Synergistic control barrier functions with application to obstacle avoidance for nonholonomic vehicles

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Abstract— **Control barrier functions (CBFs) have recently emerged as a means to ensure safety of controlled dynamical systems. CBFs are suitable for obstacle avoidance, where the CBF is often constructed from the distance and relative velocity between the vehicle and the obstacle. For vehicles required to maintain non-zero forward speed, ordinary (non-hybrid) CBFs cannot ensure safety due to vanishing control authority when the vehicle is oriented directly towards the obstacle. In this paper, synergistic CBFs are proposed, which is an intuitive extension of CBFs using ideas from synergistic Lyapunov functions. A synergistic CBF for obstacle avoidance for nonholonomic vehicles is constructed by shifting the orientations with vanishing control authority. This induces a penalty for traversing the obstacle in the counterclockwise or clockwise direction, where a logic variable is used to determine the preferred direction. The performance of the CBF is illustrated by a case study.**

I. Introduction

The notion of control barrier functions (CBFs) was first introduced in [1], as a means to ensure safety of controlled dynamical systems. An overview of recent developments is given in [2], along with application examples. CBFs are used in [3] for collision avoidance of swarm robots, where the robot agents are modeled as point masses with the linear accelerations as control input. The CBFs are constructed from the relative distance and velocity between robot agents. CBFs based on a similar idea are used in [4] and [5] for obstacle avoidance for fully-actuated autonomous marine vessels. The short-comings of CBFs for collision avoidance are noted in [3]; in perfectly symmetric cases the robot agents, or autonomous vehicles, may enter deadlock situations, thus precluding any guarantees for achieving some nominal control objective. This is analogous to the fact that global asymptotic stabilization to a point while avoiding an obstacle is impossible by continuous feedback control (see e.g. [6] for proof). Hybrid feedbacks for global asymptotic stabilization of compact sets in presence of obstacles are proposed in [7], [8], and [9].

In this paper we consider CBFs for collision avoidance of a vehicle with unicycle dynamics; $\dot{x} = v\cos(\psi)$, $\dot{y} = v\sin(\psi)$, $\dot{\psi} = \omega$, with non-zero forward speed $v > 0$. Since ω does not appear in the derivative of the position $p := [x \;\; y]^\top$, this results in limited control authority [10]. The challenge of constructing CBFs for vehicles required to maintain non-zero forward speed is addressed in [11], where the authors propose to construct barrier functions by calculating the future evolution of the system subject to pre-determined nominal evading maneuvers.

Inspired by synergistic Lyapunov functions for robust global stabilization of continuous-time systems [12] [13], we propose a type of hybrid CBFs intended for continuous-time systems with limited control authority in certain configurations. Adopting the terminology from [12], [13], we refer to these as synergistic CBFs (SCBFs). The usefulness of SCBFs is illustrated by applying it to the problem of obstacle avoidance for a vehicle with unicycle dynamics, with constant forward speed and turning rate as control input.

Hybrid CBFs appear in many contexts. In [14] and [15] walking robots are controlled using CBFs, where the robot dynamics are modeled as a hybrid system. Hybrid (nonsmooth) CBFs for obstacle avoidance are proposed in [10], where the hybrid formulation is used to accommodate instantaneous changes in the constraints (e.g. when an obstacle is detected). In this paper, we use hybrid CBFs to address the problem of vanishing control authority of the CBF. Earlier references for CBFs assume uniform relative degree one (see for instance [16]), which was extended to systems with uniform higher relative degree in [14]. In [17] the assumption of uniform relative degree is relaxed – however, the definitions therein assume that the uncontrolled dynamics of the system are safe (in the appropriate sense) whenever the control authority vanishes. SCBFs ensure safety of systems with non-uniform relative degree one by appropriate switching between overlapping non-hybrid CBFs, without requiring that the uncontrolled dynamics are safe when the control authority vanishes.

The remainder of this paper is organized as follows: Section II describes the framework of hybrid systems. Section III contains a brief review of barrier certificates for hybrid systems and CBFs for continuous-time systems. SCBFs are proposed in Section IV, which is the main contribution of this paper. In Section V SCBFs are applied for obstacle avoidance for a vehicle with unicycle dynamics. The results from Section V is supported by numerical simulations in Section VI. Finally, Section VII concludes the paper.

Notation: $\mathbb{R}$ is the set of real numbers and $\mathbb{R}^n$ is the n-dimensional Euclidean space. $\mathbb{R}_{\geq 0}$ and $\mathbb{R}_{>0}$ are the set of non-negative and positive numbers, respectively. $\mathbb{N}$ is the set of non-negative integers. The Euclidean norm of a vector $x \in \mathbb{R}^n$ is denoted $|x|$. For two vectors $x \in \mathbb{R}^n, y \in \mathbb{R}^n$, we define $\langle x, y\rangle := x^\top y$. For a function $f : \mathbb{R}^n \times Y \to \mathbb{R}^m$, the Jacobian matrix with respect to $x \in \mathbb{R}^n$ is denoted $J_x(f(x, y)) \in$

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$\mathbb{R}^{n\times m}$. For a function $f:\mathbb{R}^n\times Y\to\mathbb{R}$, $\nabla_x(f(x,y)) := J_x^\top(f(x,y))$ is a column vector. The subscript x is omitted when the argument is clear from context. When convenient we use the Lie derivative notation: $L_f B(x) := \langle\nabla_x B(x), f(x)\rangle$, where $f:\mathbb{R}^n\to\mathbb{R}^n$ is a vector field and $B:\mathbb{R}^n\to\mathbb{R}$ is a scalar function. For a set X, ∂X is the boundary and $\mathrm{Int}X$ is the interior. An extended class-$\mathcal{K}$ function is a continuous and strictly increasing function $\alpha:\mathbb{R}\to\mathbb{R}$, defined on the entire real number line, with $\alpha(0)=0$. Finally, $\dot{x}$ is the time derivative of x, and x^+ is the value of x after an instantaneous change.

