$$\min_{u} \mathsf{E}(|S_T - f^{\mathsf{T}}X_T|^2)$$

The solution S_T^* coincides with the minimum-variance estimator of $f^{\mathsf{T}}X_T$.

This stochastic optimization problem is converted to a deterministic optimal control problem via duality.

Dual optimal control problem:

Minimize
$$J(u) = \frac{1}{2} y_0^T \Sigma_0 y_0 + \int_0^T \frac{1}{2} u_t^T R u_t + \frac{1}{2} y_t^T Q y_t dt$$

Subject to $\frac{dy_t}{dt} = -Ay_t - Hu_t, \quad y_T = f$

The process $\{y_t : t \in [0,T]\}$ is referred to as the dual process. The solution of the optimal control problem yields the optimal control input, along with the vector y_0 that determines the minimum-variance estimator S_T^* .

The Kalman filter is obtained by expressing $\{S_t^*(f) : t \ge 0, f \in \mathbb{R}^d\}$ as the solution to a linear SDE [5, Ch. 7].

III. DUALITY FOR NONLINEAR FILTERING: THE FINITE STATE SPACE CASE

Nonlinear filtering model: The finite state-space filtering problem is considered, in which the state-space is the canonical basis $\mathbb{S} = \{e_1, e_2, \dots, e_d\}$ in \mathbb{R}^d .

The Markovian state process $X = \{X_t : t \in [0, T]\}$ evolves in continuous time, taking values in S. This and the observation process $Z = \{Z_t : t \in [0, T]\}$ are modeled by the SDE,

Signal $dX_t = A^{\top} X_t dt + dB_t$ (1a)

 $\mathrm{d}Z_t = H^\top X_t \,\mathrm{d}t + \mathrm{d}W_t$

(1b)

Observation

where $A \in \mathbb{R}^{d \times d}$ is the *rate matrix*, $H \in \mathbb{R}^{d \times m}$, W is an *m*-dimensional w.p. with covariance R > 0. $B = \{B_t : t \in [0, T]\}$ is defined by

$$B_t = X_t - \int_0^t A^\top X_\tau \,\mathrm{d}\tau$$

and it is a martingale since *A* is the generator of the Markov process. The initial distribution for X_0 is denoted $\pi_0 \in \mathcal{P}(\mathbb{S})$ where $\mathcal{P}(\mathbb{S})$ denotes the probability simplex in \mathbb{R}^d . It is assumed that *X*, *W* are mutually independent.

The linear observation model is chosen without loss of generality: for any function $h : \mathbb{S} \to \mathbb{R}$ we have $h(x) = H^{\top}x$ for $x \in \mathbb{S}$, with $H_i = h(e_i)$.

Two filtrations are required in this work: $\mathcal{F} = \{\mathcal{F}_t : t \ge 0\}$ and $\mathcal{Z} = \{\mathcal{Z}_t : t \ge 0\}$ where

$$\mathcal{F}_t \coloneqq \sigma(X_\tau, W_\tau : 0 \le \tau \le t), \quad \mathcal{Z}_t = \sigma(Z_\tau : 0 \le \tau \le t)$$

Let $C_{\mathcal{Z}}^p$ denote the family of \mathbb{R}^p -valued, continuous, and \mathcal{Z} -adapted functions of time (the superscript "*p*" is omitted in the special case p = 1).

The filtering problem is to compute the posterior distribution $P(X_t \in \cdot | Z_t)$ [6]. The solution is derived here through duality, very much like in the classical linear setting.

The dual system: A backward ordinary differential equation (ODE) on \mathbb{R}^d ,

$$\frac{\mathrm{d}Y_t}{\mathrm{d}t} = -AY_t - HU_t, \quad Y_T = f \tag{2}$$

whose solution is

$$Y_t = e^{A(T-t)} f + \int_t^T e^{A(\tau-t)} H U_\tau \,\mathrm{d}\tau, \quad 0 \le t \le T$$

An optimal control problem is posed for the dual system (2) whose solution yields the nonlinear filter. This requires some restrictions on the class of control inputs. The set of *admissible control inputs* is defined as follows:

$$\mathcal{U} \coloneqq \left\{ U_t = \mathsf{K}_t^\top Y_t + V_t : \mathsf{K} \in C_{\mathcal{Z}}^{d \times m}, \, V \in C_{\mathcal{Z}}^m, \, t \in [0, T] \right\}$$
(3)

We denote $U = \{U_t : t \in [0,T]\}$, $\mathsf{K} = \{\mathsf{K}_t : t \in [0,T]\}$ and $V = \{V_t : t \in [0,T]\}$. By construction, K and V and \mathcal{Z} -adapted processes but U may not be \mathcal{Z} -adapted because of the backward nature of the ODE (2).

The following proposition provides explicit representations for the solution of the backward ODE (2). Its proof appears in Appendix A. Proposition 1: Consider the backward ODE (2) with control input $U_t = \mathsf{K}_t^\top Y_t + V_t$ where $\{\mathsf{K}_t : t \in [0,T]\}$ and $\{V_t : t \in [0,T]\}$ are given \mathcal{Z} -adapted processes. Then there exist \mathcal{Z} adapted processes $\{\Phi_t, \eta_t, \kappa_t, \gamma_t : t \in [0,T]\}$, and $Y_0 \in \mathcal{Z}_T$, such that for each $t \in [0,T]$,

$$Y_t = \Phi_t Y_0 + \eta_t, \quad U_t = \kappa_t^{\mathsf{T}} Y_0 + \gamma_t$$

This proposition is used to define stochastic integral being used throughout the paper which is illustrated in the Appendix B.

Minimum-variance estimator: The problem of interest is precisely as in the linear Gaussian case: given a fixed time T > 0, and $f \in \mathbb{R}^d$, the goal is to obtain a representation for the minimum variance estimator for the random variable $f^{\top}X_T$.

Given observations $Z = \{Z_t : 0 \le t \le T\}$ defined according to the model (1b), the following linear structure for the estimator will be justified:

$$S_T = Y_0^{\mathsf{T}} \pi_0 - \int_0^T U_t^{\mathsf{T}} \,\mathrm{d}Z_t \tag{4}$$

The vector Y_0 is obtained from the solution to (2).

