Adding virtual measurements by PWM-induced signal injection

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Abstract—We show that for PWM-operated devices, it is possible to benefit from signal injection without an external probing signal, by suitably using the excitation provided by the PWM itself. As in the usual signal injection framework conceptualized in [1], an extra "virtual measurement" can be made available for use in a control law, but without the practical drawbacks caused by an external signal.

I. INTRODUCTION

Signal injection is a control technique which consists in adding a fast-varying probing signal to the control input. This excitation creates a small ripple in the measurements, which contains useful information if properly decoded. The idea was introduced in [2], [3] for controlling electric motors at low velocity using only measurements of currents. It was later conceptualized in [1] as a way of producing "virtual measurements" that can be used to control the system, in particular to overcome observability degeneracies. Signal injection is a very effective method, see e.g. applications to electromechanical devices along these lines in [4], [5], but it comes at a price: the ripple it creates may in practice yield unpleasant acoustic noise and excite unmodeled dynamics, in particular in the very common situation where the device is fed by a Pulse Width Modulation (PWM) inverter; indeed, the frequency of the probing signal may not be as high as desired so as not to interfere with the PWM (typically, it can not exceed 500 Hz in an industrial drive with a 4 kHz-PWM frequency).

The goal of this paper is to demonstrate that for PWMoperated devices, it is possible to benefit from signal injection *without an external probing signal*, by suitably using the excitation provided by the PWM itself, as e.g. in [6]. More precisely, consider the Single-Input Single-Output system

$$\dot{x} = f(x) + g(x)u, \tag{1a}$$

$$y = h(x), \tag{1b}$$

where u is the control input and y the measured output. We first show in section II that when the control is impressed through PWM, the dynamics may be written as

$$\dot{x} = f(x) + g(x)\left(u + s_0(u, \frac{t}{\varepsilon})\right),\tag{2}$$

with s_0 1-periodic and zero-mean in the second argument, i.e. $s_0(u, \sigma+1) = s_0(u, \sigma)$ and $\int_0^1 s_0(u, \sigma) d\sigma = 0$ for all $u; \varepsilon$ is the PWM period, hence assumed small. The difference with

usual signal injection is that the probing signal s_0 generated by the modulation process now depends not only on time, but also on the control input u. This makes the situation more complicated, in particular because s_0 can be discontinuous in both its arguments. Nevertheless, we show in section III that the second-order averaging analysis of [1] can be extended to this case. In the same way, we show in section IV that the demodulation procedure of [1] can be adapted to make available the so-called virtual measurement

$$y_v := H_1(x) := \varepsilon h'(x)g(x),$$

in addition to the actual measurement $y_a := H_0(x) := h(x)$. This extra signal is likely to simplify the design of a control law, as illustrated on a numerical example in section V.

Finally, we list some definitions used throughout the paper; S denotes a function of two variables, which is T-periodic in the second argument, i.e. $S(v, \sigma + T) = S(v, \sigma)$ for all v:

- the mean of S in the second argument is the function (of one variable) $\overline{S}(v) := \frac{1}{T} \int_0^T S(v, \sigma) d\sigma$; S has zero mean in the second argument if \overline{S} is identically zero
- if S has zero mean in the second argument, its zeromean primitive in the second argument is defined by

$$S_1(v,\tau) := \int_0^\tau S(v,\sigma) d\sigma - \frac{1}{T} \int_0^T \int_0^\tau S(v,\sigma) d\sigma d\tau;$$

notice S_1 is T-periodic in the second argument because S has zero mean in the second argument

• the moving average M(k) of k is defined by

$$M(k)(t) := \frac{1}{\varepsilon} \int_{t-\varepsilon}^{t} k(\tau) d\tau$$

 O_∞ denotes the uniform "big O" symbol of analysis, namely f(z, ε) = O_∞(ε^p) if |f(z, ε)| ≤ Kε^p for ε small enough, with K > 0 independent of z and ε.

II. PWM-INDUCED SIGNAL INJECTION

When the control input u in (1a) is impressed through a PWM process with period ε , the resulting dynamics reads

$$\dot{x} = f(x) + g(x)\mathcal{M}\left(u, \frac{t}{\varepsilon}\right),\tag{3}$$

with \mathcal{M} 1-periodic and mean u in the second argument; the detailed expression for \mathcal{M} is given below. Setting $s_0(u, \sigma) := \mathcal{M}(u, \sigma) - u$, (3) obviously takes the form (2), with s_0 1-periodic and zero-mean in the second argument.

Classical PWM with period ε and range $[-u_m, u_m]$ is obtained by comparing the input signal u to the ε -periodic sawtooh carrier defined by

$$c(t) := \begin{cases} u_m + 4 \operatorname{w}(\frac{t}{\varepsilon}) & \text{if } -\frac{u_m}{2} \le \operatorname{w}(\frac{t}{\varepsilon}) \le 0\\ u_m - 4 \operatorname{w}(\frac{t}{\varepsilon}) & \text{if } 0 \le \operatorname{w}(\frac{t}{\varepsilon}) \le \frac{u_m}{2}; \end{cases}$$

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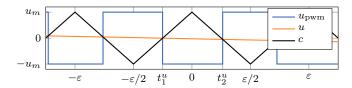


Fig. 1: PWM: u is compared to c to produce u_{pwm} .

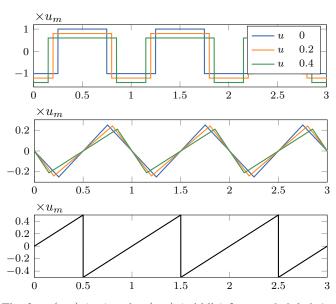


Fig. 2: $s_0(u, \cdot)$ (top) and $s_1(u, \cdot)$ (middle) for u = 0, 0.2, 0.4; w (bottom).

the 1-periodic function $w(\sigma) := u_m \mod(\sigma + \frac{1}{2}, 1) - \frac{u_m}{2}$ wraps the normalized time $\sigma = \frac{t}{\varepsilon}$ to $[-\frac{u_m}{2}, \frac{u_m}{2}]$. If u varies slowly enough, it crosses the carrier c exactly once on each rising and falling ramp, at times $t_1^u < t_2^u$ such that

$$u(t_1^u) = u_m + 4 \operatorname{w}\left(\frac{t_1^u}{\varepsilon}\right)$$
$$u(t_2^u) = u_m - 4 \operatorname{w}\left(\frac{t_2^u}{\varepsilon}\right).$$