II. Hybrid dynamical systems

A. Modeling framework

A hybrid inclusion is modeled as [18]

$$\mathcal{H}:\begin{cases}\dot{x}\in F(x) & x\in C,\\ x^+\in H(x) & x\in D.\end{cases} \tag{1}$$

The state $x\in\mathbb{R}^n$ may evolve both in continuous time, referred to as flow, and in discrete time, referred to as jumps. As such, $C\subset\mathbb{R}^n$ is the flow set, $F:\mathbb{R}^n\rightrightarrows\mathbb{R}^n$ is the flow map, $D\subset\mathbb{R}^n$ is the jump set, and $H:\mathbb{R}^n\rightrightarrows\mathbb{R}^n$ is the jump map. $\mathcal{H}$ satisfies the hybrid basic conditions [18, Assumption 6.5] if: 1) C and D are closed, 2) F is non-empty, locally bounded, outer semicontinuous and convex on C, 3) H is non-empty, locally bounded and outer semicontinuous on D.

B. Forward invariance of hybrid systems

A solution to $\mathcal{H}$ exists on a hybrid time domain $\mathrm{dom}\,x\subset\mathbb{R}_{\geq 0}\times\mathbb{N}$, parametrized by an ordinary time variable $t\in\mathbb{R}_{\geq 0}$ and a jump variable $j\in\mathbb{N}$. If a solution cannot be extended, it is said to be maximal, and if it exists on an unbounded hybrid time domain it is said to be complete. Due to space constraints we refer to [18, Definitions 2.3 – 2.7]. Central to this paper is the notion of forward invariance, which for hybrid dynamical systems may be stated as [19, Definition 1]:

Definition 1. *The set $\mathcal{K}\subset\mathbb{R}^n$ is forward invariant if for each $x_0\in\mathcal{K}$, each maximal solution x starting from x_0 satisfies $x(t,j)\in\mathcal{K}$ for all $(t,j)\in\mathrm{dom}\,x$.*

We will refer to $\mathcal{K}$ as the safe set, while forward invariance of $\mathcal{K}$ is referred to as safety. In [19], Definition 1 is referred to as forward pre-invariance, while forward invariance has the additional requirement of maximal solutions being complete. In this paper we do not make that distinction, since we are mainly concerned with safety in the sense of solutions not entering the complement of $\mathcal{K}$, that is, ensuring $x_0\in\mathcal{K}\implies x(t,j)\notin(C\cup D)\setminus\mathcal{K}\,,\forall(t,j)\in\mathrm{dom}\,x$. In particular, the conditions of Proposition 1 below does not rule out escape to infinity in finite time. We also note that Definition 1 does not require that solutions are unique, and is sometimes referred to as strong forward (pre-)invariance. Forward invariance, and controlled forward invariance, of hybrid systems is studied in [20] and [21], respectively.

III. Barrier certificates and CBFs

A. Barrier certificates for hybrid systems

Barrier certificates are used to establish forward invariance of sets in nonlinear systems and hybrid systems. For forward invariance of sets in hybrid systems, we will employ the following proposition, as a special case of [19, Theorem 1]:

Proposition 1. *Assume $\mathcal{H}$ satisfies the hybrid basic conditions. Let $B:\mathbb{R}^n\to\mathbb{R}$ be a continuously differentiable function that defines the closed set $\mathcal{K}:=\{x\in\mathbb{R}^n : B(x)\leq 0\}$. Let $\mathcal{U}$ be an open neighborhood of $\mathcal{K}$. If the following holds:*

$$\langle\nabla B(x), y\rangle\leq 0\quad\forall x\in(\mathcal{U}\setminus\mathcal{K})\cap C\quad\forall y\in F(x), \tag{2}$$

$$B(h)\leq 0,\quad\forall x\in\mathcal{K}\cap D\quad\forall h\in H(x), \tag{3}$$

$$H(x)\subset C\cup D\quad\forall x\in D, \tag{4}$$

then B is a barrier function and $\mathcal{K}$ is forward invariant.

See [19, Theorem 1] for proof. Condition (2) ensures that x cannot leave $\mathcal{K}$ through flow; whenever $x\in\partial\mathcal{K}$ it is only allowed to flow along the boundary or towards the interior of $\mathcal{K}$. Condition (3) and (4) ensures that x cannot leave $\mathcal{K}$ through jumps; whenever $x\in\mathcal{K}$ it is only allowed to jump to somewhere within $\mathcal{K}$.

Remark 1. *In some earlier literature, barrier functions are defined as functions that approach infinity on the boundary of the safe set. The barrier certificates for continuous-time systems discussed in [2] follow a similar logic as the one outlined above.*

B. Control barrier functions for continuous-time systems

Whereas barrier certificates can be used to show safety of closed-loop systems, CBFs are used to derive a set of control inputs that render controlled systems safe. Define the affine control system

$$\dot{x}=f(x)+g(x)u, \tag{5}$$

with state $x\in\mathbb{R}^n$ and input $u\in U\subset\mathbb{R}^m$.

