

Regret-Optimal Full-Information Control

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Abstract—We consider the infinite-horizon, discrete-time full-information control problem. Motivated by learning theory, as a criterion for controller design we focus on regret, defined as the difference between the LQR cost of a causal controller (that has only access to past and current disturbances) and the LQR cost of a clairvoyant one (that has also access to future disturbances). In the full-information setting, there is a unique optimal non-causal controller that in terms of LQR cost dominates all other controllers, and we focus on the regret compared to this particular controller. Since the regret itself is a function of the disturbances, we consider the worst-case regret over all possible bounded energy disturbances, and propose to find a causal controller that minimizes this worst-case regret. The resulting controller has the interpretation of guaranteeing the smallest possible regret compared to the best non-causal controller that can see the future, no matter what the disturbances are. We show that the regret-optimal control problem can be reduced to a Nehari extension problem, i.e., to approximate an anticausal operator with a causal one in the operator norm. In the state-space setting we obtain explicit formulas for the optimal regret and for the regret-optimal controller (in both the causal and the strictly causal settings). The regret-optimal controller is the sum of the classical H_2 state-feedback law and an n -th order controller (where n is the state dimension of the plant) obtained from the Nehari problem. The controller construction simply requires the solution to the standard LQR Riccati equation, in addition to two Lyapunov equations. Simulations over a range of plants demonstrates that the regret-optimal controller interpolates nicely between the H_2 and the H_∞ optimal controllers, and generally has H_2 and H_∞ costs that are simultaneously close to their optimal values. The regret-optimal controller thus presents itself as a viable option for control system design.

I. INTRODUCTION

In this paper, we consider control through the lens of *regret minimization*. While the literature on control is vast, control theorists have largely studied control in two distinct settings. In one setting, we assume that the disturbances are generated by random processes whose statistics we know (in the Gaussian case this is LQG control, in the iid case with linear controllers it is H_2 control), and the goal is to design a control policy which minimizes the expected control cost. In robust control, there are no distributional assumptions about the noise and we seek to minimize the worst-case gain across all bounded disturbances (for bounded energy or power this is H_∞ control [2], [3], for bounded amplitude it is ℓ_1 control [4]). In the regret minimization framework, instead of trying to design controllers that achieve optimal performance relative to a certain class of disturbances, we seek to design controllers that closely track the performance of some benchmark non-causal controller, *irrespective of how the disturbances are*

generated. The motivation behind regret as performance metric is that the resulting controllers are adaptive: they should obtain good performance regardless of whether the disturbances are stochastic, adversarial, etc. This stands in stark contrast to H_2 and H_∞ control, which generally yield controllers that perform well in the environments they are designed for but whose performance can degrade badly when placed in different environments. This is made transparent in (8).

Regret minimization in control has attracted much recent interest (see, e.g., [5]–[16] and the references therein). In this paper, we will study the so-called full-information control problem, where at every time instant the causal controller has access to the current and past states. Most papers in this area try to design causal controllers that compete with the best static linear state feedback controller selected in hindsight; in other words, they compete with the controller which in every round sets the control action u_t to be Kx_t where x_t is the state and K is a *fixed* matrix selected with full clairvoyant knowledge of the disturbances. We believe this choice of non-causal controller to be rather unnatural:

Why restrict to static linear state feedback?

It is not clear when or whether such a non-causal policy outperforms a causal controller that is allowed to be an arbitrary causal function of the current and past states. We therefore would like to contend that it is more natural to design controllers which compete with the best sequence of control actions selected in hindsight, not just those generated by static linear state feedback. In other words, we seek to design controllers that compete with the optimal sequence of control actions $u_1^* \dots u_{T-1}^*$, without imposing the restriction that the benchmark sequence satisfy $u_t^* = -Kx_t$ for some fixed matrix K .

We utilize the regret metric to design a causal controller by comparing its performance with the unique optimal non-causal controller. We then define the optimal regret as the largest difference between the costs of the causal and the non-causal controller among all bounded energy disturbances. The motivation behind the regret definition is to construct a causal controller aims to mimic the behavior of the non-causal controller by minimizing the regret distance. At the operator level, we show that the regret problem can be reduced to the classical Nehari problem [17]. For the state-space setting, we derive the optimal regret as a simple formula and provide an explicit regret-optimal controller that is given by a simple state-space realization. The resulting controller inherits a finite-dimensional state-space structure from the underlying system and its implementation requires the computation of the standard LQR Riccati equation and two additional Lyapunov equations.

The rest of the paper is organized as follows. In Section

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A short version of this paper has been accepted to the ACC 2021 [1].

II, we present the problem formulation. Section III includes our main results regarding the causal and the strictly causal scenario. In Section IV, we present numerical simulations and Section V includes the proofs outline. Finally, the paper is concluded in Section VI and technical proofs appear in the appendices.

II. THE SETTING AND PROBLEM FORMULATION

In this section, we present the setting and the regret problem in its state-space representation. We then proceed to discuss the regret problem from a general operational theory perspective to expose the fundamental difference between H_∞ and regret-optimal controls.

A. The setting

A time-invariant linear dynamical system is given by

$$x_{t+1} = Ax_t + B_u u_t + B_w w_t. \quad (1)$$

The vector $x_t \in \mathbb{R}^n$ is the state variable, $u_t \in \mathbb{R}^m$ is the control variable which we can dynamically adjust to influence the evolution of the system, and $w_t \in \mathbb{R}^p$ is the disturbance process. It is also assumed that the pair (A, B_u) is stabilizable.

A policy π is defined as a mapping of noise sequences $w = \{w_t\}$ to control sequences $u = \{u_t\}$. At this point, it is convenient to define policy as a mapping of the disturbance sequence and not the states sequence which is a more natural assumption in practice. However, it will be shown that the regret-optimal policy can be implemented as a mapping of the system states only. We mostly focus on the doubly infinite-horizon regime where, for a fixed policy π , the linear-quadratic regulator (LQR) cost is given by

$$\text{cost}(\pi; w) = \sum_{t=-\infty}^{\infty} (x_t^* Q x_t + u_t^* R u_t), \quad (2)$$

where $Q, R \succ 0$ are weight matrices. We remark that for the above infinite sum to have any chance of being finite (and therefore meaningful) it is necessary for the disturbance to be square-summable, i.e., $w = \{w_t\} \in \ell_2$.

In the classical H_2 and H_∞ control problems, the objective is to design a controller which minimizes the LQR cost under different assumptions on w which in turn imply the optimization of some norm. Our approach is different since we aim to design a causal controller (policy) based on a comparison criteria with an unrealizable controller. Specifically, we design a causal controller that minimizes the norm of its LQR cost subtracted from the LQR cost of the best non-causal controller. In other words, the regret-optimal controller aims to mimic the behaviour of a non-causal controller by minimizing a *regret objective*.

More formally, let the set of offline (non-causal) policies be Π^{OFF} which corresponds to policies that have clairvoyant access to the entire noise sequence w . Then, the regret of a policy π is defined as

$$\text{Regret}(\pi) = \sup_{\|w\|_2 \leq 1} \left(\text{cost}(\pi; w) - \inf_{\pi' \in \Pi^{\text{OFF}}} \text{cost}(\pi'; w) \right). \quad (3)$$

The regret criterion is now made clear; the performance of a controller, defined by a policy π , is measured by the worst-case disturbance to the best offline policy. The comparison between the two policies becomes meaningful since the disturbance w plays a role in both costs. Finally, note that the offline policy is optimized over the set of all offline policies and is not restricted to being linear or time-invariant.

Here, we assumed that the disturbance w has bounded energy, i.e., that w is an ℓ_2 sequence. An alternative formulation is to define the infinite horizon cost as $\lim_{T \rightarrow \infty} \frac{1}{T} \text{cost}_T(\pi; w)$ where cost_T is cumulative cost and to assume that the sequence w has bounded power, i.e., $\lim_{T \rightarrow \infty} \frac{1}{T} \sum_{t=0}^{T-1} w_t^* w_t < \infty$. It is known in H_∞ theory that both formulations lead to the same optimal controller

We consider two scenarios for the regret optimization in (3): the causal and the strictly causal controllers. The causal policy is a one that chooses the control signal u_t based on $\{w_i\}_{i \leq t}$, while the strictly causal controller has access to $\{w_i\}_{i < t}$ only. The sets of the causal and strictly causal policies are denoted by Π^C and $\Pi^{\text{S.C.}}$ respectively. Moreover, we restrict our attention in both scenarios to linear policies¹.

Let us summarize the problem formulation as follows:

Problem 1 (Full Information Regret-Optimal Control). *Find a linear causal policy π that solves the optimization problem*

$$\text{Regret}^* = \inf_{\pi \in \Pi^C} \text{Regret}(\pi). \quad (4)$$

Similarly, for the case of a strictly causal policy, the regret is defined as

$$\overline{\text{Regret}}^* = \inf_{\pi \in \Pi^{\text{S.C.}}} \text{Regret}(\pi). \quad (5)$$

In section III, we solve Problem 1 for both scenarios. Specifically, we will provide an explicit formula for the optimal regret value and construct a causal/strictly-causal controllers that achieve the optimal regrets.