The PWM-encoded signal is therefore given by

$$u_{\rm pwm}(t) = \begin{cases} u_m & \text{if } -\frac{u_m}{2} < w\left(\frac{t}{\varepsilon}\right) \le w\left(\frac{t_1}{\varepsilon}\right) \\ -u_m & \text{if } w\left(\frac{t_1}{\varepsilon}\right) < w\left(\frac{t}{\varepsilon}\right) \le w\left(\frac{t_2}{\varepsilon}\right) \\ u_m & \text{if } w\left(\frac{t_2}{\varepsilon}\right) < w\left(\frac{t}{\varepsilon}\right) \le \frac{u_m}{2}. \end{cases}$$

Fig. 1 illustrates the signals u, c and u_{pwm} . The function

$$\mathcal{M}(u,\sigma) := \begin{cases} u_m & \text{if } -2u_m < 4 \operatorname{w}(\sigma) \le u - u_m \\ -u_m & \text{if } u - u_m < 4 \operatorname{w}(\sigma) \le u_m - u \\ u_m & \text{if } u_m - u < 4 \operatorname{w}(\sigma) \le 2u_m \end{cases}$$
$$= u_m + \operatorname{sign}(u - u_m - 4 \operatorname{w}(\sigma))$$
$$+ \operatorname{sign}(u - u_m + 4 \operatorname{w}(\sigma)).$$

which is obviously 1-periodic and with mean u with respect to its second argument, therefore completely describes the PWM process since $u_{\text{pwm}}(t) = \mathcal{M}(u(t), \frac{t}{\epsilon})$.

Finally, the induced zero-mean probing signal is

$$s_0(u,\sigma) := \mathcal{M}(u,\sigma) - u$$

= $u_m - u + \operatorname{sign}\left(\frac{u - u_m}{4} - w(\sigma)\right)$
+ $\operatorname{sign}\left(\frac{u - u_m}{4} + w(\sigma)\right),$

and its zero-mean primitive in the second argument is

$$s_1(u,\sigma) := \left(1 - \frac{u}{u_m}\right) w(\sigma) - \left|\frac{u - u_m}{4} - w(\sigma)\right| \\ + \left|\frac{u - u_m}{4} + w(\sigma)\right|.$$

Remark 1: As s_0 is only piecewise continuous, one might expect problems to define the "solutions" of (2). But as noted above, if the input u(t) of the PWM encoder varies slowly enough, its output $u_{pwm}(t) = \mathcal{M}(u(t), \frac{t}{\varepsilon})$ will have exactly two discontinuities per PWM period. Chattering is therefore excluded, which is enough to ensure the existence and uniqueness of the solutions of (2), see [7], without the need for the more general Filipov theory [8]. Of course, we assume (without loss of generality in practice) that f, g and h in (1) are smooth enough.

Notice also s_1 is continuous and piecewise C^1 in both its arguments. The regularity in the second argument was to be expected as $s_1(u, \cdot)$ is a primitive of $s_0(u, \cdot)$; on the other hand, the regularity in the first argument stems from the specific form of s_0 .

III. AVERAGING AND PWM-INDUCED INJECTION

Section III-A outlines the overall approach and states the main Theorem 1, which is proved in the somewhat technical section III-B. As a matter of fact, the proof can be skipped without losing the main thread; suffice to say that if s_0 were Lipschitz in the first argument, the proof would essentially be an extension of the analysis by "standard" second-order averaging of [1], with more involved calculations

A. Main result

Assume we have designed a suitable control law

$$\overline{u} = \alpha(\overline{\eta}, \overline{Y}, t)$$
$$\dot{\overline{\eta}} = a(\overline{\eta}, \overline{Y}, t),$$

where $\overline{\eta} \in \mathbb{R}^q$, for the system

$$\begin{split} \dot{\overline{x}} &= f(\overline{x}) + g(\overline{x})\overline{u}, \\ \overline{Y} &= H(\overline{x}) := \begin{pmatrix} h(\overline{x}) \\ \varepsilon h'(\overline{x})g(\overline{x}) \end{pmatrix} \end{split}$$

By "suitable", we mean the resulting closed-loop system

$$\dot{\overline{x}} = f(\overline{x}) + g(\overline{x})\alpha(\overline{\eta}, H(\overline{x}), t)$$
(4a)

$$\dot{\overline{\eta}} = a(\overline{\eta}, H(\overline{x}), t)$$
 (4b)

has the desired exponentially stable behavior. We have changed the notations of the variables with $\overline{}$ to easily distinguish between the solutions of (4) and of (7) below. Of course, this describes an unrealistic situation:

• PWM is not taken into account

• the control law is not implementable, as it uses not only the actual output $\overline{y}_a = h(\overline{x})$, but also the a priori not available virtual output $\overline{y}_v = \varepsilon h'(\overline{x})g(\overline{x})$.

Define now (up to $\mathcal{O}_{\infty}(\varepsilon^2)$) the function

$$\overline{H}(x,\eta,\sigma,t) := H\left(x - \varepsilon g(x)s_1\left(\alpha(\eta,H(x),t),\sigma\right)\right) + \mathcal{O}_{\infty}(\varepsilon^2), \quad (5)$$

where s_1 is the zero-mean primitive of s_0 in the second argument, and consider the control law

$$u = \alpha \left(\eta, \overline{H}(x, \eta, \frac{t}{\varepsilon}, t), t \right)$$
(6a)

$$\dot{\eta} = a\left(\eta, \overline{H}(x, \eta, \frac{t}{\varepsilon}, t), t\right). \tag{6b}$$

The resulting closed-loop system, including PWM, reads

$$\dot{x} = f(x) + g(x)\mathcal{M}\left(\alpha\left(\eta, \overline{H}(x, \eta, \frac{t}{\varepsilon}, t), t\right), \frac{t}{\varepsilon}\right)$$
(7a)

$$\dot{\eta} = a \left(\eta, \overline{H}(x, \eta, \frac{t}{\varepsilon}, t), t \right). \tag{7b}$$

Though PWM is now taken into account, the control law (6) still seems to contain unknown terms. Nevertheless, it will turn out from the following result that it can be implemented.