Assumption 1. *$f:\mathbb{R}^n\to\mathbb{R}^n$ and $g:\mathbb{R}^n\to\mathbb{R}^{n\times m}$ are locally Lipschitz. U is non-empty, convex, and bounded.*

The following definition of CBFs is modified from [22, Definition 5]:

Definition 2. *Let $B:X\to\mathbb{R}$, where $X\subset\mathbb{R}^n$, be a continuously differentiable function that defines the sets*

$$\mathcal{K}:=\{x\in\mathrm{Int}X : B(x)\leq 0\} \tag{6}$$

$$\partial\mathcal{K}:=\{x\in\mathrm{Int}X : B(x)=0\}. \tag{7}$$

B is a CBF for (5) if there exists an extended class-$\mathcal{K}$ function $\alpha:\mathbb{R}\to\mathbb{R}$ such that

$$\inf_{u\in U}\Big[L_f B(x)+L_g B(x)u\Big]\leq-\alpha(B(x))\quad\forall x\in X. \tag{8}$$

Let $U_B: X\rightrightarrows\mathbb{R}^m$ be the admissible inputs defined by

$$U_B(x):=\{u\in U : L_f B(x)+L_g B(x)u+\alpha(B(x))\leq 0\}. \tag{9}$$

Corollary 2 in [22] (see also [2, Theorem 2]) states forward invariance of $\mathcal{K}$ for any Lipschitz continuous feedback controller $u(x) \in U_B(x)$, provided that $\nabla B(x) \neq 0\ \forall x \in \partial\mathcal{K}$. In [23, Theorem 3] safety of the system

$$\dot{x} \in F_B(x) := \{f(x) + g(x)u : u \in U_B(x)\} \tag{10}$$

is established, provided that the set-valued mapping $F_B : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is Lipschitz continuous and $\nabla B(x) \neq 0\ \forall x \in \partial\mathcal{K}$. This allows discontinuous u, as long as U_B is Lipschitz. Since U_B is defined on a neighborhood of $\mathcal{K}$, forward invariance of $\mathcal{K}$ may be established from Proposition 1, without requiring that F_B is Lipschitz, and omitting the regularity property $\nabla B(x) \neq 0\ \forall x \in \partial\mathcal{K}$.

Proposition 2. *If B is a CBF on X defining $\mathcal{K}$, then $\mathcal{K}$ is forward invariant for (10).*

Proof. Since B is a CBF on X, and U is bounded and convex by assumption, F_B is non-empty, locally bounded, outer semicontinuous, and convex on X. Since $L_fB(x) + L_gB(x)u \leq -\alpha(B(x))\ \forall u \in U_B$, $\alpha(B(x)) > 0\ \forall x \in X \setminus \mathcal{K}$, and $\mathcal{K} \subset \mathrm{Int}X$, there exists an open neighborhood $\mathcal{U}$ of $\mathcal{K}$ such that

$$\langle \nabla B(x), y\rangle < 0,\ \forall x \in \mathcal{U} \setminus \mathcal{K},\ \forall y \in F_B(x). \tag{11}$$

Forward invariance follows from Proposition 1, with $C = \mathbb{R}^n$, $D = \emptyset$ and arbitrary G. ☐

IV. Synergistic control barrier functions

We now introduce SCBFs, which is an intuitive extension of CBFs following a similar idea as synergistic Lyapunov functions [12], [13]. SCBFs are motivated by a desire to ensure safety of systems where the influence of the control input on the CBF vanishes in certain configurations.

A. Synergistic CBF candidate

Augmenting (5) with a logic variable $q \in Q$, where Q is a discrete set, results in

$$\dot{x} = f(x) + g(x)u, \quad \dot{q} = 0. \tag{12}$$

Let $X \subset \mathbb{R}^n$ be a closed set, and define $\mathcal{X} := X \times Q$. Let $B : \mathcal{X} \to \mathbb{R}$ be a continuously differentiable function that defines the sets

$$\mathcal{K} := \{(x, q) \in \mathrm{Int}\mathcal{X} : B(x, q) \leq 0\} \tag{13}$$

$$\partial\mathcal{K} := \{(x, q) \in \mathrm{Int}\mathcal{X} : B(x, q) = 0\}. \tag{14}$$

Define $\Psi \subset \mathcal{X}$ as

$$\Psi := \{(x, q) \in \mathcal{X} : L_gB(x, q) = 0 \text{ and } L_fB(x, q) \geq 0\}, \tag{15}$$

which is the set with vanishing control authority and non-decreasing B. Let $M(x) := \min_{q\in Q} B(x, q)$ and define the synergy gap

$$\mu := \inf_{(x,q)\in\Psi} (B(x, q) - M(x)). \tag{16}$$

If the synergy gap is non-zero, that is $\mu > 0$, B is an SCBF candidate for the system (12).

B. Guaranteed safety using Synergistic CBFs

Select a $\delta > 0$, satisfying $\delta < \mu$, and define

$$D := \{(x, q) \in \mathcal{X} : M(x) - B(x, q) \leq -\delta\} \tag{17}$$

$$C := ((\mathbb{R}^n \times Q) \setminus D) \cup \partial D. \tag{18}$$

Definition 3. *Let B be an SCBF candidate for (12), with synergy gap $\mu > 0$. B is an SCBF for (12) if there exists an extended class-$\mathcal{K}$ function $\alpha : \mathbb{R} \to \mathbb{R}$ such that, $\forall (x, q) \in \mathrm{Int}\mathcal{X} \cap C$,*

$$\inf_{u\in U} \Big[L_fB(x, q) + L_gB(x, q)u\Big] \leq -\alpha(B(x, q)). \tag{19}$$

Note that (19) needs to hold on the flow set only. By design, $(\Psi \cap C) \setminus \partial\mathcal{X} = \emptyset$. In other words, the only way solutions can flow in $\mathrm{Int}\mathcal{X}$ with $L_gB(x, q) = 0$, is if B is strictly decreasing. Let $U_B : \mathcal{X} \rightrightarrows \mathbb{R}^m$ be defined by

$$U_B(x, q) := \{u \in U : L_fB(x, q) + L_gB(x, q)u + \alpha(B(x, q)) \leq 0\}. \tag{20}$$