The optimal control input is chosen as the solution to the optimization problem:

$$\min_{U \in \mathcal{U}} \mathsf{E}[|S_T - f^{\mathsf{T}} X_T|^2]$$

Justification for the form (4) is provided through the formulation of the dual control problem.

Remark 1: The stochastic integral $\int_0^T U_t^{\mathsf{T}} dZ_t$ in (4) is defined as a forward integral. Formally, for a given admissible choice of \mathcal{Z} -adapted processes K and V, upon using the representation in Prop. 1,

$$\int_0^T U_t^\top \, \mathrm{d} Z_t = Y_0^\top \int_0^T \kappa_t \, \mathrm{d} Z_t + \int_0^T \gamma_t^\top \, \mathrm{d} Z_t$$

where { $\kappa_t : t \in [0,T]$ }, { $\gamma_t : t \in [0,T]$ } are adapted processes and therefore the associated integrals are well-defined as standard Itô-integrals. A self-contained background on interpreting stochastic integrals for the *non-adapted* processes considered in this paper appears in Appendix B.

Dual optimal control problem:

$$\begin{array}{ll}
\underset{U \in \mathcal{U}}{\operatorname{Min}} & J(U) = \mathsf{E}\left(\frac{1}{2}|Y_0^{\mathsf{T}}X_0 - Y_0^{\mathsf{T}}\pi_0|^2 + \int_0^T \frac{1}{2}U_t^{\mathsf{T}}RU_t \, \mathrm{d}t \\
& + \int_0^T \frac{1}{2}Y_t^{\mathsf{T}} \, \mathrm{d}\langle X, X^{\mathsf{T}} \rangle_t Y_t + \mathcal{E}_t U_t^{\mathsf{T}} \, \mathrm{d}W_t + \mathcal{E}_t Y_t^{\mathsf{T}} \, \mathrm{d}B_t\right) \quad (5a)
\end{array}$$

Subject to
$$\frac{\mathrm{d}Y_t}{\mathrm{d}t} = -AY_t - HU_t, \quad Y_T = f$$
 (5b)

where $\langle X, X^{\top} \rangle$ denotes the quadratic variation of the Markov process *X*, and the *error process* $\mathcal{E} = \{\mathcal{E}_t : t \in [0, T]\}$ is defined as follows:

$$\mathcal{E}_{t} \coloneqq Y_{0}^{\top} (X_{0} - \pi_{0}) + \int_{0}^{t} U_{\tau}^{\top} dW_{\tau} + \int_{0}^{t} Y_{\tau}^{\top} dB_{\tau}$$
(6)

As in Remark 1, the four stochastic integrals appearing above are defined also as forward integrals (see Appendix B).

The relationship between the optimal control objective $J(\cdot)$ and the minimum variance objective (III) is illustrated in the following proposition. The proof appears in the Appendix C. *Proposition 2:* Consider the state-observation model (1), the linear estimator (4) and the dual optimal control problem (5). For any arbitrary choice of an admissible control input,

$$J(U) = \frac{1}{2} \mathsf{E}[|S_T - f^{\top} X_T|^2]$$

This provides a justification for the objective function (5a) and moreover shows that $J(U) \ge 0$ for any admissible control.

Remark 2: Consider a deterministic control input of the form $U_t = k_t^{\mathsf{T}} Y_t + v_t$ where $\{k_t\}$, $\{v_t\}$ are deterministic functions of time (in particular, they do not depend upon the observations). Such a control is trivially admissible. In this case, $\{Y_t\}$ is a deterministic function of time and the error process \mathcal{E} is a \mathcal{F} -martingale. Consequently,

$$\mathsf{E}\Big(\int_0^T \mathcal{E}_t U_t^{\mathsf{T}} \,\mathrm{d}W_t + \mathcal{E}_t Y_t^{\mathsf{T}} \,\mathrm{d}B_t\Big) = 0$$

and the objective function in (5a) simplifies to

$$J(U) = \frac{1}{2}Y_0^{\mathsf{T}}\Sigma_0 Y_0 + \int_0^T \frac{1}{2}U_t^{\mathsf{T}} R U_t + \frac{1}{2}Y_t^{\mathsf{T}} \mathsf{E}(Q(X_t))Y_t \, \mathrm{d}t$$

where $\Sigma_0 := \mathsf{E}((X_0 - \pi_0)(X_0 - \pi_0)^{\top})$ and $Q(\cdot)$ is a $\mathbb{S} \to \mathbb{R}^{d \times d}$ map defined as follows:

$$Q(e_i) \coloneqq \sum_{j \neq i} A_{ij} (e_j - e_i) (e_j - e_i)^{\mathsf{T}}, \quad i = 1, \dots, d$$

The resulting problem is a deterministic LQ problem whose optimal solution $\{U_t^* : t \in [0,T]\}$ will (in general) yield a sub-optimal estimate S_T^* using (4). The general problem considered here is much tougher because \mathcal{E} is *not* a \mathcal{F} -martingale: Under arbitrary admissible controls, it is not even adapted to this filtration.

We have now set the stage to derive the nonlinear filter via the solution to the dual optimal control problem.

IV. DERIVATION OF THE NONLINEAR FILTER

Recall that an admissible input has the form $U_t = K_t^T Y_t + V_t$ where $t \in [0, T]$. The goal is to obtain a formula for the gain process $K = \{K_t : t \in [0, T]\}$ such that the best choice of $V = \{V_t : t \in [0, T]\}$ is zero.

This choice of input class can be regarded as an instance of the method of "change of control" because V represents the new variable for control [6, Ch. 3.1].

If $V_t \equiv 0$ then $\overline{Y} = {\overline{Y}_t : t \in [0, T]}$ solves the backward ODE

$$\frac{\mathrm{d}Y_t}{\mathrm{d}t} = -A\bar{Y}_t - H\mathsf{K}_t^\top \bar{Y}_t, \quad \bar{Y}_T = f$$

and the associated control is denoted $\overline{U}_t = K_t^{\top} \overline{Y}_t$ for $t \in [0, T]$. With an arbitrary V, the solution is expressed

$$Y_t = \bar{Y}_t + \tilde{Y}_t, \quad U_t = \bar{U}_t + \tilde{U}_t$$

where $\tilde{Y} = {\tilde{Y}_t : t \in [0, T]}$ also solves a backward ODE:

$$\frac{\mathrm{d}\tilde{Y}_t}{\mathrm{d}t} = -A\tilde{Y}_t - H\mathsf{K}_t^{\top}\tilde{Y}_t - HV_t, \quad \tilde{Y}_T = 0 \tag{7}$$

with $\tilde{U}_t = \mathsf{K}_t^\top \tilde{Y}_t + V_t$ for $t \in [0, T]$.