Theorem 1: Let $(x(t), \eta(t))$ be the solution of (7) starting from (x_0, η_0) , and define $u(t) := \alpha(\eta(t), H(x(t)), t)$ and y(t) := H(x(t)); let $(\overline{x}(t), \overline{\eta}(t))$ be the solution of (4) starting from $(x_0 - \varepsilon g(x_0)s_1(u(0), 0), \eta_0)$, and define $\overline{u}(t) := \alpha(\overline{\eta}(t), H(\overline{x}(t)), t)$. Then, for all $t \ge 0$,

$$x(t) = \overline{x}(t) + \varepsilon g(\overline{x}(t)) s_1(\overline{u}(t), \frac{t}{\varepsilon}) + \mathcal{O}_{\infty}(\varepsilon^2)$$
(8a)

$$\eta(t) = \overline{\eta}(t) + \mathcal{O}_{\infty}(\varepsilon^2) \tag{8b}$$

$$y(t) = H_0(\overline{x}(t)) + H_1(\overline{x}(t)) s_1(\overline{u}(t), \frac{t}{\varepsilon}) + \mathcal{O}_{\infty}(\varepsilon^2).$$
 (8c)

The practical meaning of the theorem is the following. As the solution $(x(t), \eta(t)))$ is piecewise C^1 , we have by Taylor expansion using (8a)-(8b) that $u(t) = \overline{u}(t) + \mathcal{O}_{\infty}(\varepsilon^2)$. In the same way, as s_1 is also piecewise C^1 , we have

$$s_1(u(t), \frac{t}{\varepsilon}) = s_1(\overline{u}(t), \frac{t}{\varepsilon}) + \mathcal{O}_{\infty}(\varepsilon^2)$$

As a consequence, we can invert (8a)-(8b), which yields

$$\overline{x}(t) = x(t) - \varepsilon g(x(t)) s_1(u(t), \frac{t}{\varepsilon}) + \mathcal{O}_{\infty}(\varepsilon^2)$$
(9a)
$$\overline{\eta}(t) = \eta(t) + \mathcal{O}_{\infty}(\varepsilon^2).$$
(9b)

Using this into (5), we then get

$$\overline{H}(x(t),\eta(t),\frac{t}{\varepsilon},t) = H\left(x(t) - \varepsilon g(x(t))s_1(u(t),\frac{t}{\varepsilon})\right) + \mathcal{O}_{\infty}(\varepsilon^2), \\
= H(\overline{x}(t)) + \mathcal{O}_{\infty}(\varepsilon^2).$$
(10)

On the other hand, we will see in section IV that, thanks to (8c), we can produce an estimate $\widehat{Y} = H(\overline{x}) + \mathcal{O}_{\infty}(\varepsilon^2)$. This means the PWM-fed dynamics (3) acted upon by the implementable feedback

$$u = \alpha(\eta, \hat{Y}, t)$$
$$\dot{\eta} = a(\eta, \hat{Y}, t).$$

behaves exactly as the "ideal" closed-loop system (4), except for the presence of a small ripple (described by (8a)-(8b)).

Remark 2: Notice that, according to Remark 1, $H_0(\overline{x}(t))$ and $H_1(\overline{x}(t))$ in (8c) may be as smooth as desired (the regularity is inherited from only f, g, h, α, a); on the other hand, $s_1(u(t), \frac{t}{\varepsilon})$ is only continuous and piecewise C^1 . Nevertheless, this is enough to justify all the Taylor expansions performed in the paper.

B. Proof of Theorem 1

Because of the lack of regularity of s_0 , we must go back to the fundamentals of the second-order averaging theory presented in [9, chapter 2] (with slow time dependence [9, section 3.3]). We first introduce two ad hoc definitions.

Definition 1: A function $\varphi(X, \sigma)$ is slowly-varying in average if there exists $\lambda > 0$ such that for ε small enough,

$$\int_{a}^{a+T} \left\|\varphi\left(p(\varepsilon\sigma) + \varepsilon^{k}q(\sigma), \sigma\right) - \varphi\left(p(\varepsilon\sigma), \sigma\right)\right\| d\sigma \leq \lambda T \varepsilon^{k},$$

where p, q are continuous with q bounded; a and T > 0 are arbitrary constants. Notice that if φ is Lipschitz in the first variable then it is slowly-varying in average. The interest of this definition is that it is satisfied by s_0 .

Definition 2: A function ϕ is $\mathcal{O}_{\infty}(\varepsilon^3)$ in average if there exists K > 0 such that $\left\|\int_0^{\sigma} \phi(q(s), s) \, ds\right\| \leq K \, \varepsilon^3 \sigma$ for all $\sigma \geq 0$. Clearly, if ϕ is $\mathcal{O}_{\infty}(\varepsilon^3)$ then it is $\mathcal{O}_{\infty}(\varepsilon^3)$ in average. The proof of Theorem 1 follows the same steps as [9,

chapter 2], but with weaker assumptions. We first rewrite (7) in the fast timescale $\sigma := t/\varepsilon$ as

$$\frac{dX}{d\sigma} = \varepsilon F(X, \sigma, \varepsilon \sigma). \tag{11}$$

where $X := (x, \eta)$ and

$$F(X,\sigma,\tau) := \begin{pmatrix} f(x) + g(x)\mathcal{M}\left(\alpha\left(\eta,\overline{H}(x,\eta,\sigma,\tau),\tau\right)\right) \\ a\left(\eta,\overline{H}(x,\eta,\sigma,\tau),\tau\right) \end{pmatrix}.$$

Notice F is 1-periodic in the second argument. Consider also the so-called averaged system

$$\frac{d\overline{X}}{d\sigma} = \varepsilon \overline{F}(\overline{X}, \varepsilon \sigma). \tag{12}$$

where \overline{F} is the mean of F in the second argument.