B. The regret problem in operator form

We introduce the operator notation for the regret problem. As we will see, despite the fact that the state-space model in (1) is a special case of this notation, it leads to clean exposition of the regret problem and an insightful comparison with the H_∞ control problem.

Consider a linear dynamical system given by

$$s = Fv + Gw,$$

where F and G are causal (lower triangular) block operators. The sequence w corresponds to the disturbance, s is the state sequence and v is a regulating sequence, i.e., a control sequence. An operator (controller) K is a mapping from a sequence w to a sequence v . For a fixed K , the cost is defined as

$$\|s\|_2^2 + \|v\|_2^2.$$

Note here that we intentionally omit the time-horizon which can be either finite, semi-infinite or the doubly-infinite regime that is required for our problem formulation.

¹It is possible to show that the optimal causal controller is linear, but it is beyond the scope of this paper.

To see that the state-space model is a special case of the setting, one can eliminate the weight matrices $Q = Q^{*/2}Q^{1/2}$ and $R = R^{*/2}R^{1/2}$ by defining the transformed state variable $s = \{Q^{1/2}x_t\}$ and the transformed control variable $v = \{R^{1/2}u_t\}$. The LQR cost now takes the required form, i.e.,

$$\text{cost}(\pi; w) = \|s\|_2^2 + \|v\|_2^2.$$

Furthermore, we can choose the causal operators F and G to be lower triangular, doubly-infinite block Toeplitz operators with Markov parameters $F_i = Q^{1/2}A^{i-1}B_uR^{-1/2}$ and $G_i = Q^{1/2}A^{i-1}B_w$, respectively, for $i > 0$.

It is convenient to define the cost operator

$$\begin{bmatrix} s \\ v \end{bmatrix} = \underbrace{\begin{bmatrix} FK + G \\ K \end{bmatrix}}_{T_K} w, \quad (6)$$

for any linear controller K . The LQR cost of a linear controller can now be compactly expressed as

$$\text{cost}(K; w) = w^* T_K^* T_K w.$$

The following result characterizes the optimal non-causal controller, K_0 , which is shown to be a linear function of w and an alternative representation for the LQR cost of a linear operator.

Theorem 1 (The non-causal controller [18]). *The optimal non-causal controller is linear and is given by $v = K_0 w$, where K_0 is the linear operator*

$$K_0 = -(I + F^*F)^{-1}F^*G.$$

Furthermore, for any linear operator K , one can write

$$T_K^* T_K = (K - K_0)^*(I + F^*F)(K - K_0) + T_{K_0}^* T_{K_0}. \quad (7)$$

The optimality of a linear non-causal controller implies that the regret problem can be formulated with respect to an arbitrary offline controller. Note that by (7), the non-causal controller outperforms any controller K disturbance for any w . The results in Theorem 1 are well-known, e.g. [18, Th. 11.2.1], but for completeness, we provide a concise proof of Theorem 1 in Appendix A.

As mentioned earlier, Problem 1 presents a new formulation of control problems that is distinct from conventional H_2 and H_∞ control. It is insightful to compare the objectives of the regret-optimal control and the classical H_∞ formulation. Recall that in both formulations there is a maximization over w that can be replaced with an operator norm. Thus, their objectives can be written as

$$\underbrace{\inf_{\text{causal } K} \|T_K^* T_K\|}_{H_\infty \text{ control}}, \quad \underbrace{\inf_{\text{causal } K} \|T_K^* T_K - T_{K_0}^* T_{K_0}\|}_{\text{regret-optimal control}} \quad (8)$$

The stark difference is now transparent; in H_∞ control, one attempts to minimize the worst-case gain from the disturbance energy to the control cost, whereas in regret-optimal control one attempts to minimize the worst-case gain from the disturbance energy to the regret. It is this latter fact that makes the regret-optimal controller more adaptive. It has as its

baseline the best that any noncausal controller can do, whereas the H_∞ controller has no baseline to measure itself against. The comparison of the regret-optimal controller and the H_2 controller will be discussed extensively after presenting the results in Section III.

C. The Nehari problem

Before proceeding to the main results, we present a fundamental problem that lies at the heart of the solution to the regret problem.

Problem 2 (Nehari Problem [17]). *Given a strictly anti-causal (strictly upper triangular) doubly-infinite block Toeplitz operator U , find a causal (lower triangular) doubly-infinite block Toeplitz operator L , such that $\|L - U\|$ is minimized.*

The Nehari problem seeks the best causal approximation to a strictly anti-causal operator in the operator norm sense. The problem has been widely investigated and the minimal norm can be characterized by the Hankel norm of an operator. As we will see in Theorem 9 and its application to our problem in the next section, when the operator has a state-space structure, the Hankel norm can be computed explicitly and we can also characterize the state-space representation of the approximation L .

Throughout the paper, we will occasionally refer to a γ -optimal solution for the Nehari problem, that is, a solution L_γ that achieves $\|L_\gamma - U\| \leq \gamma$, when such a solution exists.

III. MAIN RESULTS

This section has three parts and contains our main results. In Section III-A, we present the reduction of the regret problem to a Nehari problem in its general operator notation. We then proceed in Section III-B to present the optimal regret value and the regret-optimal controller for the causal scenario in the frequency and time domains. Lastly, in Section III-C results for the strictly causal scenario.

A. Reduction to a Nehari problem

The following theorem presents the relation of the regret and the Nehari problems.

Theorem 2 (Reduction to the Nehari problem). *The optimal regret can be formulated as the Nehari problem*

$$\inf_{\text{causal } K} \|T_K^* T_K - T_{K_0}^* T_{K_0}\| = \inf_{\text{causal } L} \|L - \{\Delta K_0\}_-, \quad (9)$$

where Δ is given by the canonical factorization $\Delta^* \Delta = I + F^*F$, K_0 is the optimal offline controller and $\{\cdot\}_-$ denotes the strictly anti-causal part of an operator.

Furthermore, let L be a solution to the Nehari problem in (9), then a regret-optimal controller is given by

$$K = \Delta^{-1} (L + \{\Delta K_0\}_+), \quad (10)$$

where $\{\cdot\}_+$ denotes the causal part of an operator.

Nehari showed that the minimal value of the approximation problem in (9) is equal to the operator norm of the Hankel operator [17]. In our case, it is the anticausal operator $\{\Delta K_0\}_-$.

However, it is rather involved to compute this minimal value and the optimal operator L unless the operators have a state-space structure [18]. In the following section, we solve the Nehari problem in (9) explicitly for the case of a state-space model. Indeed, Theorem 2 already reveals the technical steps required to derive the controller explicitly. Specifically, we will need to perform a canonical factorization of the positive operator $I + F^*F$ and a decomposition of ΔK_0 into its causal and strictly anticausal parts. The proof of Theorem 2 appears in Section V-D.

B. Solution for Problem 1 in the state-space setting

We proceed to present our main results regarding the optimal regret value, the solution to the Nehari problem in our setting and the causal, regret-optimal controller both in frequency and in state-space representations. To preserve the paper flow, proofs of the results in this section appear in Section V.

It is useful to define $P \succeq 0$ as the unique stabilizing solution to the standard LQR Riccati equation

$$P = Q + A^*PA - A^*PB_u(R + B_u^*PB_u)^{-1}B_u^*PA, \quad (11)$$

and $K_{lqr} = (R + B_u^*PB_u)^{-1}B_u^*PA$ be the LQR controller and $A_K \triangleq A - B_uK_{lqr}$.

The first result is an explicit formula for the optimal regret value using the maximal singular value of a matrix denoted by $\bar{\sigma}(\cdot)$.

Theorem 3 (The optimal regret value). *The optimal regret for the causal scenario is given by*

$$\text{Regret}^* = \bar{\sigma}(Z\Pi), \quad (12)$$

where Z and Π are the unique solutions for the Lyapunov equations

$$\begin{aligned} Z &= A_K Z A_K^* + B_u(R + B_u^*PB_u)^{-1}B_u^* \\ \Pi &= A_K^* \Pi A_K + A_K^* P B_w B_w^* P A_K. \end{aligned} \quad (13)$$

The characterization of the optimal regret here follows from the explicit formulas for the Hankel norm of an operator when it has a state-space structure. More specifically, when the operator can be represented with a state-space model, its Hankel norm can be computed as the maximal singular value of its controllability and observability Gramians product.

Before presenting the regret-optimal controller for the causal scenario, we present a technical result that lies at the heart of the regret-optimal controller. Specifically, we will provide the solution to the Nehari problem for our problem, i.e., the one with the anticausal part of $\Delta(z)K_0(z)$. For convenience, we denote the strictly anticausal part of $\Delta(z)K_0(z)$ as $T(z)$. The explicit expression for $T(z)$ is not necessary here and will be provided in the sequel.