Further define $F_B : \mathcal{X} \rightrightarrows \mathbb{R}^n$ as

$$F_B(x, q) := \{f(x) + g(x)u : u \in U_B(x, q)\}. \tag{21}$$

Let $H(x) := \{q \in Q : B(x, q) = M(x)\}$, and define

$$\mathcal{H}_B : \begin{cases} \dot{x} \in F_B(x),\ \dot{q} = 0 & (x, q) \in C \\ q^+ \in H(x),\ x^+ = x, & (x, q) \in D. \end{cases} \tag{22}$$

We now state our main theorem on safety of $\mathcal{H}_B$.

Theorem 1. *If B is an SCBF on $\mathcal{X}$ defining $\mathcal{K}$, then $\mathcal{K}$ is forward invariant for $\mathcal{H}_B$ defined in (22).*

Proof. We prove the theorem by showing that the conditions of Proposition 1 are satisfied. First note that $x^+ = x \implies B(x^+, q) = B(x, q)$, and $\dot{q} = 0 \implies \langle \nabla B_q(x, q), \dot{q}\rangle = 0$. Since B is a CBF on $\mathcal{X}$, and U is bounded and convex by assumption, F_B is non-empty, locally bounded, outer semicontinuous and convex on $\mathrm{Int}\mathcal{X} \cap C$. Condition (2) is satisfied by similar arguments as Proposition 2. D and H are designed such that, $\forall (x, q) \in D$, $\forall h \in H(x)$,

$$B(x, h) - B(x, q) = M(x) - B(x, q) \leq -\delta < 0, \tag{23}$$

which shows that (3) is satisfied. (4) is satisfied since $C \cup D = \mathbb{R}^n \times Q$. Forward invariance follows from Proposition 1. ☐

An important feature of SCBFs is that U_B, and consequently F_B, is only required to be non-empty on a subset of $\mathrm{Int}\mathcal{X}$, namely, the subset where flow is allowed. In the next section we show that an SCBF exists for a system where ordinary CBFs do not exist.

V. Case: Safety of nonholonomic vehicle

A. Vehicle model and problem statement

The unit circle and planar rotations are given by [24]

$$\mathcal{S}^1 := \{z \in \mathbb{R}^2 : z^\top z = 1\}, \tag{24}$$

$$SO(2) := \{R \in \mathbb{R}^{2\times 2} : R^\top R = I, det(R) = 1\}. \tag{25}$$

We denote the unit vector corresponding to an angle a as

$$z^a := \begin{bmatrix} z_1^a \\ z_2^a \end{bmatrix} = \begin{bmatrix} \cos a \\ \sin a \end{bmatrix} \in \mathcal{S}^1, \tag{26}$$

while the corresponding map $R : \mathcal{S}^1 \to SO(2)$ is given by

$$R(z^a) := \begin{bmatrix} z_1^a & -z_2^a \\ z_2^a & z_1^a \end{bmatrix} \in SO(2). \tag{27}$$

Let S be the rotation matrix corresponding to the 90 degree counterclockwise rotation;

$$S := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in SO(2). \tag{28}$$

The kinematic equation for motion along the unit circle with angular velocity $\omega_a = \dot{a}$ is given by $\dot{z}^a = \omega_a S z^a$.

We will consider a vehicle with unicycle dynamics. Let $p := (x, y) \in \mathbb{R}^2$ be the position in the plane, and $z \in \mathcal{S}^1$ the unit heading vector, where the superscript is omitted for compactness. Using this notation the unicycle kinematics are given by $\dot{p} = zv$, $\dot{z} = Sz\omega$, where $v \in \mathbb{R}_{>0}$ is the forward speed assumed constant. The turning rate $\omega \in U \subset \mathbb{R}$ is considered the control input. Let ω be saturated by $|\omega| \leq \omega_{sat} \in \mathbb{R}_{>0}$, resulting in $U := [-\omega_{sat}, \omega_{sat}]$. Augmenting the system with logic variables $q \in Q$, we obtain the affine control system

$$\begin{bmatrix} \dot{p} \\ \dot{z} \end{bmatrix} = f(z) + g(z)\omega, \quad \dot{q} = 0, \tag{29}$$

where

$$f(z) := \begin{bmatrix} zv \\ 0 \end{bmatrix}, \; g(z) := \begin{bmatrix} 0 \\ Sz \end{bmatrix}. \tag{30}$$

The states of the system are $(p, z, q) \in \mathbb{R}^2 \times \mathcal{S}^1 \times Q$, where q is available for control system design. We consider a circular obstacle with radius $r_o \in \mathbb{R}_{>0}$. Without loss of generality, we define a coordinate system with the obstacle centered at the origin. Select a safety radius $r_s > r_o$ and define the unsafe set $\mathcal{K}_u := \{(p, z, q) \in \mathbb{R}^2 \times \mathcal{S}^1 \times Q : |p| < r_s\}$. Define $X := \{p \in R^2 : |p| \geq \epsilon\}$, where $\epsilon < r_s$ is an arbitrarily small positive number, and let $\mathcal{X} = X \times \mathcal{S}^1 \times Q$. We are now ready to state the control task.

Problem statement: Consider the system (29). Design a set $\mathcal{K} \subset \operatorname{Int}\mathcal{X}$ and a corresponding set of control inputs $U_B : \mathcal{X} \rightrightarrows \mathbb{R}$ such that $\omega \in U_B$ ensures forward invariance of the set $\mathcal{K} \setminus \mathcal{K}_u$.