Define the near-identity transformation

$$X = \widetilde{X} + \varepsilon W(\widetilde{X}, \sigma, \varepsilon \sigma), \tag{13}$$

where $\widetilde{X} := (\widetilde{x}, \widetilde{\eta})$ and

$$W(\widetilde{X},\sigma,\tau) := \begin{pmatrix} g(\widetilde{x}) \\ 0 \end{pmatrix} s_1 \Big(\alpha \big(\widetilde{\eta}, H(\widetilde{x},\widetilde{\eta},\sigma,\tau),\tau \big), \sigma \Big).$$

Inverting (13) yields

$$\widetilde{X} = X - \varepsilon W(X, \sigma, \varepsilon \sigma) + \mathcal{O}_{\infty}(\varepsilon^2).$$
(14)

By lemma 1, this transformation puts (11) into

$$\frac{d\widetilde{X}}{d\sigma} = \varepsilon \overline{F}(\widetilde{X}, \varepsilon\sigma) + \varepsilon^2 \Phi(\widetilde{X}, \sigma, \varepsilon\sigma) + \phi(\widetilde{X}, \sigma, \varepsilon\sigma); \quad (15)$$

 Φ is periodic and zero-mean in the second argument, and slowly-varying in average, and ϕ is $\mathcal{O}_{\infty}(\varepsilon^3)$ in average.

By lemma 2, the solutions $\overline{X}(\sigma)$ and $\widetilde{X}(\sigma)$ of (12) and (15), starting from the same initial conditions, satisfy

$$\widetilde{X}(\sigma) = \overline{X}(\sigma) + \mathcal{O}_{\infty}(\varepsilon^2)$$

As a consequence, the solution $X(\sigma)$ of (11) starting from X_0 and the solution $\overline{X}(\sigma)$ of (12) starting from $X_0 - \varepsilon W(X_0, 0, 0)$ are related by $X(\sigma) = \overline{X}(\sigma) + \varepsilon W(\overline{X}(\sigma), \sigma, \varepsilon \sigma) + \mathcal{O}_{\infty}(\varepsilon^2)$, which is exactly (8a)-(8b). Inserting (8a) in y = h(x) and Taylor expanding yields (8c).

Remark 3: If s_0 were differentiable in the first variable, Φ would be Lipschitz and ϕ would be $\mathcal{O}_{\infty}(\varepsilon^3)$ in (15), hence the averaging theory of [9] would directly apply.

Remark 4: In the sequel, we prove for simplicity only the estimation $\widetilde{X}(\sigma) = \overline{X}(\sigma) + \mathcal{O}(\varepsilon^2)$ on a timescale $1/\varepsilon$. The continuation to infinity follows from the exponential stability of (4), exactly as in [1, Appendix].

In the same way, lemma 2 is proved without slowtime dependence, the generalization being obvious as in [9, section 3.3].

Lemma 1: The transformation (13) puts (11) into (15), where Φ is periodic and zero-mean in the second argument, and slowly-varying in average, and ϕ is $\mathcal{O}_{\infty}(\varepsilon^3)$ in average.

Proof: To determine the expression for $d\tilde{X}/d\sigma$, the objective is to compute $dX/d\sigma$ as a function of \tilde{X} with two different methods. On the one hand we replace X with its transformation (13) in the closed-loop system (11), and on the other hand we differentiate (13) with respect to σ .

We first compute $s_0(\alpha(\eta, \overline{H}(x, \eta, \sigma, \varepsilon\sigma), \varepsilon\sigma), \sigma)$ as a function of $\widetilde{X} = (\widetilde{x}, \widetilde{\eta})$. Exactly as in (10), with $(\widetilde{x}, \widetilde{\eta})$ replacing $(\overline{x}, \overline{\eta})$, and (14) replacing (9), we have

$$\overline{H}(x,\eta,\sigma,\varepsilon\sigma) = H(\widetilde{x}) + \mathcal{O}_{\infty}(\varepsilon^2).$$

Therefore, by Taylor expansion

$$\alpha(\eta, \overline{H}(x, \eta, \sigma, \varepsilon\sigma), \varepsilon\sigma) = \alpha(\widetilde{\eta}, H(\widetilde{x}), \varepsilon\sigma) + \varepsilon^2 K_\alpha(\widetilde{X}, \sigma),$$

with K_{α} bounded. The lack of regularity of s_0 prevents further Taylor expansion; nonetheless, we still can write

$$s_0(\alpha(\eta, \overline{H}(x, \eta, \sigma, \varepsilon\sigma), \varepsilon\sigma), \sigma) = s_0(\alpha(\tilde{\eta}, H(\tilde{x}), \varepsilon\sigma) + \varepsilon^2 K_\alpha(\tilde{X}, \sigma), \sigma).$$

Finally, inserting (13) into (11) and Taylor expanding, yields after tedious but straightforward computations,

$$\frac{dX}{d\sigma} = \varepsilon \overline{F}(\widetilde{X}, \varepsilon \sigma) + \varepsilon G(\widetilde{X}) s_0^{\alpha, +}(\widetilde{\cdot})
+ \varepsilon^2 \overline{F}(\widetilde{X}, \varepsilon \sigma) G(\widetilde{X}) s_1^{\alpha}(\widetilde{\cdot})
+ \varepsilon^2 G'(\widetilde{X}) G(\widetilde{X}) s_1^{\alpha}(\widetilde{\cdot}) s_0^{\alpha, +}(\widetilde{\cdot}) + \mathcal{O}_{\infty}(\varepsilon^3); \quad (16)$$

we have introduced the following notations

$$\begin{split} &(\widetilde{\cdot}) := (X, \sigma, \varepsilon \sigma) \\ &s_i^{\alpha}(\widetilde{\cdot}) := s_i \big(\alpha(\widetilde{\eta}, H(\widetilde{x}), \varepsilon \sigma), \sigma \big), \\ &s_0^{\alpha, +}(\widetilde{\cdot}) := s_0 \big(\alpha(\widetilde{\eta}, H(\widetilde{x}), \varepsilon \sigma) + \varepsilon^2 K_{\alpha}(\widetilde{X}, \sigma), \sigma \big) \\ &\Delta s_0^{\alpha}(\widetilde{\cdot}) := s_0^{\alpha, +}(\widetilde{\cdot}) - s_0(\widetilde{\cdot}) \\ &G(X) := \binom{g(x)}{0} \\ &\overline{F}(X, \varepsilon \sigma) := \binom{f(x) + g(x)\alpha(\eta, H(x), \varepsilon \sigma)}{a(\eta, H(x), \varepsilon \sigma)} \Big). \end{split}$$