The error term is analogously split as $\mathcal{E}_t = \overline{\mathcal{E}}_t + \widetilde{\mathcal{E}}_t$, with

$$\begin{split} \bar{\mathcal{E}}_t &= \bar{Y}_0(X_0 - \pi_0) + \int_0^t \bar{U}_\tau^\top dW_\tau + \int_0^t \bar{Y}_\tau^\top dB_\tau \\ \bar{\mathcal{E}}_t &= \tilde{Y}_0(X_0 - \pi_0) + \int_0^t \tilde{U}_\tau^\top dW_\tau + \int_0^t \tilde{Y}_\tau^\top dB_\tau \end{split}$$

The optimal gain is described in the following theorem. *Theorem 1:* Consider the optimal control problem (5). For any non-zero $V \in C_{\mathcal{Z}}^m$,

$$J(U) \ge J(\bar{U})$$

where the optimal gain is defined as following:

$$\mathrm{d}\bar{\pi}_t = A^{\top}\bar{\pi}_t \,\mathrm{d}t - \mathsf{K}_t^{\top} \big(\,\mathrm{d}Z_t - H^{\top}\bar{\pi}_t \,\mathrm{d}t \,\big), \quad \bar{\pi}_0 = \pi_0 \qquad (8a)$$

$$\mathsf{K}_{t} = -\mathsf{E}\big((X_{t} - \bar{\pi}_{t})(X_{t} - \bar{\pi}_{t})^{\top} | \mathcal{Z}_{t}\big) H R^{-1}, \quad t \in [0, T]$$
(8b)

A. Proof of Thm. 1

It is simple calculation to see that

$$J(U) = J(\bar{U}) + J(\tilde{U}) + \mathsf{E}(\mathcal{C})$$

where the cross-term C is defined by

$$\mathcal{C} = \underbrace{\tilde{Y}_{0}^{\top}(X_{0} - \pi_{0})(X_{0} - \pi_{0})^{\top}\bar{Y}_{0}}_{\text{term (i)}} + \underbrace{\int_{0}^{T} \tilde{U}_{t}^{\top}R\bar{U}_{t} dt + \tilde{Y}_{t}^{\top} d\langle X, X^{\top} \rangle_{t} \bar{Y}_{t}}_{\text{term (ii)}} + \underbrace{\int_{0}^{T} (\tilde{\mathcal{E}}_{t}\bar{U}_{t}^{\top} + \bar{\mathcal{E}}_{t}\tilde{U}_{t}^{\top}) dW_{t} + \int_{0}^{T} (\tilde{\mathcal{E}}_{t}\bar{Y}_{t}^{\top} + \bar{\mathcal{E}}_{t}\tilde{Y}_{t}^{\top}) dB_{t}}_{\text{term (iii)}}$$

The strategy now is to choose K such that E(C) = 0 for all possible choices of Z-adapted V.

Term (i): A standard technique of optimal control theory dictates that the terminal condition term be expressed as an integral by introducing a dual variable. Towards this goal, we introduce a vector-valued stochastic process $\bar{\pi} = \{\bar{\pi}_t : t \in [0,T]\}$ with $\bar{\pi}_0 = \pi_0$ (the prior). At this point of time, we only require that $\bar{\pi}$ is a \mathbb{Z} -adapted process. The dynamics of this process will be defined later.

Using the process $\bar{\pi}$, together with the requirement (7) that $\tilde{Y}_T = 0$, we obtain

$$\tilde{Y}_{0}^{\top}(\pi_{0} - X_{0})(\pi_{0} - X_{0})^{\top}\bar{Y}_{0} = -\int_{0}^{T} \mathrm{d}(\tilde{Y}_{t}^{\top}(\bar{\pi}_{t} - X_{t})(\bar{\pi}_{t} - X_{t})^{\top}\bar{Y}_{t})$$

The differential is evaluated by an application of the product formula:¹

$$\begin{aligned} &d\left(\tilde{Y}_{t}^{\top}(\bar{\pi}_{t}-X_{t})(\bar{\pi}_{t}-X_{t})^{\top}\bar{Y}_{t}\right) \\ &=\tilde{Y}_{t}^{\top}\left\{\left(d\bar{\pi}_{t}-A^{\top}\bar{\pi}_{t}\,dt+\mathsf{K}_{t}H^{\top}(X_{t}-\bar{\pi}_{t})\,dt-dB_{t}\right)(\bar{\pi}_{t}-X_{t})^{\top}\right. \\ &\left.+\left(\bar{\pi}_{t}-X_{t}\right)\left(d\bar{\pi}_{t}-A^{\top}\bar{\pi}_{t}\,dt+\mathsf{K}_{t}H^{\top}(X_{t}-\bar{\pi}_{t})\,dt-dB_{t}\right)^{\top}\right. \\ &\left.+d\left\langle(\bar{\pi}-X),(\bar{\pi}-X)^{\top}\right\rangle_{t}\right\}\bar{Y}_{t}-V_{t}^{\top}H^{\top}(X_{t}-\bar{\pi}_{t})(X_{t}-\bar{\pi}_{t})^{\top}\bar{Y}_{t}\,dt \end{aligned}$$

¹See Appendix B for a justification of the product formula for the class of (non-adapted) stochastic processes arising in this paper.

where $\langle (\bar{\pi} - X), (\bar{\pi} - X)^{\top} \rangle$ denotes the quadratic variation of the process $\bar{\pi} - X$. It is noted that each of the term in the integral is a quadratic either in \tilde{Y}_t and \bar{Y}_t or in V_t and \bar{Y}_t .