Lemma 1. *For any $\gamma \geq \sqrt{\text{Regret}^*}$, a γ -optimal solution to the Nehari problem with the anticausal transfer function $T(z)$ (given in (39)) is*

$$L_\gamma(z)$$

$$= -(R + B_u^*PB_u)^{-*/2} B_u^* \Pi (I + F_\gamma(zI - F_\gamma)^{-1}) K_\gamma, \quad (14)$$

where

$$\begin{aligned} K_\gamma &= (I - A_K Z_\gamma A_K^* \Pi)^{-1} A_K Z_\gamma A_K^* P B_w \\ F_\gamma &= A_K - K_\gamma B_w^* P A_K, \end{aligned} \quad (15)$$

Π is given in (13) and Z_γ is obtained as the unique solution for the Lyapunov equation

$$Z_\gamma = A_K Z_\gamma A_K^* + \gamma^{-2} B_u(R + B_u^*PB_u)^{-1} B_u^*. \quad (16)$$

Recall that a γ -optimal solution to the Nehari problem implies that the causal approximation L_γ induces an operator norm at most γ . Thus, any γ induces a solution to the Nehari problem which in turn induces a controller that achieves a regret γ . Obviously, the regret-optimal controller is revealed when $\gamma = \sqrt{\text{Regret}^*}$. With some abuse of notation, from here, γ refers to the minimal value such that there exists a solution, i.e., $\sqrt{\text{Regret}^*}$.

We are now ready to present the regret-optimal controller using the solution to the Nehari problem. The first representation of the optimal controller is given in the frequency domain. Its main purpose is to reveal the close relation with the H_2 controller, while a more explicit regret-optimal controller is given in Theorem 5.

Theorem 4 (Regret-optimal controller in frequency domain). *Given $\gamma \geq \sqrt{\text{Regret}^*}$, a γ -optimal regret controller for the causal scenario is given by*

$$K(z) = \Delta^{-1}(z)(L_\gamma(z) + S(z)), \quad (17)$$

where

$$\Delta^{-1}(z) = R^{1/2}(I - K_{lqr}(zI - A_K)^{-1}B_u)(R + B_u^*PB_u)^{-1/2}$$

$$S(z) = -(R + B_u^*PB_u)^{-*/2} B_u^*(PA(zI - A)^{-1} + P)B_w,$$

$L_\gamma(z)$ is given in (14) and the triplet (P, K_{lqr}, A_K) is given in (11).

The regret-optimal controller can also be expressed as

$$K(z) = H(zI - F)^{-1}G + J, \quad (18)$$

where

$$\begin{aligned} H &= -R^{1/2}((R + B_u^*PB_u)^{-1}B_u^*\Pi F_\gamma \quad K_{lqr}) \\ F &= \begin{pmatrix} F_\gamma & 0 \\ -B_u(R + B_u^*PB_u)^{-1}B_u^*\Pi F_\gamma & A_K \end{pmatrix} \\ G &= \begin{pmatrix} K_\gamma \\ B_w - B_u(R + B_u^*PB_u)^{-1}B_u^*(PB_w + \Pi K_\gamma) \end{pmatrix} \\ J &= -R^{1/2}(R + B_u^*PB_u)^{-1}B_u^*(PB_w + \Pi K_\gamma). \end{aligned} \quad (19)$$

To assess the essential difference between the regret-optimal controller and the classical H_2 controller, let us write the non-causal controller as

$$\Delta^{-1}(z)\Delta(z)K_0(z) = \Delta^{-1}(z)(S(z) + T(z)), \quad (20)$$

where $S(z)$ and $T(z)$ represent the decomposition of $\Delta(z)K_0(z)$ to its causal and strictly anticausal transfer functions, respectively. It can be shown that the expression $\Delta^{-1}(z)S(z)$ is the classical H_2 controller. It is now evident

that in H_2 control the anticausal part $T(z)$ is being omitted from the optimal controller. This fact should be of no surprise since in the H_2 norm (the Frobenius norm), the controller has no effect on the cost induced by the anticausal part $T(z)$.

However, in the regret-optimal control, the transfer function $T(z)$ is meaningful and is replaced with its causal approximation $L_\gamma(z)$. Thus, the controller may be viewed as a correction to the classical H_2 controller. Following this relation with the H_2 controller, we proceed to show that the regret-optimal controller has a state-feedback law of the H_2 controller with a correction that is driven by a *Nehari state space*.

Theorem 5 (Regret-optimal controller in time domain). *For any $\gamma \geq \sqrt{\text{Regret}^*}$, a γ -optimal regret controller for the causal scenario is given by*

$$u_t = \hat{u}_t - K_{lqr}x_t - (R + B_u^*PB_u)^{-1}B_u^*PB_w w_t, \quad (21)$$

where (P, K_{lqr}) is given in (11) and \hat{u}_t is the (scaled) solution to the Nehari problem, that is,

$$\begin{aligned} \xi_{t+1} &= F_\gamma \xi_t + K_\gamma w_t \\ \hat{u}_t &= -(R + B_u^*PB_u)^{-1}B_u^*\Pi(F_\gamma \xi_t + K_\gamma w_t) \end{aligned} \quad (22)$$

with (F_γ, K_γ) given in (15).

Recall that the optimal H_2 (causal) controller has the state-feedback law

$$u_t^{H_2} = -K_{lqr}x_t - (R + B_u^*PB_u)^{-1}B_u^*PB_w w_t. \quad (23)$$

Thus, the regret-optimal controller is the classical H_2 state-feedback law with an additional state \hat{u}_t driven by the state-space in (22). The causal approximation results in an increased dimension due to the tracking of the hidden state ξ_t . However, from computational complexity perspective, the regret-optimal controller requires a solution to the standard LQR Riccati equation with two additional Lyapunov equations to obtain Z_γ and Π . Compared to H_∞ problem, there are no indefinite factorizations. By examining the structure of the controller (22), one can conclude the practical implications that the controller does not need a direct access to the disturbance but to the state. This is states in the following result.

Corollary 1. *The regret-optimal controller can be implemented with access to the state x_t (rather than the disturbance w_t).*

C. Solution for the the strictly causal scenario

In this section, we present the optimal regret value and the regret-optimal controller both in frequency domain and in state-space representation. The structure of the solutions is similar to the causal scenario. To distinguish from the causal scenario but be consistent, we add an overline over constants that correspond to the strictly causal scenario.

Theorem 6. *The optimal regret for strictly causal scenario is*

$$\overline{\text{Regret}^*} = \bar{\sigma}(Z\bar{\Pi}), \quad (24)$$

where Z and $\bar{\Pi}$ are obtained as the solutions to the Lyapunov equations

$$Z = A_K Z A_K^* + B_u(R + B_u^*PB_u)^{-1}B_u^*$$

$$\bar{\Pi} = A_K^* \bar{\Pi} A_K + P B_w B_w^* P. \quad (25)$$

Note that Z is the same as the one used to compute the regret in the causal scenario.

Theorem 7 (Strictly causal regret-optimal controller (Z-domain)). *For any $\gamma \geq \overline{\text{Regret}^*}$, a γ -optimal regret controller for the strictly causal scenario is given by*

$$\bar{K}(z) = \Delta^{-1}(z)(\bar{S}(z) + \bar{L}_\gamma(z)), \quad (26)$$

with

$$\begin{aligned} \Delta^{-1}(z) &= R^{1/2}(I - K_{lqr}(zI - A_K)^{-1}B_u)(R + B_u^*PB_u)^{-1/2} \\ \bar{S}(z) &= -(R + B_u^*PB_u)^{-*/2}B_u^*PA(zI - A)^{-1}B_w \\ \bar{L}_\gamma(z) &= -(R + B_u^*PB_u)^{-*/2}B_u^*\bar{\Pi}(zI - \bar{F}_\gamma)^{-1}\bar{K}_\gamma, \end{aligned} \quad (27)$$

where the triplet (P, K_{lqr}, A_K) is given in (11),

$$\begin{aligned} \bar{K}_\gamma &= (I - A_K Z_\gamma A_K^* \bar{\Pi})^{-1} A_K Z_\gamma P B_w \\ \bar{F}_\gamma &= A_K - \bar{K}_\gamma B_w^* P, \end{aligned} \quad (28)$$

$\bar{\Pi}$ is given in (25) and Z_γ is the solution to the Lyapunov equation

$$Z_\gamma = A_K Z_\gamma A_K^* + \gamma^{-2} B_u (R + B_u^* P B_u)^{-1} B_u^*. \quad (29)$$

Theorem 8 (The strictly causal regret-optimal controller in time-domain). *For any $\gamma \geq \overline{\text{Regret}^*}$, a γ -optimal regret controller strictly causal regret-optimal controller is given by*

$$u_t = \hat{u}_t - K_{lqr}x_t, \quad (30)$$

where \hat{u}_t is given by

$$\begin{aligned} \xi_{t+1} &= \bar{F}_\gamma \xi_t + \bar{K}_\gamma w_t \\ \hat{u}_t &= -(R + B_u^*PB_u)^{-1}B_u^*\bar{\Pi}\xi_t, \end{aligned} \quad (31)$$

and (P, K_{lqr}) is given in (11) and $(\bar{F}_\gamma, \bar{K}_\gamma, \bar{\Pi})$ are given in (28).