Remark 2. *By requiring safety only for solutions starting in $\mathcal{K} \setminus \mathcal{K}_u$, and not for all $\mathcal{K}$, we implicitly allow $\mathcal{K} \cap \mathcal{K}_u \neq \emptyset$.*

B. Logic variables

We will first construct a non-hybrid CBF, which fails to ensure safety when the vehicle is oriented directly towards the origin. Next, we will construct a synergistic CBF with the critical orientations shifted in the clockwise or counter-clockwise direction, resulting in a preferred turning direction. This is desired when approaching the obstacle, but may result in undesired behavior when the vehicle is in the interior of $\mathcal{K}_u$. Finally, the non-hybrid CBF and the synergistic CBF are combined into a new synergistic CBF, ensuring both safety when approaching the obstacle and mitigating any undesired behavior if disturbances pushes the vehicle far into $\mathcal{K}_u$. To this end we define the logic variables

$$q := \begin{bmatrix} q_0 \\ q_1 \end{bmatrix} \in Q_0 \times Q_1 =: Q, \quad \begin{matrix} Q_0 := \{0, 1\} \\ Q_1 := \{-1, 1\}. \end{matrix} \tag{31}$$

Here, $q_0 = 1$ will indicate that the vehicle is in evasive mode, where evasive mode is loosely defined as the vehicle approaching the obstacle. When in evasive mode, q_1 will be used to assign the preferred turning direction.

C. Non-hybrid CBF

Let $B_0 : X \times \mathcal{S}^1 \to \mathbb{R}$ be given by

$$B_0(p, z) := r_s - |p| - t_0 v \frac{p^\top}{|p|} z, \tag{32}$$

where the last term is the relative velocity between the vessel and the origin weighted by a time constant $t_0 \in \mathbb{R}_{>0}$. B_0 has similar form as the non-hybrid CBFs used in [3], [4]. The Lie derivatives of B_0 are given by

$$L_g B_0(p, z) = -\, t_0 v \frac{p^\top}{|p|} S z, \tag{33}$$

$$L_f B_0(p, z) = -\, v \frac{p^\top}{|p|} z - t_0 \frac{v^2}{|p|} \left(\frac{p^\top}{|p|} S^\top z \right)^2. \tag{34}$$

Define the set of critical orientations

$$\Psi_0 := \{(p, z, q) \in \mathcal{X} \\ L_g B_0(p, z) = 0 \text{ and } L_f B_0(p, z) \geq 0\}. \tag{35}$$

The last term of (34) vanishes when $L_g B_0(p, z) = 0$. It follows that

$$\frac{p^\top}{|p|} z = -1, \quad \forall (p, z, q) \in \Psi_0. \tag{36}$$

Assume B_0 defines $\mathcal{K}_0 = \{(p, z, q) \in \operatorname{Int}\mathcal{X} : B_0(p, z) \leq 0\}$. We note that $\Psi_0 \cap \mathcal{K}_0 \neq \emptyset$, which motivates the use of a hybrid CBF.

D. Synergistic CBF: Part I

1) Shifting the critical orientations: We seek a function $B_1 : X \times \mathcal{S}^1 \times Q_1 \to \mathbb{R}$ that shifts the orientations with vanishing control authority. To this end, define $P : X \times \mathcal{S}^1 \times Q_1 \to [-1, 1]$ as

$$\begin{aligned} P(p, z, q_1) :=& \frac{p^\top}{|p|} R(z^{q_1 k_1}) z \\ =& \cos(q_1 k_1) \frac{p^\top}{|p|} z + \sin(q_1 k_1) \frac{p^\top}{|p|} S z, \end{aligned} \tag{37}$$

with $k_1 \in (0, \frac{\pi}{2})$. The rotation matrix $R(z^{q_1 k_1})$ shifts the critical orientations $q_1 k_1$ radians in the counterclockwise direction. The Lie derivatives of P are given by

$$L_g P(p, z, q_1) = \frac{p^\top}{|p|} R(z^{q_1 k_1}) S z, \tag{38}$$

$$L_f P(p, z, q_1) = \frac{v}{|p|} \frac{p^\top}{|p|} S z \frac{p^\top}{|p|} R(z^{q_1 k_1}) S z. \tag{39}$$

When the vehicle is oriented directly towards the center of the obstacle we have, $\forall(p,z,q) \in \Psi_0, \forall q_1 \in Q_1$,

$$P(p,z,q_1) = -z^\top R(z^{k_1})z = -\cos(k_1) < 0 \tag{40}$$
$$|L_g P(p,z,q_1)| = |z^\top R(z^{k_1})Sz| = \sin(k_1) > 0. \tag{41}$$

We are now ready to define B_1 as

$$B_1(p,z,q_1) := r_s - |p| - t_1 v\left(P(p,z,q) - \sin(k_1)\right), \tag{42}$$

where $t_1 \in \mathbb{R}_{>0}$ is a time constant. Assume that B_1 defines

$$\mathcal{K}_1 := \{(p,z,q) \in \mathrm{Int}\mathcal{X} : B_1(p,z,q_1) \leq 0\}. \tag{43}$$

The constant $\sin(k_1)$ ensures that forward invariance of $\mathcal{K}_1$ implies safety, as stated in the following proposition.