We now time-differentiate (13), which reads with the previous notations

$$X = X + \varepsilon G(X) s_1^{\alpha}(\widetilde{\cdot}).$$

This yields

$$\frac{dX}{d\sigma} = \frac{d\widetilde{X}}{d\sigma} + \varepsilon G'(\widetilde{X}) \frac{d\widetilde{X}}{d\sigma} s_1^{\alpha}(\widetilde{\cdot}) + \varepsilon G(\widetilde{X}) \partial_1 s_1^{\alpha}(\widetilde{\cdot}) \frac{d\widetilde{X}}{d\sigma} + \varepsilon G(\widetilde{X}) s_0^{\alpha}(\widetilde{\cdot}) + \varepsilon^2 G(\widetilde{X}) \partial_3 s_1^{\alpha}(\widetilde{\cdot}), \quad (17)$$

since $\partial_2 s_1^{\alpha} = s_0^{\alpha}$. Now assume \widetilde{X} satisfies

$$\frac{d\widetilde{X}}{d\sigma} = \varepsilon \overline{F}(\widetilde{X}, \varepsilon \sigma) + \varepsilon G(\widetilde{X}) \Delta s_0^{\alpha}(\widetilde{\cdot}) + \varepsilon^2 \Psi(\widetilde{\cdot}), \quad (18)$$

where $\Psi(\tilde{\cdot})$ is yet to be computed. Inserting (18) into (17),

$$\frac{dX}{d\sigma} = \varepsilon \overline{F}(\widetilde{X}, \varepsilon \sigma) + \varepsilon G(\widetilde{X}) \Delta s_0^{\alpha}(\widetilde{\cdot}) + \varepsilon^2 \Psi(\widetilde{\cdot})
+ \varepsilon^2 G'(\widetilde{X}) \overline{F}(\widetilde{X}, \varepsilon \sigma) s_1^{\alpha}(\widetilde{\cdot})
+ \varepsilon^2 G'(\widetilde{X}) G(\widetilde{X}) \Delta s_0^{\alpha}(\widetilde{\cdot}) s_1^{\alpha}(\widetilde{\cdot})
+ \varepsilon^2 G(\widetilde{X}) \partial_1 s_1^{\alpha}(\widetilde{\cdot}) \overline{F}(\widetilde{X}, \varepsilon \sigma)
+ \varepsilon^2 G(\widetilde{X}) \partial_1 s_1^{\alpha}(\widetilde{\cdot}) G(\widetilde{X}) \Delta s_0^{\alpha}(\widetilde{\cdot})
+ \varepsilon G(\widetilde{X}) s_0^{\alpha}(\widetilde{\cdot})
+ \varepsilon^2 G(\widetilde{X}) \partial_3 s_1^{\alpha}(\widetilde{\cdot})
+ \mathcal{O}_{\infty}(\varepsilon^3).$$
(19)

Next, equating (19) and (16), Ψ satisfies

$$\Psi(\widetilde{\cdot}) = [\overline{F}, G](\widetilde{X}, \varepsilon\sigma)s_1^{\alpha}(\widetilde{\cdot}) + G'(\widetilde{X})G(\widetilde{X})s_0^{\alpha}(\widetilde{\cdot})s_1^{\alpha}(\widetilde{\cdot}) - G(\widetilde{X})\partial_1s_1^{\alpha}(\widetilde{\cdot})\overline{F}(\widetilde{X}, \varepsilon\sigma) - G(\widetilde{X})\partial_3s_1^{\alpha}(\widetilde{\cdot}) - G(\widetilde{X})\partial_1s_1^{\alpha}(\widetilde{\cdot})G(\widetilde{X})\Delta s_0^{\alpha}(\widetilde{\cdot}).$$
(20)

This gives the expressions of Φ and ϕ in (15),

$$\begin{split} \Phi(\widetilde{\cdot}) &:= [\overline{F}, G](\widetilde{X}, \varepsilon \sigma) s_1^{\alpha}(\widetilde{\cdot}) + G'(\widetilde{X}) G(\widetilde{X}) s_0^{\alpha}(\widetilde{\cdot}) s_1^{\alpha}(\widetilde{\cdot}) \\ &- G(\widetilde{X}) \partial_1 s_1^{\alpha}(\widetilde{\cdot}) \overline{F}(\widetilde{X}, \varepsilon \sigma) - G(\widetilde{X}) \partial_3 s_1^{\alpha}(\widetilde{\cdot}), \\ \phi(\widetilde{\cdot}) &:= \varepsilon^2 \Psi_1(\widetilde{\cdot}) + \varepsilon G(\widetilde{X}) \Delta s_0^{\alpha}, \end{split}$$

with

$$\Psi_1(\widetilde{\cdot}) := -G(\widetilde{X})\partial_1 s_1^{\alpha}(\widetilde{\cdot})G(\widetilde{X})\Delta s_0^{\alpha}(\widetilde{\cdot})$$

The last step is to check that Φ and ϕ satisfy the assumptions of the lemma. Since s_0^{α} , s_1^{α} , $\partial_1 s_1^{\alpha}$ and $\partial_3 s_1^{\alpha}$ are periodic and zero-mean in the second argument, and slowly-varying

in average, so is Φ . There remains to prove that $\phi = \mathcal{O}_{\infty}(\varepsilon^3)$ in average. Since Δs_0^{α} is slowly-varying in average,

$$\int_0^\sigma \|\Delta s_0^{\alpha}(\widetilde{\cdot}(s))\| \, ds \le \lambda_0 \sigma \varepsilon^2$$

with $\lambda_0 > 0$. G being bounded by a constant c_g , this implies

$$\left\|\int_0^{\sigma} \varepsilon G(\widetilde{X}(s)) \Delta s_0^{\alpha}(\widetilde{\cdot}(s)) \, ds\right\| \le c_g \lambda_0 \sigma \varepsilon^3.$$

Similarly, $\partial_1 s_1$ being bounded by c_{11} , Ψ_1 satisfies

$$\left\|\int_0^{\sigma} \varepsilon^2 \Psi_1(\widetilde{\cdot}(s)) \, ds\right\| \le c_g^2 c_{11} \lambda_0 \sigma \varepsilon_0 \varepsilon^3.$$

Summing the two previous inequalities yields

$$\left\|\int_0^{\sigma} \phi(\widetilde{\cdot}(s)) \, ds\right\| \leq \lambda_0 c_g (1 + c_{11} c_g \varepsilon_0) \sigma \varepsilon^3,$$

which concludes the proof.