In a very similar fashion to the causal scenario, the controller inherits the state-feedback law of the LQR problem. Also, the controller can be implemented using the state only since $\bar{K}_\gamma w_t$ can be easily expressed as a function of the tuple (x_t, x_{t-1}, u_{t-1}) . The proof of the strictly causal scenario appear in Section V-C.

IV. NUMERICAL EXAMPLES

In this section, we empirically study the performance of the regret-optimal controller compared to the traditional H_2 and H_∞ controllers. As mentioned earlier, the performance of any (linear) controller is governed by the transfer operator T_K that maps the disturbance sequence w to the sequences s and v . It will be useful to represent this operator via its transfer function in the z -domain, i.e.,

$$T_K(z) = \begin{bmatrix} F(z)K(z) + G(z) \\ K(z) \end{bmatrix},$$

The squared Frobenius norm of T_K , which is what H_2 controllers minimize, is given by

$$\|T_K\|_F^2 = \frac{1}{2\pi} \int_0^{2\pi} \text{trace}(T_K^*(e^{j\omega})T_K(e^{j\omega})) d\omega,$$

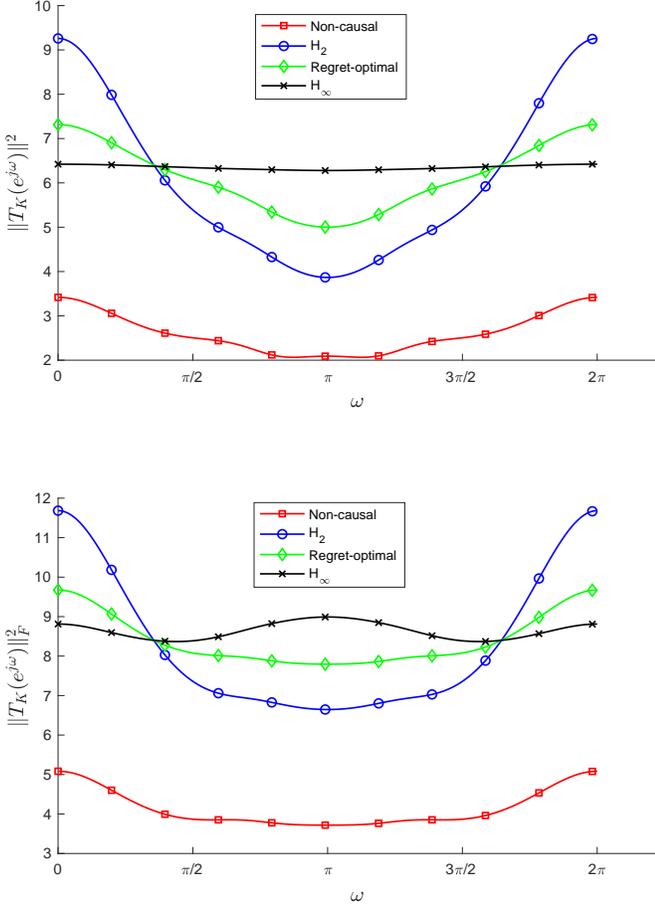


Fig. 1. The cost operator as a function of the frequency for a randomly generated linear dynamical system with six-dimensional state vector and two-dimensional control vector. In the upper figure, the squared operator norm for each of the controller is illustrated while the bottom figure describes the squared Frobenius norm.

and the squared operator norm of T_K , which is what H_∞ controllers minimize, by

$$\sigma_{\max}^2(T_K) = \max_{0 \leq \omega \leq 2\pi} \sigma_{\max}^2(T_K(e^{j\omega})).$$

First, we consider the performance in a randomly generated time-invariant linear dynamical system with $n = 6$ and $m = 2$. All matrices, *i.e.*, A, B_u, B_w, Q, R , are randomly generated and where A is unstable but the pair (A, B_u) is stabilizable. For this setting, we construct the optimal non-causal, H_2 , H_∞ and regret-optimal controllers and establish the transfer operators T_K for each of them. In order to assess and compare the performance of respective controllers across the full range of input disturbances, we plot $\|T_K(e^{j\omega})\|^2$ and $\|T_K(e^{j\omega})\|_F^2$ as a function of frequency in Figure 1.

As can be seen, the non-causal controller outperforms the other three controllers across all frequencies. The H_2 controller minimizes the Frobenius norm, *i.e.*, the average performance over iid w , which is the area under the curve of the bottom figure in Fig. 1. However, in doing so, it sacrifices the worst-case performance and so has a relatively large peak at low frequencies. The H_∞ controller minimizes the operator norm, *i.e.*, the worst-case performance, which is the peak of

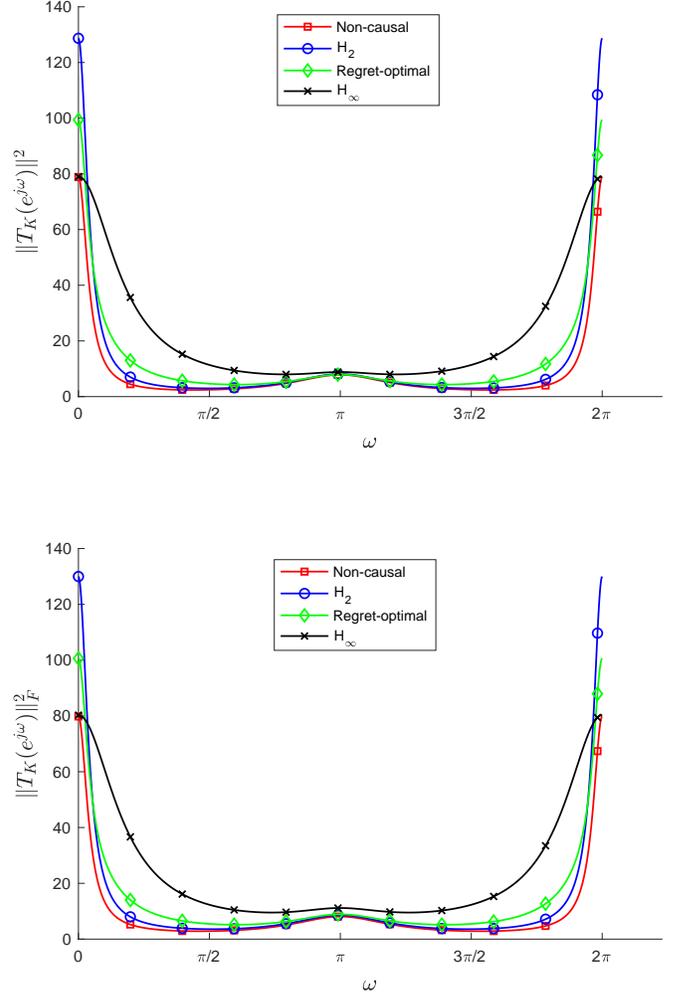


Fig. 2. The cost operator as a function of the frequency for Boeing 747. In the upper figure, the squared operator norm for each of the controller is illustrated while the bottom figure describes the squared Frobenius norm.

the curve. However, in doing so, it sacrifices the average performance and has a relatively large area under the curve. On the other hand, regret-optimal controller nicely finds the best of both worlds between H_2 and H_∞ , *i.e.* stays close to average performance over all frequencies of H_2 and close worst-case performance of H_∞ .

Next, we consider the longitudinal flight control of Boeing 747 with linearized dynamics [19]. For level flight of Boeing 747 at the altitude of 40000ft with the speed of 774ft/sec, for a discretization of 1 second, the dynamics can be represented with a linear dynamical system with

$$A = \begin{bmatrix} .99 & .03 & -.02 & -.32 \\ .01 & .47 & 4.7 & 0 \\ .02 & -.06 & .40 & 0 \\ .01 & -.04 & .72 & .99 \end{bmatrix} \quad B_u = \begin{bmatrix} 0.01 & 0.99 \\ -3.44 & 1.66 \\ -0.83 & 0.44 \\ -0.47 & 0.25 \end{bmatrix}$$

and $B_w = I$. We take $Q = I$ and $R = I$ with appropriate dimensions. For this dynamical system, we also construct the optimal H_2 , H_∞ , and non-causal controllers, as well as the regret-optimal controller. Figure 2 presents $\|T_K(e^{j\omega})\|^2$ and $\|T_K(e^{j\omega})\|_F^2$ for this system, as a function of frequency.

TABLE I
PERFORMANCE OF THE CONTROLLERS IN NUMERICAL EXPERIMENT

	$\ T_K\ _F^2$	$\ T_K\ ^2$	Regret
Noncausal Controller	1.33	78.8	0
Regret-optimal controller	2.16	99.4	20.6
H_2 controller	1.91	128.7	49.95
H_∞ controller	3.73	79	47.97

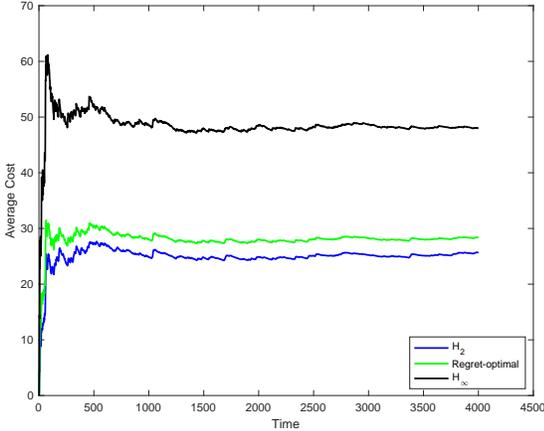


Fig. 3. No-DC Time-domain evaluation of the numerical example: The system is excited with a white Gaussian noise. It can be noted that the H_2 controller, the one designed for such a disturbance characteristics outperforms all causal controllers.