Proposition 3. *For the system (29), if the set $\mathcal{K}_1$ is forward invariant, so is the set $\mathcal{K}_1 \setminus \mathcal{K}_u$. Moreover, any solution starting in $\mathcal{K}_1 \cap \mathcal{K}_u$ will converge towards $\mathcal{K}_1 \setminus \mathcal{K}_u$.*

Proof. The only way a solution can leave or enter the set $\mathcal{K}_u$ is by evolution of p. We prove the proposition by showing that $|p|$ is non-decreasing in $\mathcal{K}_1 \cap \partial\mathcal{K}_u$, and strictly increasing in $\mathcal{K}_1 \cap \mathcal{K}_u$. The time derivative of $|p|$ is given by

$$\frac{d}{dt}|p| := \nabla_p(|p|)\dot{p} = v\frac{p^\top}{|p|}z. \tag{44}$$

Whenever $|p|$ is increasing we have

$$\frac{p^\top}{|p|}z \geq 0 \implies \frac{p^\top}{|p|}z \geq P(p,z,q_1) - \sin(k_1). \tag{45}$$

This can be realized from some manipulations on (37). We continue by establishing lower bounds for $P(p,z,q_1) - \sin(k_1)$. By definition, $B_1(p,z,q_1) \leq 0,\ \forall(p,z,q) \in \mathcal{K}_1$. Furthermore, $|p| - r_s = 0,\ \forall(p,z,q) \in \partial\mathcal{K}_u$ and $|p| - r_s < 0,\ \forall(p,z,q) \in \mathcal{K}_u$. This implies that

$$P(p,z,q_1) - \sin(k_1) \geq 0,\ \forall(p,z,q) \in \mathcal{K}_1 \cap \partial\mathcal{K}_u, \tag{46}$$
$$P(p,z,q_1) - \sin(k_1) > 0,\ \forall(p,z,q) \in \mathcal{K}_1 \cap \mathcal{K}_u. \tag{47}$$

It follows that $|p|$ is non-decreasing for $(p,z,q) \in \mathcal{K}_1 \cap \partial\mathcal{K}_u$, and strictly increasing for $(p,z,q) \in \mathcal{K}_1 \cap \mathcal{K}_u$. $\square$

2) Synergy gap for B_1: We have that

$$L_g B_1(p,z,q_1) = -t_1 v L_g P(p,z,q_1) \tag{48}$$
$$L_f B_1(p,z,q_1) = -v\frac{p^\top}{|p|}z - t_1 v L_f P(p,z,q_1). \tag{49}$$

As before, define the set of critical orientations

$$\Psi_1 := \{(p,z,q) \in \mathcal{X} : L_g B_1(p,z,q_1) = 0 \text{ and } L_f B_1(p,z,q_1) \geq 0\}. \tag{50}$$

The following proposition states that the critical orientations correspond to $P(p,z,q) = -1$.

Proposition 4. *For the set Ψ_1, the following holds:*

$$P(p,z,q_1) = -1,\ \forall(p,z,q) \in \Psi_1. \tag{51}$$

Proof. It is trivial to verify that $L_g B_1(p,z,q_1) = 0 \implies L_g P(p,z,q_1) = 0 \implies P(p,z,q_1) = \pm 1$. It remains to show that $P(p,z,q_1) \neq 1,\ \forall(p,z,q) \in \Psi_1$. Since

$$L_f P(p,z,q_1) = \frac{v}{|p|}\frac{p^\top}{|p|}Sz L_g P(p,z,q_1) = 0, \tag{52}$$

we have that $L_f B_1(p,z,q_1) = -v\frac{p^\top}{|p|}z,\ \forall(p,z,q) \in \Psi_1$. Inserting $P(p,z,q_1) = 1$ into (45) yields $\frac{p^\top}{|p|}z \geq 1 - \sin(k_1) > 0$, which implies $L_f B_1(p,z,q_1) < 0$, which is not contained in Ψ_1. It follows that $P(p,z,q_1) \neq 1, \forall(p,z,q) \in \Psi_1$. $\square$

We can now show that B_1 has a non-zero synergy gap. Define

$$M_1(p,z) := \min_{q_1 \in Q_1} B_1(p,z,q_1), \tag{53}$$

$$\begin{aligned}\mu_1 &:= \inf_{(p,z,q)\in\Psi_1}\{B_1(p,z,q_1) - M_1(p,z)\} \\ &= t_1 v(1 - \cos(k_1)) > 0.\end{aligned} \tag{54}$$

Since $\mu_1 > 0$, B_1 is an SCBF candidate.

Remark 3. *If X contained the origin, the synergy gap would be zero, since $B_1(0,z,q_1) - M(0,z) = 0\ \forall(z,q_1) \in \mathcal{S}^1 \times Q_1$. Moreover, $L_f P \to \pm\infty$ as $|p| \to 0$. Since X does not contain the origin this does not pose a problem.*

E. Synergistic CBF: Part II

Define $B : \mathcal{X} \to \mathbb{R}$ as

$$B(p,z,q) := (1 - q_0)B_0(p,z) + q_0 B_1(p,z,q_1). \tag{55}$$

Note that $B = B_0$ when $q_0 = 0$, and $B = B_1$ when $q_0 = 1$. Assume that B defines the set

$$\mathcal{K} := \{(p,z,q) \in \mathrm{Int}\mathcal{X} : B(p,z,q) \leq 0\}. \tag{56}$$

Further define

$$\Psi := \{(p,z,q) \in \mathcal{X} : L_g B(p,z,q) = 0 \text{ and } L_f B(p,z,q) > 0\}, \tag{57}$$
$$M(p,z) := \min_{q \in Q} B(p,z,q) \tag{58}$$
$$\mu := \inf_{(p,z,q)\in\Psi}\{B(p,z,q) - M(p,z)\}. \tag{59}$$

B is a synergistic CBF candidate for (29) if $\mu > 0$. Sufficient conditions for $\mu > 0$ is given in the following proposition.