Term (ii): The second term is expressed as:

$$\int_0^T \tilde{U}_t^\top R \bar{U}_t \, \mathrm{d}t + \tilde{Y}_t^\top \, \mathrm{d}\langle X, X^\top \rangle_t \bar{Y}_t$$
$$= \int_0^T \left(\tilde{Y}_t^\top \left(\mathsf{K}_t R \mathsf{K}_t^\top \, \mathrm{d}t + \mathrm{d}\langle X, X^\top \rangle_t \right) \bar{Y}_t + V_t^\top R \mathsf{K}_t^\top \bar{Y}_t \, \mathrm{d}t \right)$$

Term (iii): It remains to tackle the two stochastic integrals involving the error processes. We begin by recalling (6):

$$\mathcal{E}_t = Y_0^\top (X_0 - \pi_0) + \int_0^t U_\tau^\top \, \mathrm{d}W_\tau + \int_0^t Y_\tau^\top \, \mathrm{d}B_\tau$$

Proceeding as in term (i), the process $\bar{\pi}$ is again used to express the terminal condition term $Y_0^{\top}(\pi_0 - X_0)$ as an integral. Once again, using the product rule

$$d(Y_t^{\top}(X_t - \bar{\pi}_t)) = -Y_t^{\top} (d\bar{\pi}_t - A^{\top} \bar{\pi}_t dt + \mathsf{K}_t H^{\top} (X_t - \bar{\pi}_t) dt) + Y_t^{\top} dB_t - V_t^{\top} H^{\top} (X_t - \bar{\pi}_t) dt$$

Therefore,

$$\begin{aligned} \mathcal{E}_{t} &= Y_{0}^{\top} \left(X_{0} - \pi_{0} \right) + \int_{0}^{t} U_{\tau}^{\top} \, \mathrm{d}W_{\tau} + \int_{0}^{t} Y_{\tau}^{\top} \, \mathrm{d}B_{\tau} \\ &= Y_{t}^{\top} \left(X_{t} - \bar{\pi}_{t} \right) + \int_{0}^{t} V_{\tau}^{\top} \left(\, \mathrm{d}W_{\tau} + H^{\top} \left(X_{\tau} - \bar{\pi}_{\tau} \right) \mathrm{d}\tau \right) \\ &+ \int_{0}^{t} Y_{\tau}^{\top} \left(\, \mathrm{d}\bar{\pi}_{\tau} - A^{\top} \pi_{\tau} \, \mathrm{d}\tau + \mathsf{K}_{\tau} \left(\, \mathrm{d}W_{\tau} + H^{\top} \left(X_{\tau} - \bar{\pi}_{\tau} \right) \mathrm{d}\tau \right) \right) \end{aligned}$$

In order to reduce the notational burden, the following differential notation is adopted for the \mathcal{Z} -adapted stochastic processes $\overline{I} = {\overline{I}_t : t \in [0,T]}$ and $\mathcal{L} = {\mathcal{L}_t : t \in [0,T]}$:

$$d\bar{I}_t := dZ_t - H^{\top} \bar{\pi}_t dt$$
$$d\mathcal{L}_t := d\bar{\pi}_t - A^{\top} \bar{\pi}_t dt + \mathsf{K}_t d\bar{I}_t$$

The notation is used to express the error succinctly as

$$\mathcal{E}_t = Y_t^{\top} (\bar{\pi}_t - X_t) - \int_0^t Y_{\tau}^{\top} \, \mathrm{d}\mathcal{L}_{\tau} + \int_0^t V_{\tau}^{\top} \, \mathrm{d}\bar{I}_{\tau}$$

In particular, upon splitting $\mathcal{E}_t = \overline{\mathcal{E}}_t + \widetilde{\mathcal{E}}_t$, we have

$$\begin{split} \bar{\mathcal{E}}_t &= \bar{Y}_t^\top (X_t - \bar{\pi}_t) + \int_0^t \bar{Y}_\tau^\top \, \mathrm{d}\mathcal{L}_\tau \\ \tilde{\mathcal{E}}_t &= \tilde{Y}_t^\top (X_t - \bar{\pi}_t) + \int_0^t \tilde{Y}_\tau^\top \, \mathrm{d}\mathcal{L}_\tau + \int_0^t V_\tau^\top \, \mathrm{d}\bar{I}_\tau \end{split}$$

We thus obtain a useful expression for term (iii):

$$\begin{split} &\int_{0}^{T} \tilde{\mathcal{E}}_{t} \bar{U}_{t}^{\top} + \bar{\mathcal{E}}_{t} \tilde{U}_{t}^{\top} \, \mathrm{d}W_{t} + \int_{0}^{T} \left(\tilde{\mathcal{E}}_{t} \bar{Y}_{t}^{\top} + \bar{\mathcal{E}}_{t} \tilde{Y}_{t}^{\top} \right) \mathrm{d}B_{t} \\ &= \int_{0}^{T} \left\{ \tilde{Y}_{t}^{\top} \left(X_{t} - \bar{\pi}_{t} \right) \bar{Y}_{t}^{\top} \mathsf{K}_{t} + \bar{Y}_{t}^{\top} \left(X_{t} - \bar{\pi}_{t} \right) \tilde{Y}_{t}^{\top} \mathsf{K}_{t} + \left(\int_{0}^{t} \tilde{Y}_{\tau}^{\top} \, \mathrm{d}\mathcal{L}_{\tau} \right) \bar{Y}_{t}^{\top} \mathsf{K}_{t} \\ &+ \left(\int_{0}^{t} \bar{Y}_{\tau}^{\top} \, \mathrm{d}\mathcal{L}_{\tau} \right) \tilde{Y}_{t}^{\top} \mathsf{K}_{t} + \left(\int_{0}^{t} V_{\tau}^{\top} \, \mathrm{d}\bar{I}_{\tau} \right) \bar{Y}_{t}^{\top} \mathsf{K}_{t} + V_{t}^{\top} \bar{Y}_{t}^{\top} \left(X_{t} - \bar{\pi}_{t} \right) \\ &+ V_{t}^{\top} \left(\int_{0}^{t} \bar{Y}_{\tau}^{\top} \, \mathrm{d}\mathcal{L}_{\tau} \right) \right\} \mathrm{d}W_{t} \\ &+ \int_{0}^{T} \left\{ \tilde{Y}_{t}^{\top} \left(X_{t} - \bar{\pi}_{t} \right) \bar{Y}_{t}^{\top} + \bar{Y}_{t}^{\top} \left(X_{t} - \bar{\pi}_{t} \right) \tilde{Y}_{t}^{\top} + \left(\int_{0}^{t} \tilde{Y}_{\tau}^{\top} \, \mathrm{d}\mathcal{L}_{\tau} \right) \bar{Y}_{t}^{\top} \\ &+ \left(\int_{0}^{t} \bar{Y}_{\tau}^{\top} \, \mathrm{d}\mathcal{L}_{\tau} \right) \tilde{Y}_{t}^{\top} + \left(\int_{0}^{t} V_{\tau}^{\top} \, \mathrm{d}\bar{I}_{\tau} \right) \bar{Y}_{t}^{\top} \right\} \mathrm{d}B_{t} \end{split}$$