Recall that the regret-optimal controller minimizes

$$\begin{aligned} & \sigma_{\max}(T_K^* T_K - T_{K_0}^* T_{K_0}) \\ & = \max_{0 \leq \omega \leq 2\pi} (T_K^*(e^{j\omega}) T_K(e^{j\omega}) - T_{K_0}^*(e^{j\omega}) T_{K_0}(e^{j\omega})) \quad (32) \end{aligned}$$

i.e., it aims to stay as close as possible to the non-causal controller across all frequencies. In doing so, similar to random system example, it achieves the best of both worlds: an area under the curve that is very close to that of the H_2 -optimal controller (1.91 vs. 2.16) and it has a peak that improves significantly upon the peak of the H_2 controller and close to H_∞ , i.e., its peak is 99.4 rather than the high peak of the H_2 controller. This demonstrates that the regret-optimal controller has a robust and satisfactory performance across a full range of input disturbances w . The precise values of the resulted norms and the regret values of the controllers are summarized in Table I which compares the squared Frobenius, the squared operator norm and the regret value.

We also include time evaluation of longitudinal flight control of Boeing 747 with two random disturbances. First, in Fig. 3, we present the controllers' performance with a white Gaussian noise. As expected, the H_2 controller outperforms the other causal controllers. However, to illustrate the merits of the regret-optimal controller, we add a DC component (i.e., a constant) to a white Gaussian disturbance. In this case, the power ratio between the Gaussian noise will govern the performance of the different controllers. In Fig. 4, we show the evaluation with *medium-range* DC component. Specifically, we extract the eigenvector that corresponds to the largest singular value of $T_K(e^{j\omega} = 1)$ and add it to a random Gaussian noise. In this case, it can be noted in Fig. 4 that

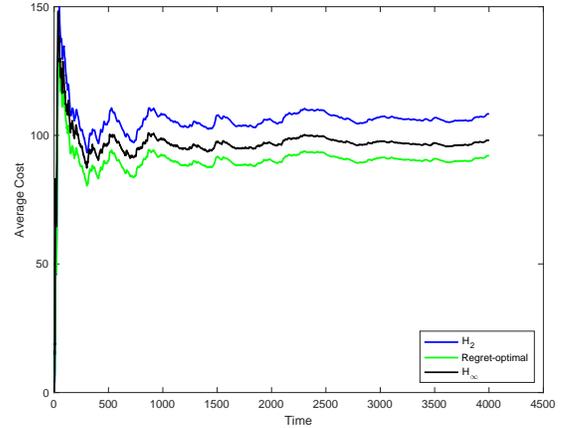


Fig. 4. Medium-DC Time-domain evaluation of the numerical example: The system is excited with a white Gaussian noise that is added to a DC component, i.e., a constant. It can be noted that in this regime where the weight on the constant is in *mid-range*, the regret-optimal controller outperforms all causal controllers.

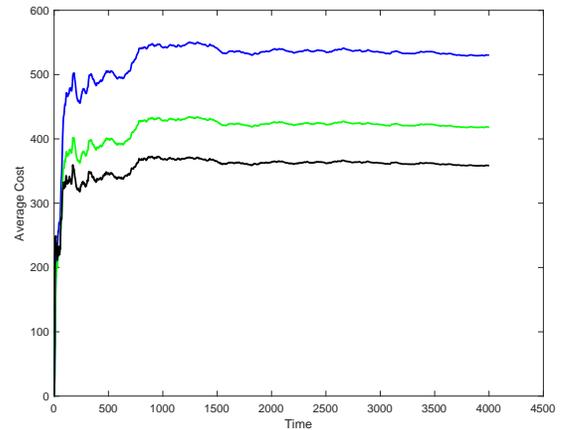


Fig. 5. Strong-DC Time-domain evaluation of the numerical example: The system is excited with a white Gaussian noise that is added to a DC component, i.e., a constant. In this regime, the weight on the constant is *larger*, which causes H_∞ to outperform all causal controller. Note that regret-optimal performs noticeably close to H_∞ controller.

the regret-optimal controller outperforms the H_2 and H_∞ controllers. It can be also seen that the H_2 and the H_∞ are close in their performance, a gap that will diminish if the DC weight is growing large. In Fig. 5, the performance of the controllers is shown when the DC component is doubled. As expected the H_∞ outperform all the causal controllers in this setting whereas H_2 performs poorly. However, note that regret-optimal controller performs remarkably close to H_∞ controller which demonstrates the best of both worlds behavior of regret-optimal controller.

V. DERIVATION OF THE MAIN RESULTS

In this section, we prove the main results in Section III. In Section V-A, we provide a solution to the general Nehari problem. In Section V-B Finally, in Section V-C we provide the proof for the strictly causal scenario.

A. The general Nehari problem

The following theorem summarizes the solution to the Nehari problem for an arbitrary anticausal transfer function.

Theorem 9 (Solution to the general Nehari problem). *Consider the Nehari problem with $T(z) = H(z^{-1}I - F)^{-1}G$ in a minimal form and stable F . The optimal norm is given by*

$$\min_{\text{causal and bounded } L(z)} \|L(z) - T(z)\| = \bar{\sigma}(Z\Pi), \quad (33)$$

where Z and Π are the unique solutions to the Lyapunov equations

$$\begin{aligned} Z &= F^*ZF + H^*H \\ \Pi &= F\Pi F^* + GG^*. \end{aligned} \quad (34)$$

Moreover, for any $\gamma \geq \sqrt{\bar{\sigma}(Z\Pi)}$, a γ -optimal solution to (33) is given by

$$L(z) = H\Pi(I + F_\gamma(zI - F_\gamma)^{-1})K_\gamma, \quad (35)$$

with

$$\begin{aligned} K_\gamma &= (I - F^*Z_\gamma F\Pi)^{-1}F^*Z_\gamma G \\ F_\gamma &= F^* - K_\gamma G^*, \end{aligned} \quad (36)$$

and Z_γ is the solution to the Lyapunov equation

$$Z_\gamma = F^*Z_\gamma F + \gamma^{-2}H^*H. \quad (37)$$

Explicit solution to the *state-space Nehari problem* are known and appear in the control literature, e.g., [20] [21]. To the best of our knowledge, the explicit solution in Theorem 9 has not appeared in the literature and might be of an independent interest. The solution follows directly from algebraic simplification of the general solution that has been characterized in [18]. A proof that is based on [18] and contains the necessary simplifications appears in Appendix B.

B. Proofs of the main results

Recall that in order to apply the solution to the Nehari problem to our problem, one should have explicit expressions for the spectral factorization $\Delta^*(z^{-*})\Delta(z) = I + F^*(z^{-*})F(z)$ and the decomposition of $\Delta(z)K_0(z)$ into its causal and anticausal transfer function. We will now present these technical lemmas and their proof will be provided in Appendix C.

The following lemma presents the required canonical spectral factorization.

Lemma 2 (Spectral factorization). *The transfer function $I + F^*(z^{-*})F(z)$ can be factored as $\Delta^*(z^{-*})\Delta(z)$, where*

$$\begin{aligned} \Delta(z) &= (R + B_u^*PB_u)^{1/2}(I + K_{\text{lqr}}(zI - A)^{-1}B_u)R^{-1/2}, \end{aligned} \quad (38)$$

P is the unique stabilizing solution to the Riccati equation

$$Q - P + A^*PA - A^*PB_u(R + B_u^*PB_u)^{-1}B_u^*PA = 0,$$

and $K_{\text{lqr}} = (R + B_u^*PB_u)^{-1}B_u^*PA$. Furthermore, $\Delta^{-1}(z)$ is causal and bounded on the unit circle.

The following lemma provides the decomposition of the transfer function $\Delta(z)K_0(z)$ to its causal and strictly anticausal counterparts.

Lemma 3 (Decomposition). *The transfer function $\Delta(z)K_0(z) = -\Delta^{-*}(z^{-*})F^*(z^{-*})G(z)$ can be written as a sum of a strictly anticausal and causal transfer functions $T(z)$ and $S(z)$ that are given by*

$$\begin{aligned} T(z) &= -(R + B_u^*PB_u)^{-*/2}B_u^*(z^{-1}I - A_K^*)^{-1}A_K^*PB_w \\ S(z) &= -(R + B_u^*PB_u)^{-*/2}B_u^*P(A(zI - A)^{-1} + I)B_w. \end{aligned} \quad (39)$$

By having these two lemmas, we are ready to prove the main results.