Proposition 5. *B defined in (55) has non-zero synergy gap if*

$$t_0 > t_1(\cos(k_1) + \sin(k_1)) > 0. \tag{60}$$

Proof. We begin by noting that $\Psi \subset \Psi_0 \cup \Psi_1$. From (53-54) we obtain $B(p,z,q) - M(p,z) = B_1(p,z,q) - M_1(p,z) = \mu_1 > 0\ \forall(p,z,q) \in \Psi_1$. It remains to verify that (60) ensures $B(p,z,q) - M(p,z) > 0,\ \forall(p,z,q) \in \Psi_0$. We have, $\forall(p,z,q) \in \Psi_0$,

$$B_0(p,z) - B_1(p,z,q_1) = t_0 v - t_1 v(\cos(k_1) + \sin(k_1)). \tag{61}$$

Thus, if (60) is satisfied, we have
$\mu = \min\{\mu_1, v(t_0 - t_1(\cos(k_1) + \sin(k_1)))\} > 0$. $\square$

F. Safety-critical controller

Select a $\delta > 0$, satisfying $\delta < \mu$, and define

$$D := \{(p, z, q) \in \mathcal{X} : M(p, z) - B(p, z, q) \leq -\delta\}, \quad (62)$$
$$C := ((\mathbb{R}^2 \times \mathcal{S}^1 \times Q) \setminus D) \cup \partial D. \quad (63)$$

An illustration of C and D is provided in Figure 1. Define

$$H(p, z) := \{q \in Q : B(p, z, q) = M(p, z)\}. \quad (64)$$

Define $U_B : \mathcal{X} \rightrightarrows \mathbb{R}$ as

$$U_B := \{\omega \in U : L_f B(p, z, q) + L_g B(p, z, q)\omega + \alpha(B(p, z, q)) \leq 0\}, \quad (65)$$

with α an extended class-$\mathcal{K}$ function. Since $\Psi \cap C \cap \mathrm{Int}\mathcal{X} = \emptyset$, the critical orientations on $\mathrm{Int}\mathcal{X}$ are fully contained in D. This implies that it is always possible to ensure that $U_B \neq \emptyset, \forall (p, z, q) \in \mathrm{Int}\mathcal{X} \cap C$, by appropriate selection of t_1, t_0, and k_1. Let $F_B : \mathcal{X} \rightrightarrows \mathbb{R}^2 \times \mathcal{S}^1$ be given by

$$F_B(p, z, q) := \{f(z) + g(z)\omega : \omega \in U_B(p, z, q)\}, \quad (66)$$

and define the system

$$\mathcal{H}_B : \begin{cases} \begin{bmatrix} \dot{p} \\ \dot{z} \end{bmatrix} \in F_B(p, z, q), \ \dot{q} = 0 & (p, z, q) \in C \\ q^+ \in H(p, z), \ p^+ = p, \ z^+ = z & (p, z, q) \in D, \end{cases} \quad (67)$$

We conclude this section with the following theorem.

Theorem 2. *For the system $\mathcal{H}_B$ defined in (67), assume that (60) is satisfied, and $U_B \neq \emptyset \ \forall (p, z, q) \in \mathrm{Int}\mathcal{X} \cap C$. Then $\mathcal{K} \setminus \mathcal{K}_u$ is forward invariant.*

Proof. Proposition 5 and the assumption that B defines $\mathcal{K}$ in (56) is sufficient to establish that B is an SCBF candidate for (29). Since U_B is non-empty on $\mathrm{Int}\mathcal{X} \cap C$ by assumption, B is an SCBF. Then, by Theorem 1, $\mathcal{K}$ is forward invariant. It remains to verify that forward invariance of $\mathcal{K}$ implies forward invariance of $\mathcal{K} \setminus \mathcal{K}_u$. This part of the proof is omitted, since it follows a similar logic as proposition 3. □

VI. Numerical simulations

In this section a small simulation study is presented. We first design a nominal controller for path-following, using the line-of-sight (LOS) algorithm [25], with desired path chosen as $x \in \mathbb{R}, y = 0$, traveling in the positive x-direction. The LOS orientation vector is then given by [26]

$$z_{LOS} := \frac{1}{\sqrt{\Delta_{LOS}^2 + y^2}} \begin{bmatrix} \Delta_{LOS} \\ -y \end{bmatrix} \in \mathcal{S}^1. \quad (68)$$

The look-ahead distance Δ_{LOS} is set to 100m. The nominal feedback control law for ω is chosen as

$$\kappa(\tilde{z}) := -\omega_{sat} \frac{\tilde{z}_2}{\sqrt{1 - \lambda^2 \tilde{z}_1^2}}, \quad \tilde{z} = R(z_{LOS})^\top z, \quad (69)$$

which is a non-hybrid version of the hybrid kinematic controller presented in [27]. We select $\omega_{sat} = 8\frac{\pi}{180}$ [rad/s] (corresponding to 8 [deg/s]) and $\lambda = 0.9$. When $\kappa(\tilde{z}) \notin U_B$, ω is set to