This concludes our program of expressing each of three terms in C as an integral with sub-terms containing $\bar{Y}_t, \tilde{Y}_t, V_t$. Now, every sub-term is a quadratic of one of the two types:

- 1) The type 1 quadratic sub-terms contain \bar{Y}_t and \bar{Y}_t . An example of this type of quadratic is $\tilde{Y}_t^\top K_t R K_t^\top \bar{Y}_t$ in the term (ii).
- 2) The type 2 quadratic sub-terms contain \bar{Y}_t and V_t . An example of this is $V_t K_t^{\top} \bar{Y}_t$ in the term (ii).

We express $C = C_1 + C_2$, where C_1 contains only the quadratic sub-terms of type 1 and C_2 contains only the quadratic sub-terms of type 2. Upon collecting terms, we obtain

$$\begin{aligned} \mathcal{C}_{1} &= \int_{0}^{T} \tilde{Y}_{t}^{\top} (\mathsf{K}_{t} R \mathsf{K}_{t}^{\top} dt + d\langle X, X^{\top} \rangle_{t}) \bar{Y}_{t} - \tilde{Y}_{t}^{\top} d\langle (\bar{\pi} - X_{t}), (\bar{\pi} - X_{t})^{\top} \rangle_{t} \bar{Y}_{t} \\ &+ \int_{0}^{T} \left(\tilde{Y}_{t}^{\top} (\bar{\pi}_{t} - X_{t}) \bar{Y}_{t}^{\top} + \bar{Y}_{t}^{\top} (\bar{\pi}_{t} - X_{t}) \tilde{Y}_{t}^{\top} \right) d\mathcal{L}_{t} \\ &+ \int_{0}^{T} \left(\left(\int_{0}^{t} \bar{Y}_{\tau}^{\top} d\mathcal{L}_{\tau} \right) \tilde{Y}_{t}^{\top} \mathsf{K}_{t} + \left(\int_{0}^{t} \tilde{Y}_{\tau}^{\top} d\mathcal{L}_{\tau} \right) \bar{Y}_{t}^{\top} \mathsf{K}_{t} \right) dW_{t} \\ &+ \int_{0}^{T} \left(\left(\int_{0}^{t} \bar{Y}_{\tau}^{\top} d\mathcal{L}_{\tau} \right) \tilde{Y}_{t}^{\top} + \left(\int_{0}^{t} \tilde{Y}_{\tau}^{\top} d\mathcal{L}_{\tau} \right) \bar{Y}_{t}^{\top} \right) dB_{t} \end{aligned}$$

and

$$\begin{aligned} \mathcal{C}_{2} &= \int_{0}^{T} V_{t}^{\top} \left(R \mathsf{K}_{t}^{\top} + H^{\top} (X_{t} - \bar{\pi}_{t}) (X_{t} - \bar{\pi}_{t})^{\top} \right) \bar{Y}_{t} \, \mathrm{d}t \\ &+ \int_{0}^{T} \left\{ V_{t}^{\top} (X_{t} - \bar{\pi}_{t})^{\top} \bar{Y}_{t} + V_{t}^{\top} \left(\int_{0}^{t} \bar{Y}_{\tau}^{\top} \, \mathrm{d}\mathcal{L}_{\tau} \right) + \left(\int_{0}^{t} V_{\tau}^{\top} \, \mathrm{d}\bar{I}_{\tau} \right) \bar{Y}_{t}^{\top} \mathsf{K}_{t} \right\} \mathrm{d}W_{t} \\ &+ \int_{0}^{T} \left(\int_{0}^{t} V_{\tau}^{\top} \, \mathrm{d}\bar{I}_{\tau} \right) \bar{Y}_{t}^{\top} \, \mathrm{d}B_{t} \end{aligned}$$

In order to have $E(C) = E(C_1) + E(C_2) = 0$ for all possible choices of $\overline{Y}, \widetilde{Y}$ and for all possible choices of \overline{Y}, V , we follow the following 2-step procedure:

1) In Step 1, we obtain an equation for $\bar{\pi}$ by setting

$$\mathsf{E}(\mathcal{C}_1) = 0, \quad \text{a.s}$$

Given π
 from Step 1, we next derive a formula for the optimal gain K by imposing the requirement

$$\mathsf{E}(\mathcal{C}_2) = 0, \quad \forall V \in C_{\mathcal{F}}^m$$

The 2-step procedure is inspired by the analogous procedure in classical LQ theory where the step 1 is used to derive the Ricatti equation and the step 2 is used to derive the formula for the optimal feedback gain; cf., [6, Ch. 7.3.1].

Step 1: By inspection, we find that upon setting

$$\mathrm{d}\bar{\pi}_t = A^\top \bar{\pi}_t \,\mathrm{d}t - \mathsf{K}_t \,\mathrm{d}\bar{I}_t \,, \quad \bar{\pi}_0 = \pi_0 \tag{9}$$

which is as presented in the theorem statement (8a), we have $d\mathcal{L}_t \equiv 0$, and \mathcal{C}_1 reduces to

$$\begin{aligned} \mathcal{C}_{1} &= \int_{0}^{T} \tilde{Y}_{t}^{\top} \big(\mathsf{K}_{t} R \mathsf{K}_{t}^{\top} dt + d \langle X, X^{\top} \rangle_{t} \big) \bar{Y}_{t} \\ &- \tilde{Y}_{t}^{\top} d \big\langle \big(\bar{\pi} - X \big), \big(\bar{\pi} - X \big)^{\top} \big\rangle_{t} \bar{Y}_{t} \end{aligned}$$

It is an easy calculation to compute the quadratic variation

$$d\langle (\bar{\pi} - X), (\bar{\pi} - X)^{\top} \rangle_{t} = \mathsf{K}_{t} \mathsf{R} \mathsf{K}_{t}^{\top} dt + d\langle X, X^{\top} \rangle_{t}$$

and therefore, upon defining the dynamics of $\bar{\pi}$ according to (9),

$$\mathcal{C}_1 = 0$$
 a.s.