Proof of Theorem 3. To derive the optimal regret value, we apply Theorem 9 with $T(z)$ from Lemma 3. Note that $T(z)$ is bounded on the unit circle since (A, B_u) is stabilizable so that the singular values of A_K are strictly smaller than 1. \square

Recall that Lemma 1 provides an explicit solution to the Nehari problem with $T(z)$ in Section III.

Proof of Lemma 1. The proof follows directly from Theorem 9 with $T(z)$ in Lemma 3 since the conditions for $T(z)$ were verified in the proof of Theorem 3. \square

Proof of Theorem 4. Recall that the optimal controller is given by

$$K(z) = \Delta^{-1}(z)(L_\gamma(z) + S(z)). \quad (40)$$

By Lemma 1, the solution to the Nehari problem is

$$L_\gamma(z) = -(R + B_u^*PB_u)^{-*/2}B_u^*\Pi(I + F_\gamma(zI - F_\gamma)^{-1})K_\gamma. \quad (41)$$

By Lemma 2, we have that

$$\Delta^{-1}(z) = R^{1/2}(I - K_{\text{lqr}}(zI - A_K)^{-1}B_u)(R + B_u^*PB_u)^{-1/2}, \quad (42)$$

and by Lemma 3 the causal part of $\Delta(z)K_0(z)$ is given by

$$S(z) = -(R + B_u^*PB_u)^{-*/2}B_u^*(P(A(zI - A)^{-1} + P)B_w). \quad (43)$$

The proof now follows by computing the products in (40) as

$$\begin{aligned} \Delta^{-1}(z)S(z) &= -R^{1/2}(R + B_u^*PB_u)^{-1}B_u^*PB_w \\ &\quad - R^{1/2}K_{\text{lqr}}(zI - A_K)^{-1}(B_w - B_u(R + B_u^*PB_u)^{-1}B_u^*PB_w), \end{aligned}$$

and

$$\begin{aligned} \Delta^{-1}(z)L_\gamma(z) &= -R^{1/2}(I - K_{\text{lqr}}(zI - A_K)^{-1}B_u)(R + B_u^*PB_u)^{-1}B_u^*\Pi \\ &\quad \cdot (I + F_\gamma(zI - F_\gamma)^{-1})K_\gamma. \end{aligned}$$

Summing these products and arranging the terms in a compact form, we get that the controller is

$$K(z) = H(zI - F)^{-1}G + J, \quad (44)$$

with

$$\begin{aligned} H &= -R^{1/2} \left((R + B_u^* P B_u)^{-1} B_u^* \Pi F_\gamma \quad K_{\text{lqr}} \right) \\ F &= \begin{pmatrix} F_\gamma & 0 \\ -B_u (R + B_u^* P B_u)^{-1} B_u^* \Pi F_\gamma & A_K \end{pmatrix} \\ G &= \begin{pmatrix} K_\gamma \\ B_w - B_u (R + B_u^* P B_u)^{-1} B_u^* (P B_w + \Pi K_\gamma) \end{pmatrix} \\ J &= -R^{1/2} (R + B_u^* P B_u)^{-1} B_u^* (P B_w + \Pi K_\gamma). \end{aligned} \quad (45)$$

□

Proof of Theorem 5. The proof begins with the state-space model in Theorem 4. We then show that one of the hidden states of the controller is the system state x_t . Finally, the proof is completed by observing that the controller is a sum of this state-feedback law and a (scaled) solution to the Nehari problem.

Let ξ^1, ξ^2 be the hidden states of the controller in (18). Then, the controller can be written as

$$\begin{aligned} \begin{pmatrix} \xi_{t+1}^1 \\ \xi_{t+1}^2 \end{pmatrix} &= F \begin{pmatrix} \xi_t^1 \\ \xi_t^2 \end{pmatrix} + G w_t \\ R^{1/2} u_t &= H \begin{pmatrix} \xi_{t+1}^1 \\ \xi_{t+1}^2 \end{pmatrix} + J w_t \end{aligned} \quad (46)$$

First, the control signal can be explicitly written as

$$\begin{aligned} u_t &= -(R + B_u^* P B_u)^{-1} B_u^* \Pi F_\gamma \xi_t^1 - K_{\text{lqr}} \xi_t^2 \\ &\quad - (R + B_u^* P B_u)^{-1} B_u^* (P B_w + \Pi K_\gamma) w_t. \end{aligned} \quad (47)$$

Now, one can show that the evolution of ξ_t^2 is

$$\begin{aligned} \xi_{t+1}^2 &= A \xi_t^2 + B_w w_t + B_u u_t \\ &\stackrel{(b)}{=} x_{t+1}, \end{aligned} \quad (48)$$

where (a) follows from the control signal in (46) and (b) follows from an inductive argument on the evolution of ξ_t^2 .

The last observation is that the evolution of ξ_t^1 is given by the solution to the Nehari problem in Lemma 1, i.e.,

$$\begin{aligned} \xi_{t+1}^1 &= F_\gamma \xi_t^1 + K_\gamma w_t \\ \tilde{u}_t &= (R + B_u^* P B_u)^{-*/2} B_u^* \Pi F_\gamma \xi_t^1 \\ &\quad + (R + B_u^* P B_u)^{-*/2} B_u^* \Pi K_\gamma w_t \end{aligned} \quad (49)$$

To simplify the presentation of the controller, we multiply scale \tilde{u}_t with $-(R + B_u^* P B_u)^{-1/2}$ as $\hat{u}_t = -(R + B_u^* P B_u)^{-1/2} \tilde{u}_t$ and omit the redundant superscript. To conclude, the control signal can be written as

$$u_t = \hat{u}_t - K_{\text{lqr}} \xi_t^2 - (R + B_u^* P B_u)^{-1} B_u^* P B_w w_t. \quad \square$$

C. The strictly causal controller

In this section, we prove the main results on the strictly causal controller.

Proof of Theorems 6-8. As in the causal scenario, the main technical steps are the factorization and the decomposition. The factorization is the one required for the causal scenario in Lemma 2 and the factorization can be deduced directly from

the derivation of the decomposition for the causal scenario. The proof then follows by showing that the strictly causal regret problem reduces to a Nehari problem with the same structure as in Theorem 9. That is, the problem is converted into the approximation of a strictly anticausal transfer function with a causal one.

Using Lemma 3, we can write the decomposition of $-\Delta^{-*}(z^{-*})F^*(z^{-*})G(z)$ as the sum of the transfer functions $\bar{T}(z) = -(R + B_u^* P B_u)^{-*/2} B_u^* (I + (z^{-1}I - A_K^*)^{-1} A_K^*) P B_w$ and $\bar{S}(z) = -(R + B_u^* P B_u)^{-*/2} B_u^* P A (zI - A)^{-1} B_w$, (50)

where $\bar{T}(z)$ is now an anticausal transfer function while $\bar{S}(z)$ is a *strictly causal* transfer function.

In the case of a strictly causal controller, following the same steps in Theorem 2, it can be shown that the regret problem is reduced to the Nehari problem

$$\min_{\text{S. Causal } \bar{L}(z)} \|\bar{L}(z) - \bar{T}(z)\| \quad (51)$$

and a regret-optimal controller is given by

$$\Delta^{-1}(z)(\bar{S}(z) + \bar{L}_\gamma(z)). \quad (52)$$

We will now show that the solution to (51) can be formulated as a solution to the general Nehari problem in Theorem 9. Consider the following chain of equalities

$$\begin{aligned} \min_{\text{S. Causal } \bar{L}(z)} \|\bar{L}(z) - \bar{T}(z)\| &= \min_{\text{S. Causal } \bar{L}(z)} \|z\bar{L}(z) - z\bar{T}(z)\| \\ &\stackrel{(a)}{=} \min_{\text{causal } L'(z)} \|L'(z) - z\bar{T}(z)\|, \end{aligned} \quad (53)$$

where (a) follows from the invertible change of variable $L'(z) = z\bar{L}(z)$.

Since the transfer function $z\bar{T}(z)$ is strictly anticausal, the regret problem is reduced to a new Nehari problem with the same structure as in Theorem 9. Specifically, we will use Theorem 9 with the strictly anticausal transfer function

$$z\bar{T}(z) = -(R + B_u^* P B_u)^{-*/2} B_u^* (z^{-1}I - A_K^*)^{-1} P B_w \quad (54)$$

to conclude the optimal regret value and the solution to the Nehari problem. The optimal regret for the strictly causal scenario is

$$\overline{\text{Regret}}^* = \bar{\sigma}(Z\bar{\Pi}), \quad (55)$$

where Z and $\bar{\Pi}$ are obtained as the solutions to the Lyapunov equations

$$\begin{aligned} Z &= A_K Z A_K^* + B_u (R + B_u^* P B_u)^{-1} B_u^* \\ \bar{\Pi} &= A_K^* \bar{\Pi} A_K + P B_w B_w^* P. \end{aligned} \quad (56)$$

Using Theorem 9, we have $L'(z)$ explicitly which gives the solution to the Nehari problem

$$\begin{aligned} &-\bar{L}_\gamma(z) \\ &= -z^{-1}L'(z) \\ &= z^{-1}(R + B_u^* P B_u)^{-*/2} B_u^* \bar{\Pi} (I + \bar{F}_\gamma (zI - \bar{F}_\gamma)^{-1}) \bar{K}_\gamma \\ &= (R + B_u^* P B_u)^{-*/2} B_u^* \bar{\Pi} (zI - \bar{F}_\gamma)^{-1} \bar{K}_\gamma, \end{aligned}$$

where

$$\begin{aligned}\bar{K}_\gamma &= (I - A_K Z_\gamma A_K^* \bar{\Pi})^{-1} A_K Z_\gamma P B_w \\ \bar{F}_\gamma &= A_K - \bar{K}_\gamma B_w^* P,\end{aligned}\quad (57)$$

$\bar{\Pi}$ is given in (56) and Z_γ is the solution for the Lyapunov equation

$$Z_\gamma = A_K Z_\gamma A_K^* + \gamma^{-2} B_u (R + B_u^* P B_u)^{-1} B_u^*. \quad (58)$$

Note that Z and Z_γ are the same as in the causal scenario.