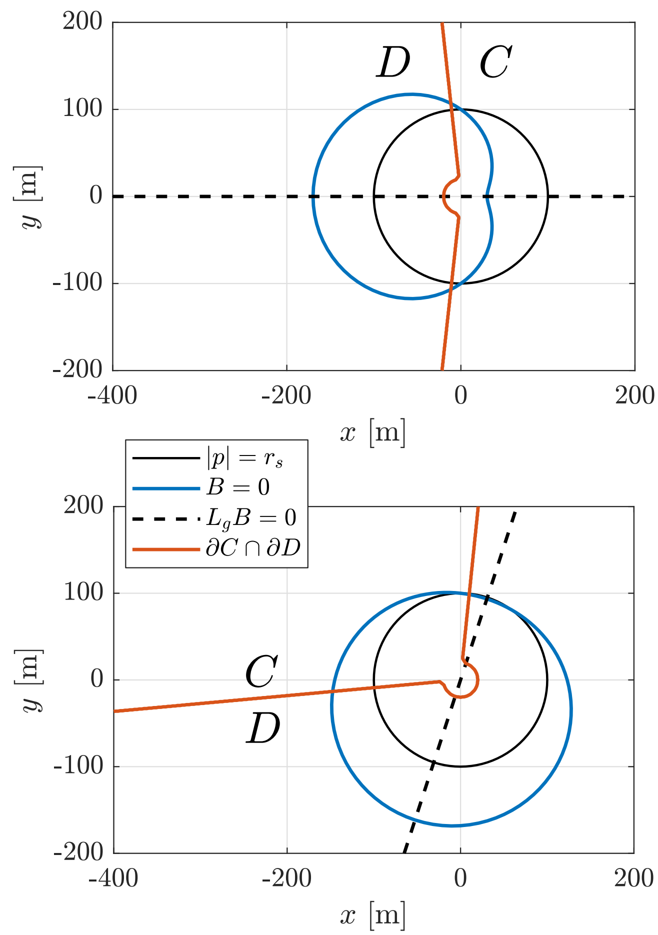

Fig. 1. Illustration of flow set C and jump set D, projected onto $\mathbb{R}^2$, for $q_0 = 0$ (top), and $q_0 = q_1 = 1$ (bottom). Fixed $z = [1 \ 0]^\top$, i.e. vehicle traveling from left to right. Parameters; $v = 5$, $r_s = 100$, $\epsilon = 20$, $k_1 = \frac{\pi}{3}$, $t_1 = 7.16$, $t_0 = t_1(1 + \sin(k_1))$ and $\delta = \frac{\mu}{4}$.

$$\omega_B := -\frac{L_f B(p, z, q) + \alpha(B(p, z, q))}{L_g B(p, z, q)}, \quad (70)$$

saturated by ω_{sat}. If $|\omega_B| \leq \omega_{sat}$, we have $\omega_B \in \partial U_B$. The safety radius about the origin is set to $r_s = 100$m. We further select $v = 5$, $k_1 = \frac{\pi}{2.5}$, $t_1 = \omega_{sat}^{-1}$, $t_0 = t_1(\sin(k_1) + 1)$, (which yields $\mu = \mu_1 > 0$), $\delta = \frac{\mu}{4}$, and $\alpha(B) = 0.5B$. A simulation is initialized with $p(0, 0) = [-250 \ 20]^\top$, $z(0, 0) = [1 \ 0]^\top$, and $q_0(0, 0) = 0$. The resulting trajectory is shown in Figure 2, while the corresponding control input and logic variable q_0 is shown in Figure 3. Since the vehicle is approaching the obstacle, q_0 immediately toggles to 1. Since q_1 also toggles to 1, the evasive maneuver is performed with an initial counterclockwise turn. Following that the vehicle travels along the boundary of $\mathcal{K}_u$, before converging to the desired path. Note that, for this example q_0, toggles back to 0 after the SCBF becomes inactive, meaning that similar performance would have been achieved with the SCBF B_1.

VII. Conclusion and future work

This paper proposed synergistic CBFs as a tool to ensure safety of continuous-time systems with limited control au-

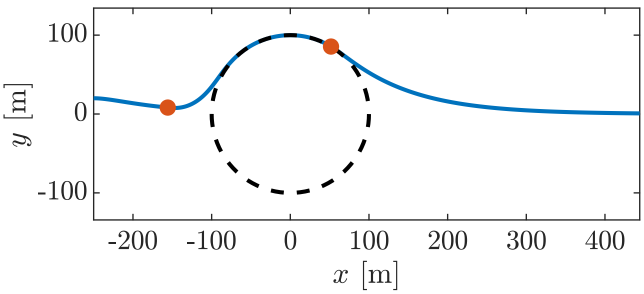

Fig. 2. Trajectory of p in the $\mathbb{R}^2$-plane. The vehicle is traveling from left to right. The red dots mark where the CBF becomes active and inactive. Boundary of $\mathcal{K}_u$ shown in dotted black lines, corresponding to $|p| = r_s$.

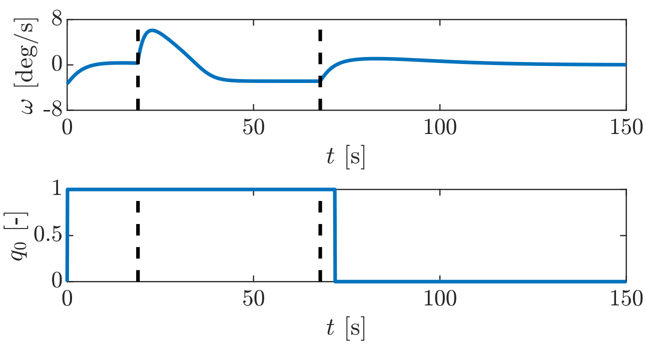

Fig. 3. ω (top) and q_0 (bottom) corresponding to the trajectory in Figure 2. The dashed horizontal lines indicate when the CBF becomes active and inactive.

thority in certain configurations. By shifting the critical orientations, an SCBF for obstacle avoidance for nonholonomic vehicles was constructed. The performance of the SCBF was illustrated by a numerical simulation. In future work, higher derivatives of the turning rate will be considered as control input, such as acceleration or torque input. Suitable applications are vehicles with limited speed envelope, e.g. underactuated ships or fixed-wing aircraft.

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