This is true for any choice of Z-adapted gain process K.

Among the consequences are the following pretty representations for the error processes:

$$\bar{\mathcal{E}}_t = \bar{Y}_t^{\top} (X_t - \bar{\pi}_t) + \int_0^t \bar{Y}_{\tau}^{\top} d\mathcal{L}_{\tau} = \bar{Y}_t^{\top} (X_t - \bar{\pi}_t)$$
(10)

and similarly,

$$\tilde{\mathcal{E}}_t = \tilde{Y}_t^{\top} (X_t - \bar{\pi}_t) + \int_0^t V_{\tau} \, \mathrm{d}\bar{I}_{\tau}$$

These expressions also hold for any Z-adapted K.

Step 2: A formula for the gain $K = \{K_t : t \in [0, T]\}$ is obtained by enforcing $E[C_2] = 0$.

We first carry out some simplifications. It is straightforward calculation that, with $\bar{\pi}$ defined according to (9), the integrand of C_2 is a perfect differential:

$$C_2 = \int_0^T d\left(\bar{\mathcal{E}}_t \int_0^t V_\tau^\top d\bar{I}_\tau\right) = \bar{\mathcal{E}}_T \int_0^T V_t^\top d\bar{I}_t \qquad (11)$$

The following orthogonality condition is thus obtained upon using the representation (10) for $\bar{\mathcal{E}}_T$:

$$f^{\mathsf{T}}\mathsf{E}\Big((\bar{\pi}_T - X_T)\int_0^T V_t^{\mathsf{T}} \mathrm{d}\bar{I}_t\Big) = \mathsf{E}(\mathcal{C}_2) = 0$$

Since the function f is arbitrary, we must have

$$\mathsf{E}\Big((X_T - \bar{\pi}_T)\int_0^T V_t^{\mathsf{T}} \,\mathrm{d}\bar{I}_t\Big) = 0$$

To obtain the formula for K, the expression inside the expectation is written as an integral—essentially by reversing the steps in finding the perfect differential. This yields

$$\mathsf{E}\Big(\int_0^T \left(\mathsf{K}_t R + (X_t - \bar{\pi}_t)(X_t - \bar{\pi}_t)^\top H\right) V_t \, \mathrm{d}t\Big) \\ - \mathsf{E}\Big(\int_0^T \left(\int_0^T V_\tau^\top \, \mathrm{d}\bar{I}_\tau\right) \left(\,\mathrm{d}\bar{\pi}_t - A^\top X_t \, \mathrm{d}t\,\right)\Big) = 0$$
(12)

For the equation to hold for arbitrary choices of V and \overline{I} (which is unrelated to the choice of V), the two terms should both be zero:

$$\mathsf{E}\Big(\int_0^T \big(\mathsf{K}_t R + (X_t - \bar{\pi}_t)(X_t - \bar{\pi}_t)^\top H\big) V_t \,\mathrm{d}t\Big) = 0 \tag{13}$$

$$\mathsf{E}\Big(\int_0^T \Big(\int_0^t V_{\tau}^{\mathsf{T}} \,\mathrm{d}\bar{I}_{\tau}\Big) \Big(\,\mathrm{d}\bar{\pi}_t - A^{\mathsf{T}} X_t \,\mathrm{d}t\,\Big)\Big) = 0 \qquad (14)$$

The formula for the optimal K is obtained by solving (13). Using the tower property of conditional expectation, because V_t and K_t are both Z_t -measurable, we have

$$E\left(\int_0^T (\mathsf{K}_t R + \mathsf{E}((X_t - \bar{\pi}_t)(X_t - \bar{\pi}_t)^\top H \mid \mathcal{Z}_t))V_t \, \mathrm{d}t\right) = 0$$

Since V is an arbitrary Z-adapted function, K_t is uniquely determined on L^2 space:

$$\mathsf{K}_t = -\mathsf{E}((X_t - \bar{\pi}_t)(X_t - \bar{\pi}_t)^\top H \mid \mathcal{Z}_t)R^{-1}, \quad t \in [0, T]$$

This gives the formula for the optimal gain K.

Remark 3: Using the optimal gain, the equation (9) for $\bar{\pi}$ becomes

$$d\bar{\pi}_{t} = A^{\top}\bar{\pi}_{t} dt + \mathsf{E}[(X_{t} - \bar{\pi}_{t})(X_{t} - \bar{\pi}_{t})^{\top}H \mid \mathcal{Z}_{t}]R^{-1} d\bar{I}_{t}, \quad \bar{\pi}_{0} = \pi_{0}$$

The equation is not closed because we do not know $E(X_t | Z_t) =: \pi_t$.

One could consider closing the equation by assuming a certainty equivalence principle that $\pi = \overline{\pi}$. In that case,

$$\mathsf{E}((X_t - \bar{\pi}_t)(X_t - \bar{\pi}_t)^{\mathsf{T}}H \mid \mathcal{Z}_t) = \operatorname{diag}(\pi_t)(H - \pi_t^{\mathsf{T}}H)^{\mathsf{T}}$$

where diag(π_t) is a diagonal matrix whose diagonal entries are the elements of the vector π_t , and one obtains the equation

$$\mathrm{d}\pi_t = A^{\mathsf{T}}\pi_t \,\mathrm{d}t + \mathrm{diag}(\pi_t)(H - \pi_t^{\mathsf{T}}H)^{\mathsf{T}}R^{-1}\,\mathrm{d}I_t, \quad \pi_0 = \pi_0$$

where $dI_t = dZ_t - H^{\top} \pi_t dt$. This is the equation for the Wonham filter.