Therefore, strictly causal regret-optimal controller is

$$\bar{K}(z) = \Delta^{-1}(z) (\bar{S}(z) + \bar{L}_\gamma(z)) \quad (59)$$

The state-space derivation follows from the products

$$\begin{aligned}\Delta^{-1}(z) \bar{S}(z) &= -R^{1/2} K_{lqr} (zI - A_K)^{-1} B_w \\ \Delta^{-1}(z) \bar{L}_\gamma(z) &= -R^{1/2} (I - K_{lqr} (zI - A_K)^{-1} B_u) \\ &\quad \cdot (R + B_u^* P B_u)^{-1} B_u^* \bar{\Pi} (zI - F_\gamma)^{-1} K_\gamma.\end{aligned}$$

□

D. Proof of Theorem 2

We are now moving to show the regret formulation as a Nehari problem in its general operator notation.

Proof of Theorem 2. Consider the following chain of equalities

$$\begin{aligned}& \inf_{\text{causal } K} \sup_{\|w\|_2 \leq 1} (w^* T_K^* T_K w - w^* T_{K_0}^* T_{K_0} w) \\ &= \inf_{\text{causal } K} \|T_K^* T_K - T_{K_0}^* T_{K_0}\| \\ &\stackrel{(a)}{=} \inf_{\text{causal } K} \|\Delta K - \Delta K_0\|^2 \\ &= \inf_{\text{causal } K} \|\Delta K - \{\Delta K_0\}_+ - \{\Delta K_0\}_-\|^2 \\ &\stackrel{(b)}{=} \inf_{\text{causal } L} \|L - \{\Delta K_0\}_-\|^2,\end{aligned}\quad (60)$$

where (a) follows from Theorem 1 and the canonical factorization

$$I + F^* F = \Delta^* \Delta, \quad (61)$$

where Δ is causal and Δ^{-1} is causal and bounded by the positive definiteness of $I + F^* F$. For step (b), note that, for any L , one can recover a causal solution K by setting $K = \Delta^{-1} L + \{\Delta K_0\}_+$. This solution is causal since Δ^{-1} and $\{\Delta K_0\}_+$ are causal operators.

□

VI. CONCLUSIONS

We derived a novel controller based on a regret criteria when compared to a clairvoyant controller with non-causal access to the entire disturbance sequence. The main difference from the classical H_∞ is its robustness against a clairvoyant controller rather than the classical robustness without a reference controller. The implementation of the regret-optimal controller is simple and is published in a public Git repository [22]. As illustrated in the numerical examples, the regret criteria is a viable approach and its potential should be assessed for

other control systems. In two subsequent works, regret-based systems design has been utilized for the filtering problem in [23] and the finite-horizon control problem studied in this paper [24]. An interesting research direction is to extend the regret-optimal controller to the case of partial observability, i.e., measurement-feedback systems. The regret can be reduced to a Nehari problem at the operator level [15], but an explicit solution for state-space systems is still under investigation.

APPENDIX A

DERIVATION OF THE NON-CAUSAL CONTROLLER

The proof is presented for completeness and appeared in previous literature, e.g. [18].

Proof of Theorem 1. For any noise realization w , the cost-minimizing non-causal sequence of control actions with respect to w is the solution of

$$\min_v \|Fv + Gw\|_2^2 + \|v\|_2^2.$$

Completing the square and applying the Matrix Inversion Lemma, the objective can be rewritten as

$$\begin{aligned}& ((I + F^* F)v + F^* Gw)^* (I + F^* F)^{-1} ((I + F^* F)v + F^* Gw) \\ &+ w^* G^* (I + F F^*)^{-1} G w.\end{aligned}\quad (62)$$

Notice that v is arbitrary; in particular we do not assume *a priori* that v is a linear function of w . The second term does not depend on v , while the first term is non-negative and equals zero when v is chosen as $v = -(I + F^* F)^{-1} F^* Gw$. Note that the matrix $(I + F^* F)$ is positive-definite and hence is invertible. Also, note that the choice of v is a linear function of w ; we have shown that the optimal offline policy is to select the control sequence $v = K_0 w$ where $K_0 = -(I + F^* F)^{-1} F^* G$.

To show the second part of the theorem, fix an arbitrary linear controller K . Then, plugging in $v = K w$ into (62) and omitting the sequence w yields

$$T_K^* T_K = (K - K_0)^* (I + F^* F) (K - K_0) + T_{K_0}^* T_{K_0}.$$

□

APPENDIX B

GENERAL SOLUTION TO THE NEHARI PROBLEM (THEOREM 9)

Proof of Theorem 9. By Theorem 12.8.2 in [18], the optimal value of a Nehari problem is the maximal singular value of the Hankel operator of $T(z)$. The Hankel operator can be computed as the product ΠZ , where Z and Π are the controllability and observability Gramians, respectively. Specifically, these can be computed as the solutions to the Lyapunov equations

$$\begin{aligned}\Pi &= F \Pi F^* + G G^* \\ Z &= F^* Z F + H^* H.\end{aligned}\quad (63)$$

The second part is the characterization of the optimal controller which achieves an approximation with norm γ . The optimal controller can be deduced from the proof of Lemma 12.8.1 in [18] when combining with Lemma 12.8.2. However, it requires some further simplifications to have the simple expression in Theorem 9.

Throughout the derivations, we will use their original notation and relate to our notation lastly.

Let P be an hermitian matrix that will be specified by the end of the proof. The transfer function of the (central) optimal approximation which solves the Nehari problem is $-L_{21}(z)L_{11}^{-1}(z)$. The following factorization appears in the proof of Lemma 12.8.1

$$\begin{pmatrix} L_{11}(z) & L_{12}(z) \\ L_{21}(z) & L_{22}(z) \end{pmatrix} = R_e^{1/2} + \begin{pmatrix} -G^* \\ H\Pi F^* \end{pmatrix} (zI - F^*)^{-1} K_p R_e^{1/2} \quad (64)$$

where

$$\begin{aligned} K_p R_e^{1/2} &= \begin{pmatrix} -F^* P G & H^* + F^* P (I + G G^* P)^{-1} F \Pi H^* \\ (I + G^* P G)^{-*/2} & 0 \\ 0 & \phi \end{pmatrix} \\ &= \begin{pmatrix} -F^* P G (I + G^* P G)^{-*/2} & \phi \end{pmatrix} \\ R_e^{1/2} &= \begin{pmatrix} (I + G^* P G)^{1/2} & 0 \\ -H \Pi F^* P G (I + G^* P G)^{-*/2} & \phi \end{pmatrix}. \end{aligned}$$

The notation ϕ denotes terms that are not relevant to the characterization of the central optimal controller.

Simplifying the relevant coordinates in the second term of (64), we obtain

$$\begin{aligned} &\begin{pmatrix} -G^* \\ H \Pi F^* \end{pmatrix} (zI - F^*)^{-1} K_p R_e^{1/2} \\ &= \begin{pmatrix} G^* (zI - F^*)^{-1} F^* P G (I + G^* P G)^{-*/2} & \phi \\ -H \Pi F^* (zI - F^*)^{-1} F^* P G (I + G^* P G)^{-*/2} & \phi \end{pmatrix}, \end{aligned}$$

so that $L_{11}(z)$ and $L_{21}(z)$ can be explicitly expressed as

$$\begin{aligned} L_{11}(z) &= [I + G^* (zI - F^*)^{-1} K_p] (I + G^* P G)^{1/2} \\ L_{21}(z) &= -H \Pi F^* P G (I + G^* P G)^{-*/2} \\ &\quad - H \Pi F^* (zI - F^*)^{-1} F^* P G (I + G^* P G)^{-*/2} \\ &= -H \Pi (I + F^* (zI - F^*)^{-1} F^*) F^* P G (I + G^* P G)^{-*/2}, \end{aligned}$$

where $K_p \triangleq F^* P G (I + G^* P G)^{-1}$.