Lemma 2: Let $\overline{X}(\sigma)$ and $X(\sigma)$ be respectively the solutions of (12) and (15) starting at 0 from the same initial conditions. Then, for all $\sigma \ge 0$

$$\begin{split} \widetilde{X}(\sigma) &= \overline{X}(\sigma) + \mathcal{O}_{\infty}(\varepsilon^2). \\ \textit{Proof:} \quad \text{Let } E(\sigma) &:= \widetilde{X}(\sigma) - \overline{X}(\sigma). \text{ Then,} \end{split}$$

$$\begin{split} E(\sigma) &= \int_0^\sigma \Big[\frac{d\widetilde{X}}{d\sigma}(s) - \frac{d\overline{X}}{d\sigma}(s) \Big] \, ds \\ &= \varepsilon \int_0^\sigma \big[F(\widetilde{X}(s)) - F(\overline{X}(s)) \big] \, ds \\ &+ \varepsilon^2 \int_0^\sigma \Phi(\widetilde{\cdot}(s)) \, ds + \int_0^\sigma \phi(\widetilde{\cdot}(s)) \, ds \end{split}$$

As F is Lipschitz with constant λ_F ,

$$\varepsilon \int_0^\sigma \left\| F(\widetilde{X}(s)) - F(\overline{X}(s)) \right\| ds \le \varepsilon \lambda_F \int_0^\sigma \|E(s)\| ds.$$

On the other hand, there exists by lemma 3 c_1 such that

$$\varepsilon^2 \left\| \int_0^\sigma \Phi(\widetilde{\cdot}(s)) \, ds \right\| \le c_1 \varepsilon^2$$

Finally, as ϕ is $\mathcal{O}_{\infty}(\varepsilon^3)$ in average, there exists c_2 such that

$$\left\|\int_0^\sigma \phi(\widetilde{\cdot}(s))\,ds\right\| \le c_2\varepsilon^3\sigma$$

The summation of these estimations yields

$$||E(\sigma)|| \le \varepsilon \lambda_F \int_0^\sigma ||E(s)|| \, ds + c_1 \varepsilon^2 + c_2 \varepsilon^3 \sigma.$$

Then by Gronwall's lemma [9, Lemma 1.3.3]

$$||E(\sigma)|| \le \left(\frac{c_2}{\lambda_F} + c_1\right) e^{\lambda_F \sigma} \varepsilon^2,$$

which means $\widetilde{X} = \overline{X} + \mathcal{O}_{\infty}(\varepsilon^2)$.

The following lemma is an extension of Besjes' lemma [9, Lemma 2.8.2] when φ is no longer Lipschitz, but only slowly-varying in average.

Lemma 3: Assume $\varphi(X, \sigma)$ is T-periodic and zero-mean in the second argument, bounded, and slowly-varying in average. Assume the solution $X(\sigma)$ of $\dot{X} = \mathcal{O}_{\infty}(\varepsilon)$ is defined for $0 \le \sigma \le L/\varepsilon$. There exists $c_1 > 0$ such that

$$\left\| \int_0^\sigma \varphi(X(s), s) \, ds \right\| \le c_1.$$

Proof: Along the lines of [9], we divide the interval [0,t] in m subintervals $[0,T], \ldots, [(m-1)T,mT]$ and a remainder [mT,t]. By splitting the integral on those intervals, we write

$$\int_0^\sigma \varphi(x(s), s) \, ds = \sum_{i=0}^m \int_{(i-1)T}^{iT} \varphi(x((i-1)T), s) \, ds$$
$$+ \sum_{i=0}^m \int_{(i-1)T}^{iT} \left[\varphi(x(s), s) - \varphi(x((i-1)T), s) \right] ds$$
$$+ \int_{mT}^\sigma \varphi(x(s), s) \, ds,$$

where each of the integral in the first sum are zero as φ is periodic with zero mean. Since φ is bounded, the remainder is also bounded by a constant $c_2 > 0$. Besides

$$\begin{aligned} x(s) &= x((i-1)T) + \int_{(i-1)T}^{s} \dot{x}(\tau) \, d\tau \\ &= x((i-1)T) + \varepsilon q(s), \end{aligned}$$

with q continuous and bounded. By hypothesis, there exists $\lambda > 0$ such that for $0 \le i \le m$,

$$\int_{(i-1)T}^{iT} \|\varphi(x(s),s) - \varphi(x((i-1)T),s)\| \ ds \le \lambda T\varepsilon$$

Therefore by summing the previous estimations,

$$\left\|\int_0^{\sigma} \varphi(x(s), s) \, ds\right\| \le m\lambda T\varepsilon + c_2,$$

with $mT \le t \le L/\varepsilon$, consequently $m\lambda T\varepsilon + c_2 \le \lambda L + c_2$; which concludes the proof.

IV. DEMODULATION

From (8c), we can write the measured signal y as

$$y(t) = y_a(t) + y_v(t)s_1\left(u(t), \frac{t}{\varepsilon}\right) + \mathcal{O}_{\infty}(\varepsilon^2),$$

where the signal u feeding the PWM encoder is known. The following result shows y_a and y_v can be estimated from y, for use in a control law as described in section III-A.

Theorem 2: Consider the estimators \hat{y}_a and \hat{y}_v defined by

$$\widehat{y}_a(t) := \frac{3}{2} M(y)(t) - \frac{1}{2} M(y)(t-\varepsilon)$$
$$k_{\Delta}(\tau) := \left(y(\tau) - \widehat{y}_a(\tau)\right) s_1\left(u(\tau), \frac{\tau}{\varepsilon}\right)$$
$$\widehat{y}_v(t) := \frac{M(k_{\Delta})(t)}{\overline{s_1^2}(u(t))},$$

where $M: \underline{y} \mapsto \varepsilon^{-1} \int_0^\varepsilon y(\tau) d\tau$ is the moving average operator, and $\overline{s_1^2}$ the mean of s_1^2 in the second argument (cf end of section I). Then,

$$\widehat{y}_a(t) = y_a(t) + \mathcal{O}_\infty(\varepsilon^2)$$
 (21a)

$$\widehat{y}_v(t) = y_v(t) + \mathcal{O}_\infty(\varepsilon^2).$$
 (21b)

Recall that by construction $y_v(t) = \mathcal{O}_{\infty}(\varepsilon)$, hence (21b) is essentially a first-order estimation; notice also that $\overline{s_1^2}(u(t))$ is always non-zero when u(t) does not exceed the range of the PWM encoder.