REFERENCES

- R. E. Kalman and R. S. Bucy, "New results in linear filtering and prediction theory," *Journal of basic engineering*, vol. 83, no. 1, pp. 95–108, 1961.
- [2] D. Ocone and E. Pardoux, "Asymptotic stability of the optimal filter with respect to its initial condition," *SIAM Journal on Control and Optimization*, vol. 34, no. 1, pp. 226–243, 1996.
 [3] E. Todorov, "Optimal control theory," *Bayesian brain: probabilistic*
- [3] E. Todorov, "Optimal control theory," Bayesian brain: probabilistic approaches to neural coding, pp. 269–298, 2006.
- [4] E. Todorov, "General duality between optimal control and estimation," in 2008 47th IEEE Conference on Decision and Control, Dec 2008, pp. 4286–4292.
- [5] K. J. Åström, Introduction to Stochastic Control Theory. Academic Press, 1970.
- [6] A. Bensoussan, Estimation and Control of Dynamical Systems. Springer, 2018, vol. 48.

Appendix

A. Proof of Proposition 1

For a given affine control law $U_t = K_t^T Y_t + V_t$, the ODE (2) is a linear system with random coefficients:

$$\frac{\mathrm{d}Y_t}{\mathrm{d}t} = -(A + H\mathsf{K}_t^{\mathsf{T}})Y_t - HV_t, \quad Y_T = f \tag{15}$$

It admits a unique solution $Y : [0,T] \to \mathbb{R}^d$. Now, because $\{K_t, V_t; t \in [0,T]\}$ are \mathbb{Z} -adapted and $Y_T = f$ is deterministic, the solution Y_0 at time t = 0 is a \mathbb{Z}_T -adapted random vector.

For $t \ge \tau$, the state transition matrix $\Phi(t, \tau)$ is defined as the solution to the matrix ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi(t,\tau) = -(A + H\mathsf{K}_t^{\mathsf{T}})\Phi(t,\tau), \quad \Phi(\tau,\tau) = I \tag{16}$$

A solution of (15) is given by

$$Y_t = \Phi(t;0)Y_0 - \int_0^t \Phi(t;\tau)HV_\tau d\tau =: \Phi_t Y_0 + \eta_t$$

Similarly,
$$U_t = \mathsf{K}_t^{\mathsf{T}} Y_t + V_t = \underbrace{(\Phi_t^{\mathsf{T}} \mathsf{K}_t)}_{\mathsf{K}_t} Y_0 + \underbrace{(\mathsf{K}_t^{\mathsf{T}} \eta_t + V_t)}_{\gamma_t}.$$

B. Stochastic integrals

Recall the filtrations: $\mathcal{F}_t := \sigma(X_0, B_\tau, W_\tau : 0 \le \tau \le t)$ and $\mathcal{Z}_t = \sigma(Z_\tau : \tau \in [0, t]), t \in [0, T]$. There are two types of stochastic processes:

- 1) Adapted stochastic processes: $W, B, X \in \mathcal{F}$ and $Z, \overline{\pi}, \overline{I}, K, V \in \mathcal{Z}$.
- 2) Non-adapted stochastic processes: Y, U, \mathcal{E} , and their optimal and perturbed counterparts, $\overline{Y}, \overline{U}, \overline{\mathcal{E}}$ and $\widetilde{Y}, \widetilde{U}, \widetilde{\mathcal{E}}$, respectively.

Now, allowing only for admissible control inputs from \mathcal{U} (see (3)), a generic stochastic process considered in this paper is expressed as $\phi_t = F^{\top} \xi_t + \alpha_t$, where $F \in \mathcal{Z}_T$ and $\xi_t, \alpha_t \in \mathcal{F}_t$ for each *t* (Prop. 1 and Prop. 4).

Definition 1: Consider two stochastic processes $\phi_t = F^{\top}\xi_t + \alpha_t$ and $\psi_t = G^{\top}\zeta_t + \beta_t$, where $\xi_t, \alpha_t, \zeta_t, \beta_t \in \mathcal{F}_t$ are piecewise continuous functions of time *t* with at most finitely many jumps and *F*, *G* are bounded random vectors. Consider a partition $\Pi_{[0,t]}^N = \{0 = t_0 < t_1 < \ldots < t_N = t\}$ with $\Delta := \max_i (t_i - t_{i-1})$. Then,

$$\int_{0}^{t} \phi_{\tau} \, \mathrm{d} \psi_{\tau} \coloneqq \lim_{\Delta \to 0} \sum_{i=1}^{N} \phi_{t_{i-1}} (\psi_{t_{i}} - \psi_{t_{i-1}})$$
$$\langle \phi, \psi \rangle_{t} \coloneqq \lim_{\Delta \to 0} \sum_{i=1}^{N} (\phi_{t_{i}} - \phi_{t_{i-1}}) (\psi_{t_{i}} - \psi_{t_{i-1}})$$

provided the respective limits exist in L^2 .

Proposition 3: Consider the two stochastic processes $\{\phi_t, \psi_t\}$ as defined in Defn. 1. Then

$$\int_{0}^{t} \phi_{\tau} \, \mathrm{d}\psi_{\tau} \stackrel{L^{2}}{=} F^{\mathsf{T}} \Big(\int_{0}^{t} \xi_{\tau} \, \mathrm{d}\zeta_{\tau}^{\mathsf{T}} \Big) G + F^{\mathsf{T}} \Big(\int_{0}^{t} \xi_{\tau} \, \mathrm{d}\beta_{\tau} \Big) \\ + G^{\mathsf{T}} \Big(\int_{0}^{t} \alpha_{\tau} \, \mathrm{d}\zeta_{\tau} \Big) + \int_{0}^{t} \alpha_{\tau} \, \mathrm{d}\beta_{\tau} \\ \langle \phi, \psi \rangle_{t} \stackrel{L^{2}}{=} F^{\mathsf{T}} \langle \xi, \zeta^{\mathsf{T}} \rangle_{t} G + F^{\mathsf{T}} \langle \xi, \beta \rangle_{t} + \langle \alpha, \zeta^{\mathsf{T}} \rangle_{t} G + \langle \alpha, \beta \rangle_{t}$$

where the integrals on the right-hand side are standard Itôintegrals. Moreover, the following Itô product formula holds:

$$\phi_t \psi_t - \phi_0 \psi_0 = \int_0^t \phi_\tau \, \mathrm{d} \psi_\tau + \int_0^t \psi_\tau \, \mathrm{d} \phi_\tau + \langle \phi, \psi \rangle_t$$