Recall that the central approximation is given by $-L_{21}(z)L_{11}^{-1}(z)$. Plugging in the expressions we obtained for $L_{21}(z)$ and $L_{11}^{-1}(z)$, we see that the controller is

$$\begin{aligned} &-L_{21}(z)L_{11}^{-1}(z) \\ &= H \Pi (I + F^* (zI - F^*)^{-1} K_p) (I + G^* (zI - F^*)^{-1} K_p)^{-1} \\ &= H \Pi z (zI - F^*)^{-1} (I + K_p G^* (zI - F^*)^{-1})^{-1} K_p \\ &\stackrel{(a)}{=} H \Pi z (zI - F_c)^{-1} K_p \\ &= H \Pi (I + F_c (zI - F_c)^{-1}) K_p, \end{aligned} \quad (65)$$

where (a) follows from $F_c \triangleq F^* - K_p G^*$.

Finally, for any valid γ , we let Z_γ be the unique solutions to the Lyapunov equation

$$Z_\gamma = F^* Z_\gamma F + \gamma^{-2} H^* H.$$

Then, by Lemma 12.8.2, the solution to the Riccati equation is given by $P = (I - Z_\gamma \Pi)^{-1} Z_\gamma$. This explicit solution can also lead to the simplification of K_p as

$$K_p = F^* (I - Z_\gamma \Pi)^{-1} Z_\gamma G (I + G^* (I - Z_\gamma \Pi)^{-1} Z_\gamma G)^{-1}$$

$$\begin{aligned} &= F^* (I - Z_\gamma \Pi + Z_\gamma G G^*)^{-1} Z_\gamma G \\ &= F^* (I - Z_\gamma F \Pi F^*)^{-1} Z_\gamma G, \end{aligned} \quad (66)$$

where (a) follows from the Lyapunov equation $\Pi = F \Pi F^* + G G^*$.

To conclude the proof with our notation, we denote $L(z)$ as the optimal $-L_{21}(z)L_{11}^{-1}(z)$, $K_\gamma = F^* (I - Z_\gamma F \Pi F^*)^{-1} Z_\gamma G$ and $F_\gamma = F^* - K_\gamma G^*$. \square

APPENDIX C TECHNICAL LEMMAS

Lemma 4. The transfer function of $F(z)$ and $G(z)$ are given by

$$\begin{aligned} F(z) &= Q^{1/2} (zI - A)^{-1} B_u R^{-1/2} \\ G(z) &= Q^{1/2} (zI - A)^{-1} B_w. \end{aligned} \quad (67)$$

Therefore, $I + F^*(z^{-*})F(z) = I + R^{-*/2} B_u^* (z^{-1}I - A^*)^{-1} Q (zI - A)^{-1} B_u R^{-1/2}$.

Proof of Lemma 4. The linear operator $\mathcal{T}_F : \mathbf{v} \rightarrow \mathbf{s}$ can be represented as the state-space model

$$\begin{aligned} x_{t+1} &= A x_t + B_u R^{-1/2} v_t \\ s_t &= Q^{1/2} x_t. \end{aligned} \quad (68)$$

Taking the z -transform, we obtain:

$$\begin{aligned} zX(z) &= AX(z) + B_u R^{-1/2} V(z) \\ S(z) &= Q^{1/2} X(z), \end{aligned} \quad (69)$$

so that $F(z) = Q^{1/2} (zI - A)^{-1} B_u R^{-1/2}$.

The transfer function $G(z) = Q^{1/2} (zI - A)^{-1} B_w$ can be obtained similarly from the state-space model

$$\begin{aligned} x_{t+1} &= A x_t + B_w w_t \\ s_t &= Q^{1/2} x_t. \end{aligned} \quad \square$$

Proof of Lemma 2. By Lemma 4, we have

$$\begin{aligned} &I + F^*(z^{-*})F(z) \\ &= I + R^{-*/2} B^* (z^{-1}I - A^*)^{-1} Q (zI - A)^{-1} B R^{-1/2}. \end{aligned} \quad (70)$$

For ease of derivation, we will factor the term $R^{*/2} (I + F^*(z^{-*})F(z)) R^{1/2}$ as $\tilde{\Delta}^*(z^{-*}) \tilde{\Delta}(z)$, and then the required factorization can be recovered as $\Delta(z) = \tilde{\Delta}(z) R^{-1/2}$.

We can express $R^{*/2} (I + F^*(z^{-*})F(z)) R^{1/2}$ in matrix form as

$$\begin{aligned} &(B^* (z^{-1}I - A^*)^{-1} \quad I) \begin{pmatrix} Q & 0 \\ 0 & R \end{pmatrix} \begin{pmatrix} (zI - A)^{-1} B \\ I \end{pmatrix} \\ &= (B^* (z^{-1}I - A^*)^{-1} \quad I) \begin{pmatrix} Q - P + A^* P A & A^* P B \\ B^* P A & R + B^* P B \end{pmatrix} \\ &\quad \cdot \begin{pmatrix} (zI - A)^{-1} B \\ I \end{pmatrix}, \end{aligned} \quad (71)$$

where the equality can be verified directly and holds for any Hermitian matrix P .

The middle matrix in (71) can be factored as

$$\begin{pmatrix} I & \Psi^*(P) \\ 0 & I \end{pmatrix} \begin{pmatrix} \Gamma(P) & 0 \\ 0 & R + B^*PB \end{pmatrix} \begin{pmatrix} I & 0 \\ \Psi(P) & I \end{pmatrix},$$

where

$$\Gamma(P) \triangleq Q - P + A^*PA - A^*PB(R + B^*PB)^{-1}B^*PA$$

and

$$\Psi(P) \triangleq (R + B^*PB)^{-1}B^*PA.$$

Suppose that (A, B) is a stabilizable pair; then the Riccati equation $\Gamma(P) = 0$ has a unique Hermitian solution. Suppose P is chosen to be this solution and define $K_{lqr} = \Psi(P)$. Finally, by defining

$$\tilde{\Delta}(z) = (R + B^*PB)^{1/2}(I + K_{lqr}(zI - A)^{-1}B),$$

we obtain the desired factorization

$$R^{*/2}(I + F^*(z^{-*})F(z))R^{1/2} = \tilde{\Delta}^*(z^{-*})\tilde{\Delta}(z).$$

Recall that $\Delta(z) = \tilde{\Delta}(z)R^{-1/2}$, so

$$\Delta(z) = (R + B^*PB)^{1/2}(I + K_{lqr}(zI - A)^{-1}B)R^{-1/2}.$$

Finally, it remains to check that this choice of $\Delta(z)$ is causal, and its inverse is causal and bounded on the unit circle.

To see that the inverse is bounded, by the Matrix Inversion Lemma, the poles are at the eigenvalues of the matrix $A - BK_{lqr}$. It is a stable since P was chosen to be the unique Hermitian solution to the Riccati equation, and hence its spectral radius is less than 1, which due to the causality of $\Delta^{-1}(z)$ guarantees the boundedness of $\Delta^{-1}(z)$ on the unit circle. \square

Proof of Theorem 3. Recall that we decompose the product $\Delta(z)K_0(z) = -\Delta^{-*}(z^{-*})F^*(z^{-*})G(z)$. The transfer functions are given by

$$\begin{aligned} \Delta^{-*}(z^{-*}) &= (R + B_u^*PB_u)^{-*/2} \\ &\quad \cdot (I + B_u^*(z^{-1}I - A^*)^{-1}K_{lqr}^*)^{-1}R^{*/2} \\ F^*(z^{-*}) &= R^{*/2}B_u^*(z^{-1}I - A^*)^{-1}Q^{*/2} \\ G(z) &= Q^{1/2}(zI - A)^{-1}B_w. \end{aligned} \quad (72)$$

First, consider the $\Delta^{-*}(z^{-*})F^*(z^{-*})$ (omitting constants on the sides)

$$\begin{aligned} &(I + B_u^*(z^{-1}I - A^*)^{-1}K_{lqr}^*)^{-1}B_u^*(z^{-1}I - A^*)^{-1} \\ &= B_u^*(I + (z^{-1}I - A^*)^{-1}K_{lqr}^*B_u^*)^{-1}(z^{-1}I - A^*)^{-1} \\ &= B_u^*(z^{-1}I - A^* + K_{lqr}^*B_u^*)^{-1} \\ &= B_u^*(z^{-1}I - A_K^*)^{-1} \end{aligned} \quad (73)$$

We now multiply $G(z)$ with (73) and apply a decomposition as appear in [18, Lemma 12.3.3],

$$\begin{aligned} &-\Delta^{-*}(z^{-*})F^*(z^{-*})G(z) \\ &= -(R + B_u^*PB_u)^{-*/2}B_u^* \\ &\quad \cdot (z^{-1}I - A_K^*)^{-1}Q(zI - A)^{-1}B_w \\ &= -(R + B_u^*PB_u)^{-*/2}B_u^* \\ &\quad \cdot [(z^{-1}I - A_K^*)^{-1}A_K^*W + WA(zI - A)^{-1} + W]B_w, \end{aligned}$$

where W solves $Q - W + A_K^*WA = 0$. Finally, note that $W = P$ solves the Lyapunov equation. \square

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