Proof: Taylor expanding y_a , y_v , u and s_1 yields

$$y_a(t-\tau) = y_a(t) - \tau \dot{y}_a(t) + \mathcal{O}_{\infty}(\tau^2)$$

$$y_v(t-\tau) = y_v(t) + \mathcal{O}_{\infty}(\varepsilon) \mathcal{O}_{\infty}(\tau)$$

$$s_1(u(t-\tau), \sigma) = s_1(u(t) + \mathcal{O}_{\infty}(\tau), \sigma)$$

$$= s_1(u(t), \sigma) + \mathcal{O}_{\infty}(\tau);$$

in the second equation, we have used $y_v(t) = \mathcal{O}_{\infty}(\varepsilon)$. The moving average of y_a then reads

$$M(y_a)(t) = \frac{1}{\varepsilon} \int_0^\varepsilon y_a(t-\tau) d\tau$$

= $\frac{1}{\varepsilon} \int_0^\varepsilon (y_a(t) - \tau \dot{y}_a(t) + \mathcal{O}_\infty(\tau^2)) d\tau$
= $y_a(t) - \frac{\varepsilon}{2} \dot{y}_a(t) + \mathcal{O}_\infty(\varepsilon^2).$ (22)

A similar computation for $k_v(t) := y_v(t)s_1(u(t), \frac{t}{\epsilon})$ yields

$$M(k_v)(t) = \frac{1}{\varepsilon} \int_0^\varepsilon y_v(t-\tau) s_1\left(u(t-\tau), \frac{t-\tau}{\varepsilon}\right) d\tau$$

= $y_v(t) \left(\overline{s_1}(u(t)) + \mathcal{O}_\infty(\varepsilon)\right) + \mathcal{O}_\infty(\varepsilon^2)$
= $\mathcal{O}_\infty(\varepsilon^2),$ (23)

since s_1 is 1-periodic and zero mean in the second argument. Summing (22) and (23), we eventually find

$$M(y)(t) = y_a(t) - \frac{\varepsilon}{2}\dot{y}_a(t) + \mathcal{O}_{\infty}(\varepsilon^2)$$

As a consequence, we get after another Taylor expansion

$$\frac{3}{2}M(y)(t) - \frac{1}{2}M(y)(t-\varepsilon) = y_a(t) + \mathcal{O}_{\infty}(\varepsilon^2),$$

which is the desired estimation (21a).

On the other hand, (21a) implies

$$k_{\Delta}(t) = y_v(t)s_1^2(u(t), \frac{t}{\varepsilon}) + \mathcal{O}_{\infty}(\varepsilon^2).$$

Proceeding as for $M(k_v)$, we find

$$M(k_{\Delta})(t) = \frac{1}{\varepsilon} \int_{0}^{\varepsilon} y_{v}(t-\tau) s_{1}^{2} \left(u(t-\tau), \frac{t-\tau}{\varepsilon} \right) d\tau$$

$$= y_{v}(t) \left(\overline{s_{1}^{2}}(u(t)) + \mathcal{O}_{\infty}(\varepsilon) \right) + \mathcal{O}_{\infty}(\varepsilon^{2})$$

$$= y_{v}(t) \overline{s_{1}^{2}}(u(t)) + \mathcal{O}_{\infty}(\varepsilon^{2}).$$

Dividing by $\overline{s_1^2}(u(t))$ yields the desired estimation (21b).

V. NUMERICAL EXAMPLE

We illustrate the interest of the approach on the system

$$\dot{x}_1 = x_2,$$

 $\dot{x}_2 = x_3,$
 $\dot{x}_3 = u + d,$
 $y = x_2 + x_1 x_3$

where d is an unknown disturbance; u will be impressed through PWM with frequency 1 kHz (i.e. $\varepsilon = 10^{-3}$) and

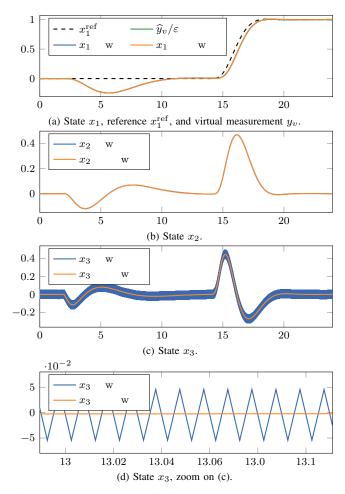


Fig. 3: States x_1, x_2, x_3 with ideal and actual control laws.

range [-20, 20]. The objective is to control x_1 , while rejecting the disturbance d, with a response time of a few seconds. We want to operate around equilibrium points, which are of the form $(x_1^{\text{eq}}, 0, 0; -d^{\text{eq}}, d^{\text{eq}})$, for x_1^{eq} and d^{eq} constant. Notice the observability degenerates at such points, which renders not trivial the design of a control law.

Nevertheless the PWM-induced signal injection makes available the virtual measurement

$$y_v = \varepsilon \begin{pmatrix} x_3 & 1 & x_1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \varepsilon x_1$$

from which it is easy to design a suitable control law, without even using the actual input $y_a = x_2 + x_1 x_3$. The system being now fully linear, we use a classical controller-observer, with disturbance estimation to ensure an implicit integral effect. The observer is thus given by

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + l_1 \left(\frac{y_v}{\varepsilon} - \hat{x}_1\right), \\ \dot{\hat{x}}_2 &= \hat{x}_3 + l_2 \left(\frac{y_v}{\varepsilon} - \hat{x}_1\right), \\ \dot{\hat{x}}_3 &= u + \hat{d} + l_3 \left(\frac{y_v}{\varepsilon} - \hat{x}_1\right), \\ \dot{\hat{d}} &= l_d \left(\frac{y_v}{\varepsilon} - \hat{x}_1\right), \end{aligned}$$

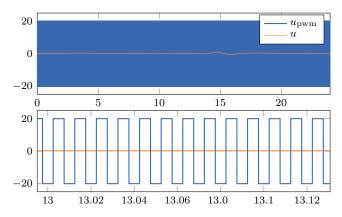


Fig. 4: Control input u and its modulation u_{pwm} ; full view (top), zoom (bottom).