Proof: The pre-limit is evaluated as

$$\begin{split} &\sum_{i=1}^{N} \phi_{t_{i-1}} \left(\psi_{t_{i}} - \psi_{t_{i-1}} \right) \\ &= F^{\top} \left(\sum_{i=1}^{N} \xi_{t_{i-1}} \left(\zeta_{t_{i}}^{\top} - \zeta_{t_{i-1}}^{\top} \right) \right) G + F^{\top} \sum_{i=1}^{N} \xi_{t_{i-1}} \left(\beta_{t_{i}} - \beta_{t_{i-1}} \right) \\ &+ G^{\top} \sum_{i=1}^{N} \alpha_{t_{i-1}} \left(\zeta_{t_{i}} - \zeta_{t_{i-1}} \right) + \sum_{i=1}^{N} \alpha_{t_{i-1}} \left(\beta_{t_{i}} - \beta_{t_{i-1}} \right) \end{split}$$

The result is obtained upon letting $\Delta \rightarrow 0$. For example,

$$\lim_{\Delta \to 0} \sum_{i=1}^{N} \xi_{t_{i-1}} \left(\zeta_{t_i}^{\top} - \zeta_{t_{i-1}}^{\top} \right) \stackrel{L^2}{=} \int_0^t \xi_{\tau} \, \mathrm{d}\zeta_{\tau}^{\top}$$

and therefore, because F, G are bounded,

$$\lim_{\Delta \to 0} F^{\mathsf{T}} \Big(\sum_{i=1}^{N} \xi_{t_{i-1}} \big(\zeta_{t_i}^{\mathsf{T}} - \zeta_{t_{i-1}}^{\mathsf{T}} \big) \Big) G \stackrel{L^2}{=} F^{\mathsf{T}} \Big(\int_0^t \xi_{\tau} \, \mathrm{d} \zeta_{\tau}^{\mathsf{T}} \Big) G$$

The calculation for the cross variation is analogous. The product rule is proved by using the following identity (which holds for arbitrary stochastic processes):

$$(\phi_{t_i} \psi_{t_i} - \phi_{t_{i-1}} \psi_{t_{i-1}}) = \phi_{t_{i-1}} (\psi_{t_i} - \psi_{t_{i-1}}) + \psi_{t_{i-1}} (\phi_{t_i} - \phi_{t_{i-1}}) + (\phi_{t_i} - \phi_{t_{i-1}}) (\psi_{t_i} - \psi_{t_{i-1}})$$

Summing over *i* and taking the limit as $\Delta \rightarrow 0$ yields the result.

Remark 4: The product rule is the *only* type of Itô formula used in the various proofs in this paper. This is because of the linear quadratic nature of the optimal control problem in finite-state-space settings. The following differential notation is frequently used:

$$d(\phi_t \psi_t) = \phi_t d\psi_t + \psi_t d\phi_t + d\langle \phi, \psi \rangle_t$$
(17)

C. Proof of Proposition 2

The following identity is established in this section for any admissible control:

$$J(U) = \frac{1}{2} \mathsf{E}[|S_T - f^{\mathsf{T}} X_T|^2]$$

The lefhand-side is the optimal control objective as defined in (5a). The righthand-side is the mean-squared error. Recall that S_T is the linear estimate as defined by (4), $f \in \mathbb{R}^d$ is deterministic, and X_T is the hidden state at time T.

The approach is to use the dual ODE (2) to express the mean-squared error as an integral. The product formula (17) is used to obtain

$$d(Y_t^{\top}X_t) = dY_t^{\top}X_t + Y_t^{\top}dX_t + d\langle Y^{\top}, X \rangle_t$$

= $(-Y_t^{\top}A^{\top} - U_t^{\top}H^{\top})X_t dt + Y_t^{\top}(A^{\top}X_t dt + dB_t)$
= $-U_t^{\top}H^{\top}X_t dt + Y_t^{\top}dB_t$

which is shorthand for the integral equation

$$Y_T^{\mathsf{T}} X_T = Y_0^{\mathsf{T}} X_0 + \int_0^T U_t^{\mathsf{T}} H^{\mathsf{T}} X_t \, \mathrm{d}t + Y_t^{\mathsf{T}} \, \mathrm{d}B_t$$

With $Y_T = f$, upon subtracting this equation from (4),

$$f^{\top}X_{T} - S_{T} = (Y_{0}^{\top}X_{0} - Y_{0}^{\top}\pi_{0}) + \int_{0}^{T} U_{t}^{\top} dW_{t} + Y_{t}^{\top} dB_{t}$$

With the definition of the error process \mathcal{E}_t in (6), the left-hand side is identified: $f^{\mathsf{T}}X_T - S_T = \mathcal{E}_T$.

The product formula (17) is then used to obtain

$$\frac{1}{2}\mathcal{E}_T^2 = \frac{1}{2}\mathcal{E}_0^2 + \int_0^T \mathcal{E}_t \,\mathrm{d}\mathcal{E}_t + \frac{1}{2}\langle \mathcal{E}, \mathcal{E} \rangle_T$$

The integral form (5a) of the objective function follows from evaluating each of the terms as summarized in the following.

Proposition 4: Consider the error process $\mathcal{E} = \{\mathcal{E}_t : t \in [0,T]\}$ defined in (6). Suppose $U = \{U_t : t \in [0,T]\}$ is any admissible control. Then

$$\mathcal{E}_{0}^{2} = |Y_{0}^{\top}X_{0} - Y_{0}^{\top}\pi_{0}|^{2}$$

$$\int_{0}^{T} \mathcal{E}_{t} \,\mathrm{d}\mathcal{E}_{t} = \int_{0}^{T} \mathcal{E}_{t}U_{t}^{\top}\,\mathrm{d}W_{t} + \int_{0}^{T} \mathcal{E}_{t}Y_{t}^{\top}\,\mathrm{d}B_{t}$$

$$\langle \mathcal{E}, \mathcal{E} \rangle_{T} = \int_{0}^{T} U_{t}^{2}\,\mathrm{d}t + Y_{t}^{\top}\,\mathrm{d}\langle X, X \rangle_{t}Y_{t}$$

The proof is the direct application of Prop. 1 and Prop. 3.