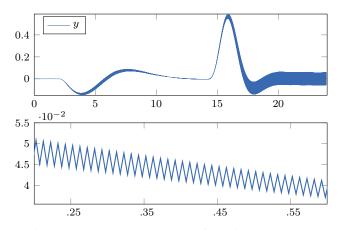


Fig. 5: Measured output y (top); full view (top), zoom (bottom).

and the controller by

$$u = -k_1 \hat{x}_1 - k_2 \hat{x}_2 - k_3 \hat{x}_3 - k_d \hat{d} + k x_1^{\text{ref}}.$$

The gains are chosen to place the observer eigenvalues at $(-1.19, -0.73, -0.49\pm0.57i)$ and the controller eigenvalues at $(-6.59, -3.30\pm5.71i)$. The observer is slower than the controller in accordance with dual Loop Transfer Recovery, thus ensuring a reasonable robustness. Setting $\eta := (\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{d})^T$, this controller-observer obviously reads

$$u = -K\eta + kx_1^{\text{ref}} \tag{24a}$$

$$\dot{\eta} = M\eta + Nx_1^{\text{ref}}(t) + Ly_v \tag{24b}$$

Finally, this ideal control law is implemented as

$$u_{\rm pwm}(t) = \mathcal{M}\left(-K\eta + kx_1^{\rm ref}, \frac{t}{\varepsilon}\right)$$
(25a)

$$\dot{\eta} = M\eta + Nx_1^{\text{ref}} + L\frac{\hat{y}_v}{2}, \qquad (25b)$$

where \mathcal{M} is the PWM function described in section II, and \hat{y}_v is obtained by the demodulation process of section IV.

The test scenario is the following: at t = 0, the system start at rest at the origin; from t = 2, a disturbance d = -0.25 is applied to the system; at t = 14, a filtered unit step is applied to the reference x_1^{ref} . In Fig. 3 the ideal control law (24), i.e. without PWM and assuming y_v known, is compared to the true control law (25): the behavior of (25) is excellent, it is nearly impossible to distinguish the two situations on the responses of x_1 and x_2 as by (8a) the corresponding ripple is only $\mathcal{O}_{\infty}(\varepsilon^2)$; the ripple is visible on x_3 , where it is $\mathcal{O}_{\infty}(\varepsilon)$. The corresponding control signals u and u_{pwm} are displayed in Fig. 4, and the corresponding measured outputs in Fig. 5.

To investigate the sensitivity to measurement noise, the same test was carried out with band-limited white noise (power density 1×10^{-9} , sample time 1×10^{-5}) added to y. Even though the ripple in the measured output is buried in noise, see Fig. 6, the virtual output is correctly demodulated and the control law (25) still behaves very well.

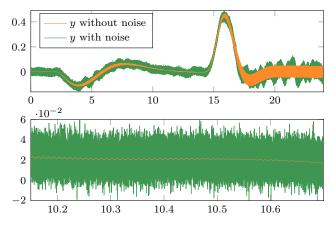


Fig. 6: Measured output y with and without noise; full view (top), zoom (bottom).

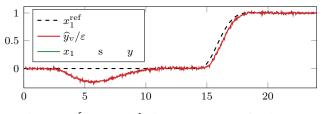


Fig. 7: x_1^{ref} , x_1 , and \hat{y}_v in the presence of noise.

CONCLUSION

We have presented a method to take advantage of the benefits of signal injection in PWM-fed systems without the need for an external probing signal. For simplicity, we have restricted to Single-Input Single-Output systems, but there are no essential difficulties to consider Multiple-Input Multiple-Output systems. Besides, though we have focused on classical PWM, the approach can readily be extended to arbitrary modulation processes, for instance multilevel PWM; in fact, the only requirements is that s_0 and s_1 meet the regularity assumptions discussed in remark 1.

REFERENCES

- P. Combes, A. K. Jebai, F. Malrait, P. Martin, and P. Rouchon, "Adding virtual measurements by signal injection," in *American Control Conference*, 2016, pp. 999–1005.
- [2] P. Jansen and R. Lorenz, "Transducerless position and velocity estimation in induction and salient AC machines," *IEEE Trans. Industry Applications*, vol. 31, pp. 240–247, 1995.

- [3] M. Corley and R. Lorenz, "Rotor position and velocity estimation for a salient-pole permanent magnet synchronous machine at standstill and high speeds," *IEEE Trans. Industry Applications*, vol. 34, pp. 784–789, 1998.
- [4] A. K. Jebai, F. Malrait, P. Martin, and P. Rouchon, "Sensorless position estimation and control of permanent-magnet synchronous motors using a saturation model," *International Journal of Control*, vol. 89, no. 3, pp. 535–549, 2016.
- [5] B. Yi, R. Ortega, and W. Zhang, "Relaxing the conditions for parameter estimation-based observers of nonlinear systems via signal injection," *Systems and Control Letters*, vol. 111, pp. 18–26, 2018.
- [6] C. Wang and L. Xu, "A novel approach for sensorless control of PM machines down to zero speed without signal injection or special PWM technique," *IEEE Transactions on Power Electronics*, vol. 19, no. 6, pp. 1601–1607, 2004.
- [7] B. Lehman and R. M. Bass, "Extensions of averaging theory for power electronic systems," *IEEE Transactions on Power Electronics*, vol. 11, no. 4, pp. 542–553, July 1996.
- [8] A. Filippov, Differential equations with discontinuous righthand sides. Control systems, ser. Mathematics and its Applications. Kluwer, 1988.
- [9] J. Sanders, F. Verhulst, and J. Murdock, Averaging methods in nonlinear dynamical systems, 2nd ed. Springer, 2005.